2 Classical Field Theory

In what follows we will consider rather general field theories. The only guiding principles that we will use in constructing these theories are (a) symmetries and (b) a generalized Least Action Principle.

2.1 Relativistic Invariance

Before we saw three examples of relativistic wave equations. They are Maxwell's equations for classical electromagnetism, the Klein-Gordon and Dirac equations. Maxwell's equations govern the dynamics of a *vector* field, the vector potentials $A^\mu(x) = (A^0, \vec{A})$, whereas the Klein-Gordon equation describes excitations of a *scalar* field $\phi(x)$ and the Dirac equation governs the behavior of the four-component spinor field $\psi_\alpha(x)(\alpha = 0, 1, 2, 3)$. Each one of these fields transforms in a very definite way under the group of Lorentz transformations, the Lorentz group. The Lorentz group is defined as a group of linear transformations Λ of Minkowski space-time $\mathcal{M}$ onto itself $\Lambda : \mathcal{M} \to \mathcal{M}$ such that

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{1}$$

The space-time components of Λ are the *Lorentz boosts* which relate inertial reference frames moving at relative velocity $\vec{v}$. Thus, Lorentz boosts along the x^1-axis have the familiar form

$$\begin{aligned} x^{0\prime} &= \frac{x^0 + vx^1/c}{\sqrt{1 - v^2/c^2}} \\ x^{1\prime} &= \frac{x^1 + vx^0/c}{\sqrt{1 - v^2/c^2}} \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned} \tag{2}$$

where $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$ (note: these are components, not powers!). If we use the notation $\gamma = (1 - v^2/c^2)^{-1/2} \equiv \cosh\alpha$, we can write the Lorentz boost as a matrix:

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{3}$$

The space components of Λ are conventional three-dimensional rotations R.

Infinitesimal Lorentz transformations are generated by the hermitian operators

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{4}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. They satisfy the algebra

$$[L_{\mu\nu}, L_{\rho\sigma}] = ig_{\nu\rho}L_{\mu\sigma} - ig_{\mu\rho}L_{\nu\sigma} - ig_{\nu\sigma}L_{\mu\rho} + ig_{\mu\sigma}L_{\nu\rho} \tag{5}$$

where $g_{\mu\nu}$ is the metric tensor for Minkowski space-time (see below). This is the algebra of the group $SO(3,1)$. Actually any operator of the form

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \tag{6}$$

where $S_{\mu\nu}$ are 4×4 matrices satisfying the algebra of Eq. 5 satisfies the same algebra. Below we will discuss explicit examples.

Lorentz transformations form a *group*, since (a) the product of two Lorentz transformations is a Lorentz transformation, (b) there exists an identity transformation, and (c) Lorentz transformations are invertible. Notice, however, that in general two transformations do not commute with each other. Hence, the lorentz group is non-Abelian.

The Lorentz group has the *defining property* that it leaves invariant the relativistic interval

$$x^2 \equiv x_0^2 - \vec{x}^{\,2} = c^2t^2 - \vec{x}^{\,2} \tag{7}$$

The group of Euclidean rotations leave invariant the Euclidean distance $\vec{x}^{\,2}$ and it is a subgroup of the Lorentz group. The rotation group is denoted by $SO(3)$, and the Lorentz group is denoted by $SO(3,1)$. This notation makes manifest the fact that the signature of the metric has one $+$ sign and three $-$ signs.

We will adopt the following conventions and definitions:

1. *Metric Tensor*: We will use the Bjorken and Drell metric in which the metric tensor $g_{\mu\nu}$ is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8}$$

   With this notation the infinitesimal relativistic interval is

$$ds^2 = dx^\mu dx_\mu = g_{\mu\nu}dx^\mu dx^\nu = dx_0^2 - d\vec{x}^{\,2} = c^2dt^2 - d\vec{x}^{\,2} \tag{9}$$

2. *4-vectors*:
   (a) x^μ is a *contravariant* 4-vector, $x^\mu = (ct, \vec{x})$
   (b) x_μ is a *covariant* 4-vector $x_\mu = (ct, -\vec{x})$
   (c) Covariant and contravariant vectors (and tensors) are related through the metric tensor $g_{\mu\nu}$

$$A^\mu = g^{\mu\nu}A_\nu \tag{10}$$

   (d) $\vec{x}$ is a vector in $\mathbb{R}^3$
   (e) $p^\mu = (\frac{E}{c}, \vec{p})$. Hence, $p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p}^{\,2}$ is a Lorentz scalar.

