Waveguides and Cavities

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also known as

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(1842 - 1919)

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Contents

In this chapter we continue with the topic of solutions of the Maxwell equations in the form of waves. This time we seek solutions in the presence of bounding surfaces which may take a variety of forms. The basic possibilities are to have boundaries in

1. one dimension only, such as a pair of parallel planes;

2. two dimensions, such as several intersecting planes forming a pipe or channel; and

3. three dimensions, such as a collection of intersecting planes that completely bound some region of space.

The materials employed to form the boundaries are usually[1] conductors. The mathematical problem is a boundary-value problem for solutions of the Maxwell equations. We shall look at harmonic solutions within the cavity or channel and must match these solutions onto appropriate ones within the walls or bounding materials. If the walls are constructed from a "good" conductor, the boundary conditions become simple and the boundary-value problem itself is not too difficult. This point is explored in the following sections.

1 Reflection and Transmission at a Conducting Wall

We consider the reflection and transmission of a harmonic plane wave incident on a conducting material at a planar surface. We let the incident wave have an arbitrary angle of incidence - which gives a hard problem to solve in the general case - and then imagine that the conductivity is very large - which simplifies the solution by allowing an expansion in a small parameter. Physically, the central point is that if $\sigma >> \omega$,

[1]Dielectric materials are also used, with conditions such that total internal reflection takes place at the surfaces in order to keep the wave within the channel or cavity.

then the skin depth $\delta = c/\sqrt{2\pi\sigma\omega\mu}$ of the wave in the conductor is much smaller than the wavelength λ of the incident wave. The distance over which the fields vary in the conductor depends on the direction. In the direction normal to the surface, this distance is δ; in directions parallel to the surface, it is λ. Thus by having $\delta << \lambda$, we can often ignore variations of the fields parallel to the surface in comparison with variations normal to the surface; in effect, the wave in the conductor travels normal to the surface no matter what the angle of incidence.

1.1 Boundary Conditions

First, let us consider the boundary or continuity conditions at the interface. We can find appropriate conditions by using the Maxwell equations and either Stokes' theorem or Gauss's Law in the usual way. Let us first do this by employing a rectangle or pillbox which has a size t normal to the interface which is much larger than δ. At the same time, the size l of these constructs parallel to the interface must be large compared to t but small compared to λ, so we have the condition

$$\lambda >> l >> t >> \delta \tag{1}$$

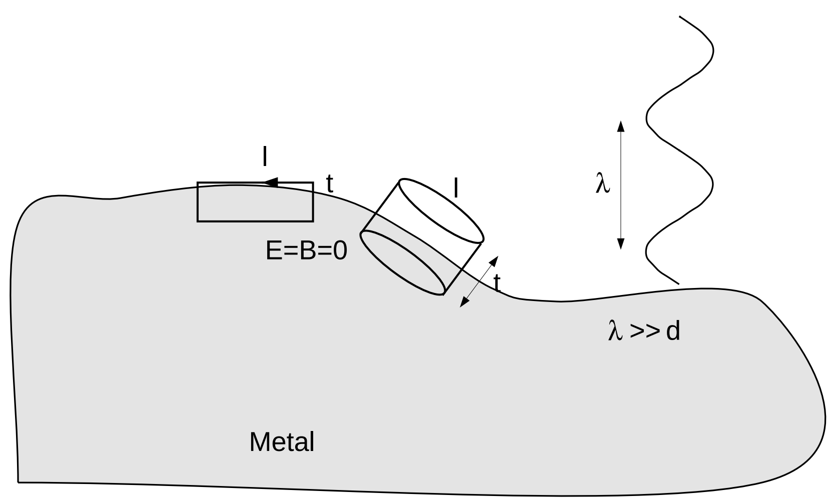

Fig.1: Integration surfaces adjacent to a good conductor.

which can be satisfied by a metal with a large enough conductivity (and an incident radiation with a small enough frequency). Then, because the side of the rectangle, or face of the pillbox, within the conductor is placed in a region where the transmitted fields have been attenuated to very small values, compared to the incident amplitudes, we can say that these fields are zero. The result is that the continuity conditions become

$$\mathbf{n}\cdot\mathbf{D} = 4\pi\sigma_q \qquad \mathbf{n}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{K} \qquad \mathbf{n}\cdot\mathbf{B} = 0 \quad \text{and} \quad \mathbf{n}\times\mathbf{E} = 0 \tag{2}$$

where the fields are those just outside of the conductor, and σ_q and $\mathbf{K}$ are the charge and current density on the conductor's surface[2]; $\mathbf{n}$ is the unit outward normal at the surface of the conductor. These relations are only approximate because we have neglected in particular the term $\partial\mathbf{B}/\partial t$ in Faraday's Law; it gives a correction of order $\omega\delta/c$ times the incident field's amplitude to the statement that the tangential electric field vanishes at the interface. To put it another way, the tangential component of the reflected wave's electric field actually differs from that of the incident wave's electric field by an amount of order $\omega\delta/c$ times the amplitude of the incident wave. In lowest order we ignore this difference. Outside of the conductor, $\partial\mathbf{B}/\partial t$ does not contribute to the integral since we assure the contour has negligible area here.

