Problem set 6

1 Calculate the specific heat (at constant density) and the linear magnetic suscetibility of a free elecron gas of constant density in the low temperature limit. For simplicity, assume that the electronic density of states is $g(\omega) = a\omega^{1/2}$, and keep only the first nonvanishing term in the low temperature expansion. *Explicitely* account for the temperature dependence of the chemical potential in each case. How good was the approximation, made in class, of ignoring the temperature dependence of μ? (The following (Somerfeld) expansion of the fermi function may be useful $f(\epsilon) \approx \theta(\mu - \epsilon) - \frac{\pi^2}{6}(k_B T)^2 \delta'(\epsilon - \mu)$.)

Solution. In order to calculate the temperature dependence of the chemical potential, we must first calculate the electron density n.

$$n = 2\int d\omega g(\omega) f(\omega) \approx 2\int_0^{\mu(T)} d\omega g(\omega) + \frac{2}{6}\pi^2 (k_B T)^2 \left. g'(\omega)\right|_{\omega=\mu} = \frac{4}{3}a\mu^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right)$$

Since the gas has constant density, n does not change. Thus to lowest order in T, we can write

$$n = n\left(\frac{\mu(T)}{\mu_0}\right)^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu_0}\right)^2\right),$$

or solving for the chemical potential

$$\mu(T) - \mu_0 = -\frac{\pi^2}{12}\mu_0\left(\frac{k_B T}{\mu_0}\right)^2$$

Now we are in a position to solve for the energy and specific heat.

$$e = \int d\omega\omega g(\omega) f(\omega) = 2a\int_0^{\infty} d\omega\omega^{3/2} f(\omega) \approx \frac{4}{5}a\mu^{5/2} + \frac{1}{2}a\pi^2 (k_B T)^2 \mu^{1/2}$$

To lowest order in T, this is

$$= \frac{4a}{5}\mu^{5/2}\left(1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right).$$

However, we cannot neglect the temperature dependence of μ

$$e = \frac{3}{5}\mu_0 n\left(1 - \frac{\pi^2}{12}\mu_0\left(\frac{k_B T}{\mu_0}\right)^2\right)^{5/2}\left(1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\mu_0}\right)^2\right)$$

Expanding, and keeping only the lowest powers in T, we obtain

$$e = \frac{3}{5}\mu_0 n\left(1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\mu_0}\right)^2\right).$$

The specific heat at constant density is then given by

$$C = k_B\frac{k_B T}{\mu_0}n\frac{\pi^2}{2} = \frac{2\pi^2}{3}k_B^2 T g(\mu_0)$$

In order to calculate the linear susceptibility, we must determine the magnetization of the gas to lowest order in the applied field H. This is given by the difference of the spin up and down densities. The spin up density is for example

$$n_\uparrow = \int d\omega g(\omega + \mu_B H) f(\omega) \approx \int d\omega \left(g(\omega) + \mu_B H g'(\omega)\right) f(\omega).$$

Then the magnetization is

$$m = \mu_B(n_\uparrow - n_\downarrow) = 2\mu_B^2 H \int g'(\omega) f(\omega) d\omega \,,$$

or, integrating by parts

$$m = -2\mu_B^2 H \int d\omega g(\omega) f'(\omega) \,.$$

Clearly the lowest nonvanishing term is the zero temperature result, for which $f'(w) = -\delta(w - \mu_0)$

$$m = 2\mu_B^2 H g(\mu_0) \quad \chi = 2\mu_B^2 g(\mu_0) = 2\mu_B^2 a \mu_0^{1/2}$$

2. (from Pines and Nozieres) As described in the assignment, this problem reduces to the calculation of $\partial N/\partial\mu$ (or its inverse). We may use Eq. 112, and the formalism leading up to it. Since the local energy of a quasiparticle at the fermi surface is always the chemical potential, when μ increases by an amount $d\mu$, the local distribution of particles changes to $n(\tilde{\epsilon}_{\mathbf{p}} - \mu - d\mu)$. Then

$$\delta\bar{n}_{\mathbf{p}} = -d\mu \frac{\partial n_{\mathbf{p}}}{\partial \epsilon_{\mathbf{p}}}$$

For an isotropic system, $\delta\bar{n}_{\mathbf{p}}$ is isotropic and spin independent (symmetric). Then using Eq. 112, we find

$$\delta n_{\mathbf{p}} = \frac{\delta\bar{n}_{\mathbf{p}}}{1+F_0^s} = -\frac{\partial n_{\mathbf{p}}/\partial\epsilon_{\mathbf{p}}}{1+F_0^s} d\mu$$

If we integrate over all momentum state, then we get

$$\frac{\partial N}{\partial \mu} = \frac{D(E_F)}{1+F_0^s}$$

Then, using the relation given in the assingment, we get for the speed of first sound

$$s^2 = \left(\frac{p_F^2}{3mm^*}\right)(1+F_0^s)$$

Using very similar arguments, we may calculate the Spin susceptibility of a fermi gas to lowest order using the Landau Fermi Liquid theory. You should get

$$\chi_P = \frac{m^* p_F}{\pi^2 \hbar^3} \frac{\beta^2}{1+F_0^a}$$