3. *Scalar Product*:

$$p \cdot q = p_\mu q^\mu = p_0 q_0 - \vec{p} \cdot \vec{q} \equiv p_\mu q_\nu g^{\mu\nu} \tag{11}$$

4. *Gradients*: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$. We define the D'Alambertian $\Box$

$$\Box \equiv \partial^\mu \partial_\mu \equiv \partial^2 \equiv \frac{1}{c^2}\partial_t^2 - \bigtriangledown^2 \tag{12}$$

   From now on we will use units of time $[T]$ and length $[L]$ such that $\hbar = c = 1$. Thus, $[T] = [L]$ and we will use units like centimeters (or any other unit of length).

5. *Interval*: The interval in Minkowski space is x^2,

$$x^2 = x_\mu x^\mu = x_o^2 - \vec{x}^2 \tag{13}$$

   Time-like intervals have $x^2 > 0$ while space-like intervals have $x^2 < 0$.

Since a field is a function (or mapping) of Minkowski space onto some other (properly chosen) space, it is natural to require that the fields should have simple transformation properties under Lorentz transformations. For example, the vector potential $A^\mu(x)$ transforms like *4-vector* under Lorentz transformations, *i.e.,* if $x'^\mu = \Lambda^\mu_\nu x^\nu \Rightarrow A'^\mu(x') = \Lambda^\mu_\nu A^\nu(x)$. In other words, A^μ transforms like x^μ. Thus, it is a vector. All vector fields have that property. A *scalar* field $\Phi(x)$, on the other hand, remains *invariant* under Lorentz transformations,

$$\Phi'(x') = \Phi(x) \tag{14}$$

A 4-spinor $\psi_\alpha(x)$ transforms under Lorentz transformations. Namely, there exists an induced 4x4 transformation matrix $S(\Lambda)$ such that

$$S(\Lambda^{-1}) = S^{-1}(\Lambda) \tag{15}$$

and

$$\Psi'(\Lambda x) = S(\Lambda)\Psi(x) \tag{16}$$

Below we will give an explicit expression for $S(\Lambda)$.

2.2 The Lagrangian, the Action and and the Least Action Principle

The evolution of any dynamical system is determined by its Lagrangian. In the Classical Mechanics of systems of particles described by the generalized coordinate q, the Lagrangian L is a differentiable function of q and its time derivatives. L must be differentiable since otherwise the equations of motion would not local in time, *i.e.* could not be written in terms of differential equations. An argument á-la Landau-Lifshitz enables us to "derive" the Lagrangian. For example, for a particle in free space, the homogeneity, uniformity and isotropy of space

and time require that L be only a function of the absolute value of the velocity $|\vec{v}|$. Since $|\vec{v}|$ is not a differentiable function of $\vec{v}$, the Lagrangian must be a function of $\vec{v}^2$. Thus, $L = L(\vec{v}^{\,2})$. In principle there is no reason to assume that L cannot be a function of the acceleration $\vec{a}$ (or rather $\vec{a}$) or of its higher derivatives. *Experiment* tells us that in Classical Mechanics it is sufficient to specify the initial position $\vec{x}(0)$ of a particle and its initial velocity $\vec{v}(0)$ in order to determine the time evolution of the system. Thus we have to choose

$$L(\vec{v}^{\,2}) = \text{const} + \frac{1}{2} m \vec{v}^{\,2} \tag{17}$$

The additive constant is irrelevant in classical physics. Naturally, the coefficient of $\vec{v}^{\,2}$ is just one-half of the inertial mass. However, in Special Relativity, the natural invariant quantity to consider is not the Lagrangian but the *action* S. For a free particle the relativistic invariant (*i.e.,* Lorentz invariant) action must involve the invariant interval or the invariant length $ds = c\sqrt{1 - \frac{\vec{v}^{\,2}}{c^2}}\, dt$, the *proper time*. Hence one writes

$$S = mc \int_{s_i}^{s_f} ds = mc^2 \int_{t_i}^{t_f} \sqrt{1 - \frac{\vec{v}^{\,2}}{c^2}} \tag{18}$$

Thus the relativistic Lagrangian is

$$L = mc^2 \sqrt{1 - \frac{\vec{v}^{\,2}}{c^2}} \tag{19}$$

and, as a power series expansion, it contains all powers of $\vec{v}^{\,2}/c^2$.