So far we don't know the surface charge and current densities, but the conditions that the tangential component of $\mathbf{E}$ and the normal component of $\mathbf{B}$ are zero at the interface are already enough to allow us to determine the reflected fields, given the incident ones. Hence we have at this point all of the information we need to obtain, to lowest order in the small parameter, the solution for the waves in the channel or cavity, i.e., the solution to the boundary-value problem posed above.

1.2 Power and Energy Loss

Before going on to look at that problem, however, let's look at the properties of the transmitted wave in the conductor. The reason for doing this is that we want to

[2]it is only appropriate to talk about surface currents or charges in the limit of a perfect conductor; otherwise, these densities will extend into the conductor to a finite extent

know how much energy is lost in the reflection process. To zero order in $\omega\delta/c$, none is lost, as is evident from the boundary condition which says that the amplitude of the reflected wave is the same as that of the incident wave. We must therefore look at the first-order corrections to this result, and that is most easily done by examining the transmitted wave.

From Faraday's Law and Ampère's Law, assuming a harmonic wave, we find that the fields in the conductor, which are identified by a subscript c, obey the relations

$$\mathbf{B}_c = -i\frac{c}{\omega}(\nabla \times \mathbf{E}_c) \qquad \text{and} \qquad \mathbf{E}_c = \frac{c}{4\pi\sigma\mu}(\nabla \times \mathbf{B}_c), \tag{3}$$

where we have ignored the displacement current term because it is of order ω/σ relative to the real current term; μ is the permeability of the conductor. Because the fields vary rapidly in the direction normal to the interface (length scale δ) and slowly in directions parallel to the interface (length scale λ), we may ignore spatial derivatives in all directions except the normal one. The conditions[3] $\nabla \cdot \mathbf{B}_c = 0$ and $\nabla \cdot \mathbf{D}_c = 0$ tell us, to lowest order, that the fields within the conductor have no components normal to the interface. Taking the curl of the second of Eqs. (3), and using the first of these equations for the curl of $\mathbf{E}_c$, we find that

$$\left(i + \frac{c^2}{4\pi\sigma\mu\omega}\nabla^2\right)\mathbf{B}_c = 0, \tag{4}$$

or

$$\left(\nabla^2 + i\frac{2}{\delta^2}\right)\mathbf{B}_c = 0, \tag{5}$$

which has the solution

$$\mathbf{B}_c(z,t) = \mathbf{B}_{c0}e^{\kappa z}e^{-i\omega t} \tag{6}$$

where

$$\kappa = \pm(1-i)/\delta. \tag{7}$$

[3]Of course, the electric displacement has a non-zero divergence if $\rho \neq 0$; as we saw in Jackson 7.7, any initial non-zero ρ dies out with some characteristic lifetime and so when the steady-state is established, $\rho = 0$.

Because the fields must vanish for $z \to \infty$, we have to choose the negative root and so find

$$\mathbf{B}_c = \mathbf{B}_{c0} e^{-z/\delta} e^{iz/\delta} e^{-i\omega t}. \tag{8}$$

Also, as is easily shown from this result and one of Eqs. (3),

$$\mathbf{E}_c = -\frac{c}{4\pi\sigma\mu\delta}(1-i)(\hat{\mathbf{z}} \times \mathbf{B}_c). \tag{9}$$

These fields are the same in form as the ones that arise in the case of normal incidence. The amplitudes are somewhat different from that case, however.

The power per unit area entering the conductor is

$$\begin{aligned} < \mathbf{S} \cdot \hat{\mathbf{z}} > |_{z=0} &= \frac{c}{8\pi} \Re(\mathbf{E}_c \times \mathbf{H}_c^*) \cdot \hat{\mathbf{z}}|_{z=0} \\ = -\frac{c}{8\pi} \Re \left\{ \frac{c}{4\pi\sigma\mu\delta}(1-i)[(\hat{\mathbf{z}} \times \mathbf{B}_c) \times (\mathbf{B}_c^*/\mu)] \cdot \hat{\mathbf{z}}|_{z=0} \right\} & \\ &= \frac{c^2}{32\pi^2\sigma\mu^2\delta} \Re[(1-i)|\mathbf{B}_c|^2]|_{z=0}. \end{aligned} \tag{10}$$

However, $c^2/2\pi\sigma\omega\mu = \delta^2$, so, writing the power per unit area as $\mathcal{P}$, we have

$$\mathcal{P} = \frac{\mu\omega\delta}{16\pi} |H_c|_{z=0}^2. \tag{11}$$

We can relate H_c at $z = 0$ to the field at the interface on the outside of the conductor by employing an appropriate continuity condition. It is not the one derived above. This time, we use a value of t which is much smaller than δ so that there is only a negligible amount of current passing through the rectangle employed in applying Stokes' Theorem. Then we find that H_c at $z = 0$ is the same as the tangential component of the magnetic field on the outside. For definiteness, let the incident field be polarized perpendicular to the plane of incidence. Then the reflected field has an equal and opposite amplitude (to lowest order) and the sum of the incident and reflected waves' magnetic field amplitudes parallel to the interface is[4] twice the

[4]We suppose that the exterior medium is vacuum, or at least has $\mu = \epsilon = 1$.

amplitude of the incident electric field times the cosine of the angle of incidence, or $2E_0 \cos\theta$.