Once the Lagrangian is found, the classical equations of motion are determined by the *Least Action Principle*. Thus, we construct the action S

$$S = \int dt\; L\left(q, \frac{\partial q}{\partial t}\right) \tag{20}$$

and demand that the physical trajectories $q(t)$ leave the action S stationary,*i.e.,* $\delta S = 0$. The variation of S is

$$\delta S = \int_{t_i}^{t_f} dt\; \left(\frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \frac{dq}{dt}}\, \delta \frac{dq}{dt} \right) \tag{21}$$

Integrating by parts, we get

$$\begin{aligned} \delta S = & \quad \int_{t_i}^{t_f} dt\; \frac{d}{dt} \left(\frac{\partial L}{\partial \frac{dq}{dt}} \delta q \right) + \int_{t_i}^{t_f} dt\; \delta q\; \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \frac{dq}{dt}} \right) \right] \\ \Rightarrow \delta S = & \quad \frac{\partial L}{\partial \frac{dq}{dt}} \delta q \,\Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} dt\; \delta q\; \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \frac{dq}{dt}} \right) \right] \end{aligned} \tag{22}$$

If we assume that the variation δq is an arbitrary function of time that vanishes at the initial and final times t_i and $t_f (\delta q(t_i) = \delta q(t_f) = 0)$, we find that $\delta S = 0$ if and only if the integrand vanishes identically. Thus,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \frac{dq}{dt}}\right) = 0 \tag{23}$$

These are the equations of motion or *Newton's equations*. In general the equations that determine the trajectories which leave the action stationary are called the *Euler-Lagrange equations*.

2.3 Scalar Field Theory

For the case of a field theory, we can proceed very much in the same way. Let us consider first the case of a scalar field $\Phi(x)$. The action S must be invariant under Lorentz transformations. Since we want to construct *local* theories it is natural to assume that S is determined by a Lagrangian *density* $\mathcal{L}$

$$S = \int d^4x \, \mathcal{L} \tag{24}$$

Since the volume element of Minkowski space d^4x is invariant, $\mathcal{L}$ can only be a local, differentiable function of Lorentz invariants that can be constructed out of the field $\Phi(x)$. Simple invariants are $\Phi(x)$ itself and all of its powers. The gradient $\partial^\mu \Phi \equiv \frac{\partial \Phi}{\partial x_\mu}$ is not an invariant but the D'alambertian $\partial^2 \Phi$ is. $\partial_\mu \Phi \partial^\mu \Phi$ is also an invariant under a change of the sign of Φ. So, we can write the following simple expression for $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - V(\Phi) \tag{25}$$

where $V(\Phi)$ is some potential (a polynomial function of Φ). Let us consider the simple choice

$$V(\Phi) = \frac{1}{2}\bar{m}^2 \Phi^2 \tag{26}$$

where $\bar{m} = mc/\hbar$. Thus,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\bar{m}^2 \Phi^2 \tag{27}$$

This is the Lagrangian density for a *free scalar field*. We will discuss later on in what sense this field is "free". Notice, in passing, that we could have added a term like $\partial^2 \Phi$. However this term, in addition of being odd under $\Phi \to -\Phi$, is a *total divergence* and, as such, it has an effect only on the boundary conditions but it does not affect the equations of motion. In what follows will will not consider surface terms.

The Least Action Principle requires that S be stationary under arbitrary variations of the field Φ and of its derivatives $\partial_\mu \Phi$. Thus, we get

$$\delta S = \int d^4x \ \left[\frac{\delta \mathcal{L}}{\delta \Phi} \ \delta \Phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \ \delta \partial_\mu \Phi \right] \tag{28}$$

Notice that since $\mathcal{L}$ is a *functional* of Φ, we have to use functional derivatives, *i.e.,* partial derivatives at each point of space-time. Upon integrating by parts, we get

$$\delta S = \int d^4x \ \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \ \delta \Phi \right) + \int d^4x \ \delta \Phi \ \left[\frac{\delta \mathcal{L}}{\delta \Phi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \right) \right] \tag{29}$$

Instead of considering initial and final conditions, we now have to imagine that the field Φ is contained inside some very large box of space-time. The term with the total divergence yields a surface contribution. We will consider field configurations such that $\delta \Phi = 0$ on that surface. Thus, the Euler-Lagrange equations are

$$\frac{\delta \mathcal{L}}{\delta \Phi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \right) = 0 \tag{30}$$

More explicitly, we find

$$\frac{\delta \mathcal{L}}{\delta \Phi} = -\frac{\partial V}{\partial \Phi} \tag{31}$$

and

$$\frac{\partial \mathcal{L}}{\delta \partial_\mu \Phi} = \partial^\mu \Phi \Rightarrow \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} = \partial_\mu \partial^\mu \Phi = \partial^2 \Phi \tag{32}$$

By direct substitution we get the equation of motion (or *field equation*)

$$\partial^2 \Phi + \frac{\partial V}{\partial \Phi} = 0 \tag{33}$$

For the choice

$$V(\Phi) = \frac{\bar{m}^2}{2} \Phi^2 \Rightarrow \frac{\partial V}{\partial \Phi} = \bar{m}^2 \Phi \tag{34}$$

the field equation is

$$\left(\Box + \bar{m}^2 \right) \Phi = 0 \tag{35}$$

where $\Box \equiv \partial^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \bigtriangledown^2$. Thus, we find that the equation of motion for the free massive scalar field Φ is

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \bigtriangledown^2 \Phi + \bar{m}^2 \Phi = 0 \tag{36}$$

This is precisely the Klein-Gordon equation if the constant $\bar{m}$ is identified with $\frac{mc}{\hbar}$. Indeed, the plane-wave solutions of these equations are

$$\Phi = \Phi_0 e^{i(p_0 x_0 - \vec{p} \cdot \vec{x})} \tag{37}$$

where p_0 and $\vec{p}$ are related through the dispersion law

$$p_0^2 = \vec{p}^{\,2}c^2 + m^2c^4 \tag{38}$$

which means that, for each momentum $\vec{p}$, there are two solutions, one with *positive frequency* and one with *negative frequency*. We will see below that, in the quantized theory, the energy of the excitation is indeed equal to $|p_0|$. Notice that $\frac{1}{\bar{m}} = \frac{\hbar}{mc}$ has units of length and it is equal to the Compton wavelength for a particle of mass m. From now on (unless it stated the contrary) I will use units in which $\hbar = c = 1$ in which $m = \bar{m}$.

The Hamiltonian for a classical field is found by a straightforward generalization of the Hamiltonian of a classical particle. Namely, one defines the canonical momentum $\Pi(x)$, conjugate to the field (the "coordinate ") $\Phi(x)$,

$$\Pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\Phi}(x)} \tag{39}$$

where

$$\dot{\Phi}(x) = \frac{\partial\Phi}{\partial x_0}(x) \tag{40}$$

In Classical Mechanics the Hamiltonian H and the Lagrangian L are related by

$$H = p\dot{q} - L \tag{41}$$

where q is the coordinate and p the canonical momentum conjugate to q. Thus, for a *scalar field theory* the Hamiltonian *density* $\mathcal{H}$ is

$$\begin{aligned} \mathcal{H} &= \Pi(x)\dot{\Phi}(x) - \mathcal{L} \\ &= \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\vec{\bigtriangledown}\Phi(x)\right)^2 + V(\Phi(x)) \end{aligned} \tag{42}$$

For a free massive scalar field the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\vec{\bigtriangledown}\Phi(x)\right)^2 + \frac{m^2}{2}\Phi^2(x) \geq 0 \tag{43}$$

which is always a positive definite quantity. Thus, the energy of a plane wave solution of a massive scalar field theory (*i.e.,* a solution of the Klein-Gordon equation) is always positive, no matter the sign of the *frequency*. In fact, the lowest energy state is simply $\Phi =$ constant. A solution made of linear superpositions of plane waves (*i.e.,* a wave packet) has positive energy. Therefore, in *Field Theory*, *the energy is always positive*. We will see that, in the quantized theory, the negative frequency solutions are identified with antiparticle states and their existence do not signal a possible instability of the theory.

2.4 Classical Field Theory in the Canonical (Hamiltonian) Formalism

In Classical Mechanics it is often convenient to use the canonical formulation in terms of a Hamiltonian instead of the Lagrangian approach. For the case of a system of particles, the canonical formalism proceeds as follows. Given a Lagrangian $L(q, \dot{q})$, a canonical momentum p is defined to be

$$\frac{\partial L}{\partial \dot{q}} = p \tag{44}$$

The classical Hamiltonian $H(p, q)$ is defined by the Legendre transformation

$$H(p, q) = p\dot{q} - L(q, \dot{q}) \tag{45}$$

If the Lagrangian L is quadratic in the velocities $\dot{q}$ and separable

$$L = \frac{1}{2} m\dot{q}^2 - V(q) \tag{46}$$

then, $H(p\dot{q})$ is simply given by

$$H(p, q) = p\dot{q} - (\frac{m\dot{q}^2}{2} - V(q)) = \frac{p^2}{2m} + V(q) \tag{47}$$

where

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{48}$$

H is identified with the total energy of the system.

In this language, the Least Action Principle becomes

$$\delta S = \delta \int L \, dt = \delta \int \left[p\dot{q} - H(p, q) \right] dt = 0 \tag{49}$$

Hence

$$\int dt \, \left(\delta p \, \dot{q} + p \, \delta \dot{q} - \delta p \, \frac{\partial H}{\partial p} - \delta q \, \frac{\partial H}{\partial q} \right) = 0 \tag{50}$$

Upon an integration by parts we get

$$\int dt \, \left[\delta p \, \left(\dot{q} - \frac{\partial H}{\partial p} \right) + \delta \dot{q} \, \left(-\frac{\partial H}{\partial q} - \dot{p} \right) \right] = 0 \tag{51}$$

which can only be satisfied if

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{52}$$

These are Hamilton's equations.