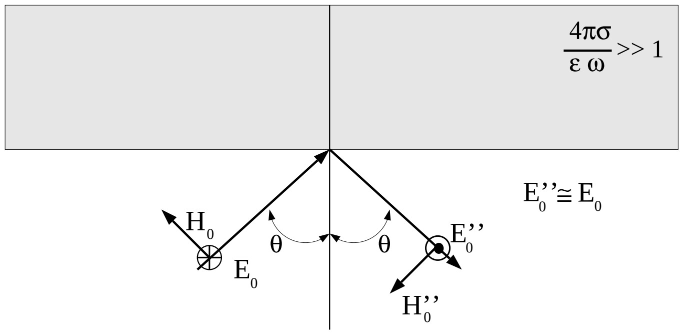

Fig.2: Wave polarized ⊥ to the plane, between vacuum and a good conductor.

Hence the power loss per unit area in the reflection process, meaning the power per unit area entering the conductor and so not reflected, is

$$\mathcal{P} = \frac{\mu\omega\delta}{4\pi}|E_0|^2 \cos^2\theta. \tag{12}$$

The ratio of the lost to incident power, which is also the transmission coefficient, is

$$T = \frac{\mu\omega\delta|E_0|^2 \cos^2\theta/4\pi}{c|E_0|^2 \cos\theta/8\pi} = \frac{2\mu\omega\delta}{c}\cos\theta. \tag{13}$$

This agrees with the result we found earlier in the case of a good conductor and for normal incidence, $\theta = 0$.

We will want to use the result for power loss later in connection with the attenuation of waves travelling along a wave guide. First we shall obtain the solution for the electromagnetic field within the waveguide in the limit of perfectly conducting walls.

2 Wave Guides

A waveguide is a hollow conducting pipe, perhaps filled with dielectric. It has a characteristic transverse size on the order of centimeters and is used to transmit electromagnetic energy (waves) from one place to another.

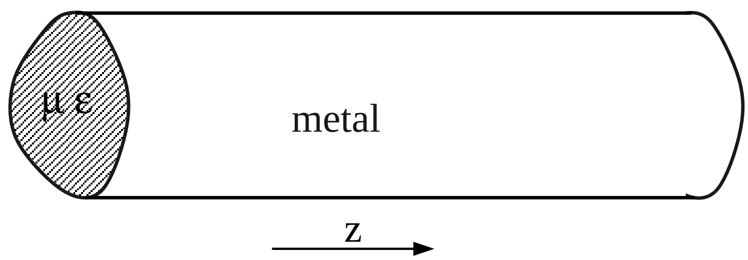

Fig.3: Wave guide of arbitrary cross-section.

The waves typically have frequencies such that the wavelength in vacuum would be comparable to the size of the waveguide. Thus $\omega = 2\pi c/\lambda$ is of order $10^{11}\, sec^{-1}$. If the walls of the guide are constructed of a good conducting material, i.e., one with $\sigma \sim 10^{17}\, sec^{-1}$, then we are in the good conducting limit so that the treatment of the previous section is valid. In particular, $T \sim 10^{-3}$ which means that some 10^3 reflections can take place before the wave is seriously attenuated. Also, we may adopt the boundary conditions that $\mathbf{E}_{tan} = 0 = B_n$ at the conducting surfaces.

2.1 Fundamental Equations

Let the wave guide have its long axis parallel to the z-direction and let its cross-section be invariant under translation along this direction. It is useful to divide operators, such as ∇ and ∇^2, and also fields into components parallel and perpendicular to the long axis. Thus we write

$$\nabla = \nabla_t + \boldsymbol{\epsilon_3}\frac{\partial}{\partial z}, \qquad \nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2},$$
$$\mathbf{E} = \mathbf{E}_t + \boldsymbol{\epsilon_3} E_z, \quad \text{and} \quad \mathbf{B} = \mathbf{B}_t + \boldsymbol{\epsilon_3} B_z. \tag{14}$$

Further we shall assume that the fields' dependence on both z and t is harmonic,

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}(x, y)e^{i(kz-\omega t)} \quad \text{and} \quad \mathbf{E}(\mathbf{x}, t) = \mathbf{E}(x, y)e^{i(kz-\omega t)}. \tag{15}$$

Given ω, we need to find k and the amplitudes $\mathbf{B}(x, y)$, $\mathbf{E}(x, y)$. Letting the material within the guide have dielectric constant ϵ and permeability μ, and assuming no macroscopic sources in this region, we can derive wave equations using familiar methods. Because of the harmonic time dependences, the Maxwell equations read

$$\nabla\times\mathbf{E}=i\frac{\omega}{c}\mathbf{B}\qquad\nabla\cdot\mathbf{B}=0\qquad\nabla\cdot\mathbf{E}=0\qquad\nabla\times\mathbf{B}=-i\frac{\omega}{c}\mu\epsilon\mathbf{E}.\tag{16}$$

Taking the curl of each curl equation and using the forms of the fields as well as the fact that both fields have zero divergence, we find that the wave equations for all Cartesian components of $\mathbf{E}(x,y)$ and $\mathbf{B}(x,y)$ have the same form; it is

$$\left(\nabla_t^2-k^2+\mu\epsilon\frac{\omega^2}{c^2}\right)\psi(x,y)=0.\tag{17}$$