Let us introduce the Poisson Bracket $\{A, B\}_{qp}$ of two functions A and B of q and p by

$$\{A, B\}_{qp} \equiv \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q} \tag{53}$$

Let $F(q,p,t)$ be some differentiable function of q, p and t. Then the total time variation of F is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\frac{dq}{dt} + \frac{\partial F}{\partial p}\frac{dp}{dt} \tag{54}$$

Using Hamilton's Equations we get the result

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} \tag{55}$$

or, in terms of Poisson Brackets,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H, \}_{qp} \tag{56}$$

In particular,

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q} = \{q, H\}_{qp} \tag{57}$$

since

$$\frac{\partial q}{\partial p} = 0 \qquad \text{and} \qquad \frac{\partial q}{\partial q} = 1 \tag{58}$$

Also the total rate of change of the canonical momentum p is

$$\frac{dp}{dt} = \frac{\partial p}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial q} \equiv -\frac{\partial H}{\partial q} \tag{59}$$

since $\frac{\partial p}{\partial q} = 0$ and $\frac{\partial p}{\partial p} = 1$. Thus,

$$\frac{dp}{dt} = \{p, H\}_{qp} \tag{60}$$

Notice that, for an isolated system, H is time-independent. So,

$$\frac{\partial H}{\partial t} = 0 \tag{61}$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\}_{qp} = 0 \tag{62}$$

since

$$\{H, H\}_{qp} = 0 \tag{63}$$

Therefore, H can be regarded as the generator of infinitesimal time translations. In passing, let us also notice that the above definition of the Poisson Bracket implies that q and p satisfy

$$\{q, p\}_{qp} = 1 \tag{64}$$

This relation is fundamental for the quantization of these systems.

Much of this formulation can be generalized to the case of fields. Let us first discuss the canonical formalism for the case of a scalar field Φ with Lagrangian

density $\mathcal{L}(\Phi, \partial_\mu, \Phi)$. We will *choose* $\Phi(x)$ to be the (infinite) set of *canonical coordinates*. The *canonical momentum* $\Pi(x)$ is defined by

$$\Pi(x) = \frac{\delta \mathcal{L}}{\delta \partial_0 \Phi(x)} \tag{65}$$

If the Lagrangian is *quadratic* in $\partial_\mu \Phi$, the canonical momentum $\Pi(x)$ is simply given by

$$\Pi(x) = \partial_0 \Phi(x) \equiv \dot{\Phi}(x) \tag{66}$$

The Hamiltonian *density* $\mathcal{H}(\Phi, \Pi)$ is a local function of $\Phi(x)$ and $\Pi(x)$ given by

$$\mathcal{H}(\Phi, \Pi) = \Pi(x)\ \partial_0 \Phi(x) - \mathcal{L}(\Phi, \partial_0 \Phi) \tag{67}$$

If the Lagrangian density $\mathcal{L}$ has the simple form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - V(\Phi) \tag{68}$$

then, the Hamiltonian density $\mathcal{H}(\Phi, \Pi)$ is

$$\mathcal{H} = \Pi \dot{\Phi} - \mathcal{L}(\Phi, \dot{\Phi}, \bigtriangledown_j \Phi) \equiv \frac{1}{2}\Pi^2(x) + \frac{1}{2}(\vec{\bigtriangledown}\Phi(x))^2 + V\left(\Phi(x)\right) \tag{69}$$

The canonical field $\Phi(x)$ and the canonical momentum $\Pi(x)$ satisfy the equal-time Poisson Bracket (PB) relations

$$\{\Phi(\vec{x}, x_0), \Pi(\vec{y}, x_0)\}_{PB} = \delta(\vec{x} - \vec{y}) \tag{70}$$

where $\delta(\vec{x})$ is the Dirac δ-function and $\{A, B\}_{PB}$ is now

$$\{A, B\}_{PB} = \int d^3x\ \left[\frac{\delta A}{\delta \Phi(\vec{x}, x_0)} \frac{\delta B}{\delta \Pi(\vec{x}, x_0)} - \frac{\delta A}{\delta \Pi(\vec{x}, x_0)} \frac{\delta B}{\delta \Phi(\vec{x}, x_0)}\right] \tag{71}$$

for any two functionals A and B of $\Phi(x)$ and $\Pi(x)$. This approach can be extended to theories other than that of a scalar field without too much difficulty. We will come back to these issues when we consider the problem of quantization.

2.5 Field Theory of the Dirac Equation

We now turn to the problem of a field theory for spinors. Let us rewrite the Dirac equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\hbar c}{i} \vec{\alpha} \cdot \vec{\bigtriangledown} \Psi + \beta m c^2\ \Psi \equiv H_{\text{Dirac}} \Psi \tag{72}$$

in a manner in which relativistic covariance is apparent. The operator H_{Dirac} defines the Dirac Hamiltonian.

We first recall that the 4×4 hermitean matrices $\vec{\alpha}$ and β should satisfy the algebra

$$\begin{aligned} \{\alpha_i, \alpha_j\} &= 2\delta_{ij} 1 \\ \{\alpha_i, \beta\} &= 0 \\ \alpha_i^2 = \beta^2 &= 1 \end{aligned} \tag{73}$$

A simple realization of this algebra is given by the 2×2 block (Dirac) matrices

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{74}$$

where the σ^i's are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{75}$$

and I is the 2×2 identity matrix. This is the *Dirac representation* of the Dirac algebra. It is now convenient to introduce the Dirac γ-matrices which are defined by the following relations:

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha^i \tag{76}$$

Thus, the matrices γ^μ are

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{77}$$

and obey the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, 1 \tag{78}$$

In terms of the γ-matrices, the Dirac equation takes the much simpler form

$$(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar})\Psi = 0 \tag{79}$$

where Ψ is a 4-spinor. It is also customary to introduce the notation (Feynman's slash)

$$\not{a} \equiv a_\mu \gamma^\mu \tag{80}$$

Using Feynman's slash, we can write the Dirac equation in the form

$$(i\not{\partial} - \frac{mc}{\hbar})\Psi = 0 \tag{81}$$

From now on I will use units in which $\hbar = c = 1$. Thus energy is measured in units of $(\text{length})^{-1}$ and time in units of length.

Notice that, if Ψ satisfies the Dirac equation, then

$$(i\not{\partial} + m)(i\not{\partial} - m)\Psi = 0 \tag{82}$$

Also,

$$\begin{aligned} \partial\!\!\!/ \cdot \partial\!\!\!/ &= \partial_\mu \partial_\nu \gamma^\mu \gamma^\nu = \partial_\mu \partial_\nu \left(\frac{1}{2} \{\gamma^\mu, \gamma^\nu\} + \frac{1}{2} [\gamma^\mu \gamma^\nu] \right) \\ &= \partial_\mu \partial_\nu g^{\mu\nu} = \partial^2 \end{aligned} \tag{83}$$

where I used the fact that the commutator $[\gamma^\mu, \gamma^\nu]$ is antisymmetric with indices μ and ν. Thus, Ψ satisfies the Klein-Gordon equation

$$\left(\Box + m^2 \right) \Psi = 0 \tag{84}$$

with $\Box = \partial^2$.

2.5.1 Solutions of the Dirac Equation

Let us briefly discuss the properties of the solutions of the Dirac equation. Let us first consider solutions representing particles at rest. Thus Ψ must be constant in space and all its space derivatives must vanish. The Dirac equation becomes

$$i\gamma^0 \frac{\partial \Psi}{\partial t} = m\Psi \tag{85}$$

where $t = x_0$ ($c = 1$). Let us introduce the *bispinors* ϕ and χ

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{86}$$

We find that the Dirac equation reduces to a simple system of two 2×2 equations

$$\begin{aligned} i\frac{\partial \phi}{\partial t} &= +m\phi \\ i\frac{\partial \chi}{\partial t} &= -m\chi \end{aligned} \tag{87}$$

The solutions are

$$\phi_1 = e^{-imt} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \phi_2 = e^{-imt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{88}$$

and

$$\chi_1 = e^{imt} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_2 = e^{imt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{89}$$

Thus, the upper component ϕ represents the solutions with positive energy while χ represents the solutions with negative energy. The additional two-fold degeneracy of the solutions is connected to the spin of the particle.

More generally, in terms of the bispinors ϕ and χ the Dirac Equation takes the form,

$$i\frac{\partial\phi}{\partial t} = m\phi + \frac{1}{i}\vec{\sigma}\cdot\vec{\bigtriangledown}\chi \tag{90}$$

$$i\frac{\partial\chi}{\partial t} = -m\chi + \frac{1}{i}\vec{\sigma}\cdot\vec{\bigtriangledown}\phi \tag{91}$$

In the limit $c \to \infty$, it reduces to the Schrödinger-Pauli equation. The slowly varying amplitudes $\tilde{\phi}$ and $\tilde{\chi}$, defined by

$$\begin{aligned} \phi &= e^{-imt}\tilde{\phi} \\ \chi &= e^{-imt}\tilde{\chi} \end{aligned} \tag{92}$$

with $\tilde{\chi}$ small and nearly static, define positive energy solutions with energies close to $+m$. In terms of $\tilde{\phi}$ and $\tilde{\chi}$, the Dirac equation becomes

$$i\frac{\partial\tilde{\phi}}{\partial t} = \frac{1}{i}\vec{\sigma}\cdot\vec{\bigtriangledown}\tilde{\chi} \tag{93}$$

$$i\frac{\partial\tilde{\chi}}{\partial t} = -2m\chi + \frac{1}{i}\vec{\sigma}\cdot\vec{\bigtriangledown}\tilde{\phi} \tag{94}$$

Indeed, in this limit, the l. h. s. of Eq. 94 is much smaller than its r. h. s. . Thus we can approximate

$$2m\tilde{\chi} \approx \frac{1}{i}\vec{\sigma}\cdot\vec{\bigtriangledown}\tilde{\phi} \tag{95}$$

We can now eliminate the "small component" $\tilde{\chi}$ from Eq. 93 to find that $\tilde{\phi}$ satisfies

$$i\frac{\partial\tilde{\phi}}{\partial t} = -\frac{1}{2m}\bigtriangledown^2\,\tilde{\phi} \tag{96}$$

which is indeed the Schrödinger-Pauli equation.