We can greatly simplify things by noting that if we find E_z and B_z first, then $\mathbf{E}_t$ and $\mathbf{B}_t$ follow. To demonstrate this statement, we shall derive explicit expressions for the latter in terms of the former. Consider the transverse components of the curl of the magnetic induction,

$$\left[\nabla\times(B_z\boldsymbol{\epsilon_3}+\mathbf{B}_t)\right]_t=(\nabla_tB_z)\times\boldsymbol{\epsilon_3}+\boldsymbol{\epsilon_3}\times\left(\frac{\partial\mathbf{B}_t}{\partial z}\right)=-i\mu\epsilon\frac{\omega}{c}\mathbf{E}_t.\tag{18}$$

Cross $\boldsymbol{\epsilon_3}$ into this equation to find

$$\nabla_tB_z-\frac{\partial\mathbf{B}_t}{\partial z}=-i\mu\epsilon\frac{\omega}{c}(\boldsymbol{\epsilon_3}\times\mathbf{E}_t).\tag{19}$$

By similar means, one finds that the transverse part of Faraday's Law can be written as

$$(\nabla_tE_z)\times\boldsymbol{\epsilon_3}+\boldsymbol{\epsilon_3}\times\left(\frac{\partial\mathbf{E}_t}{\partial z}\right)=i\frac{\omega}{c}\mathbf{B}_t.\tag{20}$$

Take the derivative with respect to z of the first of these equations and substituted the result into the second equation; the result is

$$\left(\mu\epsilon\frac{\omega^2}{c^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{B}_t=\nabla_t\left(\frac{\partial B_z}{\partial z}\right)+i\epsilon\mu\frac{\omega}{c}\boldsymbol{\epsilon_3}\times(\nabla_tE_z).\tag{21}$$

Because the fields' z-dependence is e^{ikz}, we may take the derivatives with respect to z and express the result as an equation for $\mathbf{B}_t$:

$$\mathbf{B}_t=\frac{1}{\mu\epsilon\omega^2/c^2-k^2}\left[\nabla_t\left(\frac{\partial B_z}{\partial z}\right)+i\mu\epsilon\frac{\omega}{c}\boldsymbol{\epsilon_3}\times(\nabla_tE_z)\right].\tag{22}$$

In the same fashion, we can take the derivative with respect to z of Eq. (20) and substitute into Eq. (19) to find a relation for $\mathbf{E}_t$,

$$\mathbf{E}_t = \frac{1}{\mu\epsilon\omega^2/c^2 - k^2}\left[\nabla_t\left(\frac{\partial E_z}{\partial z}\right) - i\frac{\omega}{c}\boldsymbol{\epsilon_3}\times(\nabla_t B_z)\right]. \tag{23}$$

Now we need only solve for the z components of the fields; from them and the preceding relations, all components are determined.

2.1.1 Boundary Conditions

The boundary condition on $E_z(x,y)$ is that it should vanish at the walls because the tangential component of the electric field is zero there. The other boundary condition, $B_n = 0$, does not put any constraint on B_z; however, there is a constraint on B_z which can be extracted from the equation for $\mathbf{E}_t$; one of the two components of $\mathbf{E}_t$ is tangential to the wall and this one must vanish next to the wall. From Eq. (23) we see that there is a contribution to that component which is proportional to $\partial B_z/\partial n$, so we conclude that the boundary condition on B_z is $\partial B_z/\partial n = 0$. The other contribution to $\mathbf{E}_t$ is proportional to the transverse gradient of E_z at the boundary; since E_z is zero at all points on the boundary, it is clear that this term will not give any tangential component of $\mathbf{E}_t$ at the boundary. Hence the tangential components of $\mathbf{E}$ are zero on the boundary provided E_z vanishes there along with the normal component of the gradient of B_z.

And what of the normal component of $\mathbf{B}$ itself? From Eq. (22) for $\mathbf{B}_t$, we see that the the normal component of $\mathbf{B}$ at the wall vanishes if, first, the gradient of B_z has zero normal component there, and, second, the component of $\nabla_t E_z$ parallel to the wall vanishes; these conditions are met if $E_z = 0$ and $\partial B_z/\partial n = 0$ everywhere on the boundary. Hence we are left with the following boundary-value, or eigenvalue, problem:

$$\left(\nabla_t^2 - k^2 + \mu\epsilon\frac{\omega^2}{c^2}\right)\left\{\begin{array}{c} E_z(x,y) \\ B_z(x,y)\end{array}\right\} = 0, \tag{24}$$

with

$$E_z(x.y) = 0 \qquad \text{and} \qquad \partial B_z(x,y)/\partial n = 0 \tag{25}$$

on the boundary.

2.2 Transverse Modes

Depending on the geometry, it may or may not be possible to find an eigenvalue k^2 such that the conditions on B_z and E_z are both satisfied. If it is not possible, then either $B_z \equiv 0$ or $E_z \equiv 0$. In the former case, $\mathbf{B}$ is purely transverse and one speaks of a *transverse magnetic mode*, often abbreviated as a TM mode; in the latter case, $\mathbf{E}$ is purely transverse and the mode is called a *transverse electric mode*, or a TE mode. For some geometries it is possible to have both E_z and B_z identically zero although the transverse fields are finite; then we have a *transverse electromagnetic mode* or a TEM mode.