Conserved Current: Finally, let us introduce one last bit of notation. We define $\bar{\Psi}$ by

$$\bar{\Psi} = \Psi^\dagger\,\gamma^0 \tag{97}$$

in terms of which we can write down the 4-vector j^μ

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{98}$$

which is conserved, *i.e.,* $\partial_\mu j^\mu = 0$. Notice that the time component of j^μ is the density

$$j^0 = \bar{\Psi}\gamma^0\Psi \equiv \Psi^\dagger\Psi \tag{99}$$

and that the space components of j^μ are

$$\vec{j} = \bar{\Psi}\vec{\gamma}\Psi = \Psi^\dagger\gamma^0\vec{\gamma}\Psi = \Psi^\dagger\vec{\alpha}\Psi \tag{100}$$

which we can regard as a (conserved) probability current.

2.5.2 Relativistic Covariance

Let Λ be a Lorentz transformation, $\Psi(x)$ the spinor field in an inertial frame and $\Psi'(x')$ be the Dirac spinor field in the transformed frame. The Dirac equation is *covariant* if the Lorentz transformation

$$x'_\mu = \Lambda^\nu_\mu x_\nu \tag{101}$$

induces a transformation $S(\Lambda)$ in *spinor space*

$$\Psi'_\alpha(x') = S(\Lambda)_{\alpha\beta}\Psi_\beta(x) \tag{102}$$

such that

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)_{\alpha\beta} \Psi_\beta(x) = 0 \Rightarrow \left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)_{\alpha\beta} \Psi'_\beta(x') = 0 \tag{103}$$

Notice two important facts: (1) both the field Ψ and the coordinate x change under the action of the Lorentz transformation, and (2) the γ-matrices and the mass m *do not* change under a Lorentz transformation. Thus, the γ-matrices are independent of the choice of a reference frame. However, they do depend on the choice of the set of basis states in spinor space.

What properties should the representation matrices $S(\Lambda)$ have? Let us first observe that if $x' = \Lambda x$, then

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \equiv \left(\Lambda^{-1}\right)^\nu_\mu \frac{\partial}{\partial x^\nu} \tag{104}$$

for

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{105}$$

Thus, $\frac{\partial}{\partial x^\mu}$ is a covariant vector. By substituting this transformation law back into the Dirac equation, we find

$$i\gamma^\mu \frac{\partial}{\partial x'^\mu}\Psi'(x') = i\gamma^\mu(\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu}\left(S(\Lambda)\Psi(x)\right) \tag{106}$$

Thus, the Dirac equation now reads

$$i\gamma^\mu \left(\Lambda^{-1}\right)^\nu_\mu S(\Lambda)\frac{\partial \Psi}{\partial x^\nu} - mS\ (\Lambda)\,\Psi = 0 \tag{107}$$

Or, equivalently

$$S^{-1}(\Lambda)\, i\gamma^\mu \left(\Lambda^{-1}\right)^\nu_\mu S(\Lambda)\frac{\partial \Psi}{\partial x^\nu} - m\Psi = 0 \tag{108}$$

This equation is covariant provided that $S(\Lambda)$ satisfies

$$S^{-1}(\Lambda)\,\gamma^\mu S(\Lambda)\left(\Lambda^{-1}\right)^\nu_\mu = \gamma^\nu \tag{109}$$

Since the set of Lorentz transformations form a group, *i.e.,* the product of two Lorentz transformations Λ^1 and Λ^2 is the new Lorentz transformation $\Lambda^1\Lambda^2$ and

the inverse of the transformation Λ is the inverse matrix Λ^{-1}, the representation matrices $S(\Lambda)$ should also form a group and obey the same properties. In particular, $S^{-1}(\Lambda) = S\left(\Lambda^{-1}\right)$ must hold. Recall that the invariance of the relativistic interval $x^2 = x_\mu x^\mu$ implies that Λ must obey

$$\Lambda^\nu_\mu \Lambda^\lambda_\nu = g^\lambda_\mu \equiv \delta^\lambda_\mu \tag{110}$$

Thus, $\Lambda^\nu_\mu = \left(\Lambda^{-1}\right)^\mu_\nu$. So we write,

$$S(\Lambda)\,\gamma^\mu S(\Lambda)^{-1} = \left(\Lambda^{-1}\right)^\mu_\nu \gamma^\nu \tag{111}$$

Eq. 111 shows that a Lorentz transformation induces a *similarity* transformation on the γ-matrices which is equivalent to (the inverse of) a Lorentz transformation. For the case of Lorentz boosts, Eq. 111 shows that the matrices $S(\Lambda)$ are *hermitean*. However, for the subgroup $SO(3)$ of rotations about a fixed origin, the matrices $S(\Lambda)$ are unitary.