MODE	CHARACTER
TM (Transverse Magnetic)	$B_z = 0$
TE (Transverse Electric)	$E_z = 0$
TEM	$E_z = B_z = 0$

2.2.1 TEM Mode

Let's briefly discuss the TEM modes first. In order to see what are the appropriate equations of motion of the fields, we have to go back to the Maxwell equations. If we look just at the z-component of Faraday's Law and of the generalized Ampère's Law, we find that

$$\boldsymbol{\epsilon_3} \cdot (\nabla_t \times \mathbf{E}_t) = 0 \qquad \text{and} \qquad \boldsymbol{\epsilon_3} \cdot (\nabla_t \times \mathbf{B}_t) = 0. \tag{26}$$

Since ∇_t and $\mathbf{E}_t$ have only x and y components, the curls lie entirely in the z direction, so we can write

$$\nabla_t \times \mathbf{E}_t = 0 \qquad \text{and} \qquad \nabla_t \times \mathbf{B}_t = 0. \tag{27}$$

From the other two Maxwell equations we find that

$$\nabla_t \cdot \mathbf{E}_t = 0 \qquad \text{and} \qquad \nabla_t \cdot \mathbf{B}_t = 0. \tag{28}$$

The two transverse fields are solutions of problems identical to statics problems in two dimensions. In particular, the transverse electric field, which must have zero tangential component at the walls of the waveguide, is found by solving an electrostatics problem. If the wave guide is composed of a single conductor, which is an equipotential in the equivalent electrostatics problem, then there is no nontrivial solution. *The conclusion is that a simple single-conductor waveguide cannot have a TEM mode to lowest order in the parameter $\delta\omega/c$.* Where are there TEM modes? These exist in guides which consist of a pair of parallel but electrically unconnected conductors that can be at different potentials. Such things are called *transmission lines.*

2.2.2 TE and TM Modes

Now let's turn to TE and TM modes. We will discuss here only TE modes; TM modes can be treated by making a simple modification of the boundary conditions when solving the eigenvalue problem. For TE modes, $E_z = 0$ and so the transverse fields are given in terms of B_z by

$$\mathbf{B}_t = \frac{1}{\gamma^2}\nabla_t\left(\frac{\partial B_z}{\partial z}\right) = \frac{ik}{\gamma^2}(\nabla_t B_z) \tag{29}$$

and

$$\mathbf{E}_t = -\frac{i}{\gamma^2}\frac{\omega}{c}\boldsymbol{\epsilon_3} \times (\nabla_t B_z) = -\frac{\omega}{ck}(\boldsymbol{\epsilon_3} \times \mathbf{B}_t), \tag{30}$$

where $\gamma^2 \equiv \mu\epsilon\omega^2/c^2 - k^2$. This parameter must be determined by solving the eigenvalue equation

$$(\nabla_t^2 + \gamma^2)B_z(x,y) = 0, \qquad \text{with } \partial B_z/\partial n = 0 \text{ on C}; \tag{31}$$

C is the boundary of the (cross-section of the) waveguide. This problem will have solutions $B_z(x,y) \rightarrow B_i(x,y)$ with eigenvalues γ_i^2, i=1,2,... . In terms of these eigenvalues, the wavenumber k_i is given by

$$k_i^2(\omega) = \mu\epsilon\omega^2/c^2 - \gamma_i^2, \tag{32}$$

which means that for given ω and i, k is determined.

The eigenvalues γ_i^2 are always positive (otherwise the boundary conditions cannot be satisfied), so we can see from the preceding equation that for a given mode i, ω^2 must be larger than $\omega_i^2 \equiv \gamma_i^2 c^2/\mu\epsilon$ in order for the squared wavenumber to be positive corresponding to a real wavenumber k. If ω^2 is smaller than this cutoff value, then the wavenumber is imaginary and the wave is attenuated as it moves in the z-direction. As a particular consequence, one can choose ω such that only some fixed number of modes (one, for example) can propagate.

There are two velocities of interest in connection with any mode; these are the phase velocity and the group velocity. The dispersion relation is

$$\omega^2 = \frac{c^2}{\mu\epsilon}(k^2 + \gamma_i^2), \tag{33}$$

which can be expressed also as

$$\frac{\omega^2}{k^2} = \frac{c^2/\mu\epsilon}{1 - \omega_i^2/\omega^2}. \tag{34}$$

From this form, one can see clearly that the phase velocity, ω/k, is always larger than the phase velocity in the absence of walls, $c/\sqrt{\epsilon\mu}$. Further, the phase velocity diverges as ω approaches the cutoff frequency. As for the group velocity, we have $v_g = d\omega/dk$, and

$$\omega\frac{d\omega}{dk} = \frac{c^2}{\mu\epsilon}k \qquad \text{or} \qquad \frac{\omega}{k}\frac{d\omega}{dk} = \frac{c^2}{\mu\epsilon} = v_p v_g. \tag{35}$$

This equation tells us that the product of the group and phase velocities is a constant, $c^2/\mu\epsilon$; the group velocity itself is

$$v_g = \frac{c^2}{\mu\epsilon}\frac{k}{\omega} = \frac{c}{\sqrt{\epsilon\mu}}\sqrt{1 - \frac{\omega_i^2}{\omega^2}} \tag{36}$$

which is always smaller than $c/\sqrt{\epsilon\mu}$.

2.3 Energy Flow

The significance of the group velocity of the mode becomes clear from a study of the energy flow in the guide. The time-average of the Poynting vector's z component is

$$< \mathbf{S} \cdot \boldsymbol{\epsilon_3} >= \frac{c}{8\pi} \Re(\mathbf{E}_t \times \mathbf{H}_t^*) \cdot \boldsymbol{\epsilon_3} = \frac{c}{8\pi} \frac{\omega k}{\mu c \gamma^4} |\nabla_t B_z|^2. \tag{37}$$

This must be integrated over the cross-section of the guide to find the power transmitted,

$$\begin{aligned} \mathcal{P} = \int_S d^2x \; < \mathbf{S} \cdot \boldsymbol{\epsilon_3} >= \frac{c}{8\pi} \frac{\omega k}{\mu c \gamma^4} \int_S d^2x \, |\nabla_t B_z|^2 \\ = \frac{\omega k}{8\pi \mu \gamma^4} \int_S d^2x \, [\nabla_t \cdot (B_z^* \nabla_t B_z) - B_z^* \nabla_t^2 B_z]. \end{aligned} \tag{38}$$

The first term in the final expression converts to a surface integral which is shown, from the boundary conditions on B_z, to be zero; the second term is made simpler in appearance by using the fact that $\nabla_t^2 B_z = -\gamma^2 B_z$. Thus,

$$\mathcal{P} = \frac{\omega k}{8\pi \mu \gamma^2} \int_S d^2x \, |B_z|^2. \tag{39}$$

Compare this with the time-averaged energy per unit length in the guide

$$\begin{aligned} U = \frac{1}{16\pi} \int_S d^2x \left(\epsilon \mathbf{E}_t \cdot \mathbf{E}_t^* + \frac{1}{\mu} [\mathbf{B}_t \cdot \mathbf{B}_t^* + B_z B_z^*] \right) \\ = \frac{1}{16\pi} \int_S d^2x \left[\epsilon \frac{\omega^2}{c^2 k^2} |\mathbf{B}_t|^2 + \frac{1}{\mu} |\mathbf{B}_t|^2 + \frac{1}{\mu} |B_z|^2 \right] \\ = \frac{1}{16\pi\mu} \int_S d^2x \left[\left(\epsilon\mu \frac{\omega^2}{c^2 \gamma^4} + \frac{k^2}{\gamma^4} \right) |\nabla_t B_z|^2 + |B_z|^2 \right] \\ = \frac{1}{16\pi\mu} \int_S d^2x \left[\left(\epsilon\mu \frac{\omega^2}{c^2} + k^2 \right) \frac{1}{\gamma^2} + 1 \right] |B_z|^2 = \frac{1}{8\pi} \left(\epsilon \frac{\omega^2}{c^2} \right) \frac{1}{\gamma^2} \int_S d^2x \, |B_z|^2. \end{aligned} \tag{40}$$

In arriving at the final result, we've used a whole collection of identities related to the eigenvalue problem.

Comparison of U and $\mathcal{P}$ shows that

$$\frac{\mathcal{P}}{U} = \frac{k}{\omega} \frac{c^2}{\mu\epsilon} \equiv v_g. \tag{41}$$

The obvious interpretation need not be stated.

2.3.1 TE Modes in Rectangular and Circular Guides

Before going on to other matters, let us look at the solutions of the eigenvalue problem for some standard waveguide shapes, i.e., rectangles and circles, assuming TE modes.

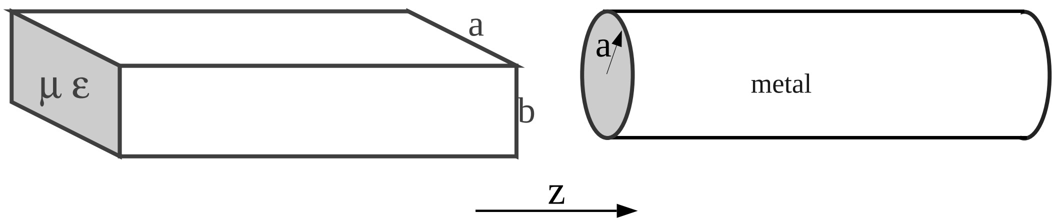

Fig.4: Geometry of Rectangular and Circular Wave Guides.

For the rectangle shown, the solution[5] for B_z is

$$B_{mn}(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{42}$$

with

$$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right). \tag{43}$$

For the circular guide of radius a, on the other hand, B_z becomes

$$B_{mn}(\rho,\phi) = B_0 e^{im\phi} J_m(y_{mn}\rho/a) \tag{44}$$

with

$$\gamma_{mn}^2 = y_{mn}^2/a^2. \tag{45}$$

Here, y_{mn} is the n^{th} zero of the derivative of the Bessel function J_m, $J_m'(y)|_{y_{mn}} = 0$.

The question of which modes will actually be excited in a waveguide for a given source can be worked out (see below); one has to address the question of how the source couples to the eigenfunctions for the different modes. Different configurations

[5]The eigenvalue problem for TE modes is formally equivalent to that of a quantum mechanical particle in a box with somewhat unusual boundary conditions; the case of the TM mode has the usual boundary conditions.

of the source will produce different superpositions of modes. A simple way to guarantee that just one propagating mode will be present is to pick ω smaller than the cutoff frequencies of all modes but one.

3 Attenuation of Modes in Waveguides

Even modes for which the wave number is real will be somewhat damped because of the finite conductivity of the walls. We have seen that the power travelling down the pipe is given by Eq. (39); also, the power lost per unit length is, from Eq. (11),

$$\frac{d\mathcal{P}}{dz} = -\frac{\mu\omega\delta}{16\pi}\oint_C dl|\mathbf{H}_{\|}|^2 \tag{46}$$

where $\mathbf{H}_{\|}$ is the component of $\mathbf{B}/\mu$ which is parallel to the boundary, and the integral is evaluated on the contour formed by the cross-section of the guide. We can evaluate $|\mathbf{H}_{\|}|^2$ up to a point. First, for TE modes

$$|\mathbf{H}_{\|}|^2 = |\mathbf{n}\times\mathbf{H}|^2 = \frac{1}{\mu^2}(|B_z|^2 + |\mathbf{n}\times\mathbf{B}_t|^2) = \frac{1}{\mu^2}\left[|B_z|^2 + \frac{k^2}{\gamma^4}|\mathbf{n}\times(\nabla_t B_z)|^2\right]. \tag{47}$$

Further, at the surface $|\mathbf{n}\times(\nabla_t B_z)|^2 = |\nabla_t B_z|^2$ (since $\partial B_z/\partial n = 0$ there). The latter can be expected to be comparable to $|B_z^*\nabla_t^2 B_z| = \gamma^2|B_z|^2$, so we write

$$|\nabla_t B_z|^2 = \xi\gamma^2|B_z|^2, \tag{48}$$

where ξ is a mode-dependent dimension-free constant of order unity that is independent of the frequency[6]. Hence,

$$|\mathbf{n}\times\mathbf{H}|^2 = \frac{1}{\mu^2}\left(1+\xi\frac{k^2}{\gamma^2}\right)|B_z|^2 \tag{49}$$

and

$$\frac{d\mathcal{P}}{dz} = -\frac{\mu\omega\delta}{16\pi}\frac{1}{\mu^2}\left(1+\xi\frac{k^2}{\gamma^2}\right)\oint_C dl|B_z|^2 \tag{50}$$

[6]For a given geometry of the guide, and a particular mode therein, one may easily calculate ξ.

This is to be compared with the power itself which is obtained by integrating $|B_z|^2$ over a cross-section of the guide. We convert the integral in the preceding equation into one over the cross-section by noticing that $|B_z|$ behaves on the boundary in much the same way as in the interior of the guide, so that

$$\oint_C dl |H_z|^2 = \frac{C}{A}\eta \int_S d^2x \, |H_z|^2 \tag{51}$$

where A and C are the cross-sectional area and circumference of the guide and η is another dimension-free constant of order unity; it depends on the shape of the guide and on the mode but not on the frequency.

Substituting this expression into Eq. (50) and dividing by the power, given by Eq. (29), we find an attenuation coefficient β,

$$\beta \equiv -\left.\frac{d\mathcal{P}}{dz}\right/ \mathcal{P} = \frac{\delta}{2k}\eta\frac{C}{A}(\gamma^2 + \xi k^2). \tag{52}$$

Substitute for k using $k^2 = \epsilon\mu\omega^2/c^2 - \gamma^2$, and replace γ using $\gamma^2 = \epsilon\mu\omega_i^2/c^2$; ω_i is the cutoff frequency of the mode. Then we find

$$\beta = \frac{\sqrt{\epsilon\mu}}{2}\sqrt{\frac{\omega_i}{\omega}}\,\frac{\omega_i}{\sqrt{\omega^2 - \omega_i^2}}\,\frac{\eta C}{A}\,\frac{\delta_i \omega_i}{c}\left[1 + \xi\left(\frac{\omega^2 - \omega_i^2}{\omega_i^2}\right)\right]; \tag{53}$$

δ_i is the penetration depth at cutoff. All of the frequency dependence of β is explicit in this result. The damping becomes very large as $\omega \to \omega_i$; for ω not too close to the cutoff for mode i, and for $\sigma \sim 10^{17}\,sec^{-1}$, we can see that the wave can travel some hundreds of meters without disastrous attenuation. At very high frequencies, $\omega >> \omega_i$, the attenuation increases once again.

The usefulness of our result is limited; in particular, it breaks down when the frequency approaches the cutoff frequency for the mode. One can do a better calculation by improving the treatment of the boundary conditions. The solution (for the fields) that we have found satisfies boundary conditions only slightly different from the correct ones; it turns out, not surprisingly, that an improved solution can be obtained by looking for small corrections to the fields we already have, with the corrections determined by demanding that the exact boundary conditions be satisfied to one higher order in ω/σ. This is a straightforward but somewhat technical calculation and we shall not spend time on it.

4 Resonating cavities

The step from a waveguide to a resonating cavity is not a large one. We need only think about what new constraints are placed on the electromagnetic fields if end walls are placed on a waveguide. Suppose that such walls are introduced at $z = 0$ and $z = d$. Then we cannot have traveling waves in the guide but must have instead standing waves; that is, in addition to the fields varying as $e^{i(kz-\omega t)}$, we must have ones that vary as $e^{-i(kz+\omega t)}$ which means that the z and t-dependent parts of the fields can be expressed as some linear combination of $\sin(kz)e^{i\omega t}$ and $\cos(kz)e^{i\omega t}$. Using such a combination, one may proceed as in the previous sections and will find in particular that $E_z(x, y)$ and $B_z(x, y)$ satisfy the same eigenequations as before.

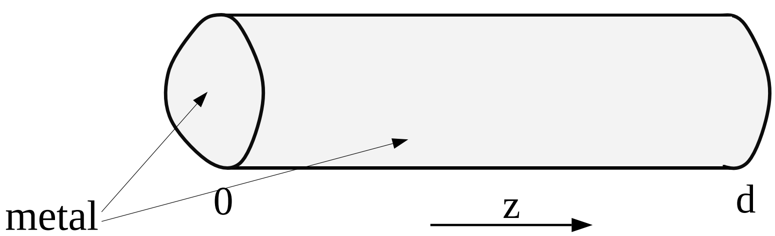

Fig.5: Geometry of a Resonant Cavity.

What is different in a cavity is that there are new boundary conditions or constraints on the fields because of the presence of end walls. If $B_z \neq 0$, then one must add the condition that $B_z = 0$ at $z = 0$ and $z = d$. Also, in order to guarantee that the tangential component of $\mathbf{E}$, (or $\mathbf{E}_t$) vanish on the end walls, it must be the case that $\partial E_z/\partial z = 0$ on the end walls; see Eq. (23). These are the additional conditions that must be satisfied for a cavity.

Following tradition established in earlier sections, we examine only TE modes.

Then B_z must vanish at the ends of the cavity, meaning that the solution we want is the one having z-dependence $\sin(kz)$; this is zero at $z = 0$, and we make it zero at $z = d$ by choosing $k = p\pi/d$ with p an integer. It is also true, as for the waveguide, that k must satisfy the equation $k^2 = \mu\epsilon\omega^2/c^2 - \gamma_{mn}^2$. Both of these conditions on k^2 can be satisfied simultaneously only for certain discrete frequencies ω_{mnp} which are given by

$$\omega_{mnp}^2 = \frac{c^2}{\mu\epsilon}\left(\frac{p^2\pi^2}{d^2} + \gamma_{mn}^2\right). \tag{54}$$

Hence a resonating cavity has a set of discrete natural "resonant" frequencies at which it can support a standing wave electromagnetic field. These frequencies can be tuned by adjusting the size of the cavity, e.g., by changing d.

A resonant cavity is useful in that if excited in a single mode, it will contain monochromatic radiation in the microwave frequency range (ie.e a maser), or would do so if it were perfect. However, the resonance is never perfectly sharp in frequency, meaning that if one Fourier transforms the fields into frequency space (instead of time) the result will not be a delta function at the resonant frequency. There are several contributing factors to the width of the resonance. One important factor is the power loss in the walls. This loss is generally characterized by the "Q" of the cavity defined by

$$Q \equiv \omega_0 \left(\frac{\text{stored energy}}{\text{power loss}}\right) \tag{55}$$

where ω_0 is the frequency of the mode in the cavity. In words, Q is 2π times the energy stored in the cavity divided by the energy loss per cycle. From this definition it follows that the connection between Q and the rate at which the energy stored in the cavity decays is $dU/dt = -U\omega_0/Q$ when no new energy is being pumped into the cavity. If Q is independent of the amount of energy in the cavity, then this differential equation has a simple exponential solution,

$$U = U_0 e^{-\omega_0 t/Q}, \tag{56}$$

and, under these circumstances, any field[7] ψ behaves in time as

$$\psi(t) = \psi_0 e^{-i\omega_0 t} e^{-\omega_0 t/2Q}. \tag{57}$$

Because of the decay, the frequency spectrum of the field contains components in addition to ω_0. Specifically,

$$\psi(\omega) \sim \int_0^\infty dt e^{i(\omega-\omega_0)t} e^{-\omega_0 t/2Q} = \frac{1}{i(\omega_0 - \omega) + \omega_0/2Q}, \tag{58}$$

or

$$|\psi(\omega)|^2 \sim \frac{1}{(\omega_0 - \omega)^2 + \omega_0^2/4Q^2} \tag{59}$$

which means that the resonance has a width in frequency space which is of order $\omega_0/2Q$.

And what is the value of Q? The energy (or fields) in the cavity decay in time because of losses in the walls. We learned in section 1 how to compute these losses. For any given mode in a particular cavity it is a straightforward matter to do the calculations, provided one can solve for the fields in that mode. One proceeds in much the same way as in the previous section where we learned how to calculate the power loss in a waveguide. Skipping over the details of the argument, which are much like the calculation of β for a mode of a waveguide, we simply state the conclusion which is that the energy lost per period is of the order of the energy in the cavity times the ratio of the volume of the walls into which the field penetrates to the volume of the cavity. If the area of the walls is A and the volume of the cavity is V, then

$$Q \sim \frac{V}{A\delta}. \tag{60}$$

Thus the relevant parameter for determining the Q of the cavity is the ratio of the penetration depth to the linear size of the cavity. For δ on the order of microns, or $10^{-4}\, cm$, and a cavity having a size on the order of a centimeter, the Q will be on the order of 10^3 to 10^4.

[7] That is, any component of $\mathbf{E}$ or $\mathbf{B}$.