We will now find the form of $S(\Lambda)$ for an infinitesimal Lorentz transformation. Since the identity transformation is $\Lambda^\mu_\nu = g^\mu_\nu$, a Lorentz transformation which is infinitesimally close to the identity should have the form

$$\Lambda^\mu_\nu = g^\mu_\nu + \omega^\mu_\nu \qquad \left(\Lambda^{-1}\right)^\mu_\nu = g^\mu_\nu - \omega^\mu_\nu \tag{112}$$

where $\omega^{\mu\nu}$ is infinitesimal and antisymmetric in its space-time indices

$$\begin{aligned} \omega^{\mu\nu} &= -\omega^{\nu\mu} \\ \omega^{\mu\nu} &= \omega^\mu_\rho g^{\rho\nu} \end{aligned} \tag{113}$$

Let us parameterize $S(\Lambda)$ in terms of a 4×4 matrix $\sigma_{\mu\nu}$ which is also antisymmetric in its indices, *i.e.,* $\sigma_{\mu\nu} = -\sigma_{\mu\nu}$. Then, we can write

$$\begin{aligned} S(\Lambda) &= I - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \ldots \\ S^{-1}(\Lambda) &= I + \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \ldots \end{aligned} \tag{114}$$

where I stands for the 4×4 identity matrix. If we substitute back, we get

$$(I - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \ldots)\gamma^\lambda(I + \frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta} + \ldots) = \gamma^\lambda - \omega^\lambda{}_\nu\gamma^\nu + \ldots \tag{115}$$

Collecting all the terms linear in ω, we find

$$\frac{i}{4}\left[\gamma^\lambda, \sigma_{\mu\nu}\right]\omega^{\mu\nu} = \omega^\lambda_\nu \gamma^\nu \tag{116}$$

Or, what is the same,

$$[\gamma^\mu, \sigma_{\nu\lambda}] = 2i(g^\mu_\nu\gamma_\lambda - g^\mu_\lambda\gamma_\nu) \tag{117}$$

This matrix equation is solved by

$$\sigma_{\nu\lambda} = \frac{i}{2}[\gamma_\nu, \gamma_\lambda] \tag{118}$$

Under a finite Lorentz transformation $x' = \Lambda x$, the 4-spinors transform as

$$\Psi'(x') = S(\Lambda)\,\Psi \tag{119}$$

with

$$S(\Lambda) = \exp[-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}] \tag{120}$$

The matrices $\sigma_{\mu\nu}$ are the *generators* of the group of Lorentz transformations in the spinor representation. While the space components σ_{jk} are *hermitean* matrices, the space-time components σ_{0j} are *antihermitean*. This feature is telling us that the Lorentz group is not a *compact* unitary group , since in that case all of its generators would be hermitean matrices, but that it is isomorphic to the *non-compact* group $SO(3,1)$. Thus, the representation matrices $S(\Lambda)$ are unitary only under space rotations with fixed origin.

The linear operator $S(\Lambda)$ gives the field in the *transformed frame* in terms of the coordinates of the *transformed frame*. However, we may also wish to ask for the transformation $U(\Lambda)$ which just *compensates* the effect of the coordinate transformation. In other words we seek for a matrix $U(\Lambda)$ such that

$$\Psi'(x) = U(\Lambda)\,\Psi(x) = S(\Lambda)\,\Psi(\Lambda^{-1}x) \tag{121}$$

For an infinitesimal Lorentz transformation, we seek a matrix $U(\Lambda)$ of the form

$$U(\Lambda) = I - \frac{i}{2}J_{\mu\nu}\omega^{\mu\nu} + \ldots \tag{122}$$

We wish to find an expression for $J_{\mu\nu}$. We find

$$\begin{aligned}\left(I - \frac{i}{2}J_{\mu\nu}\omega^{\mu\nu} + \ldots\right)\Psi &= (I - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \ldots)\Psi\,(x^\rho - \omega^\rho_\nu x^\nu + \ldots) \\ &\cong \left(I - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \ldots\right)(\Psi - \partial_\rho\Psi\;\omega^\rho_\nu x^\nu + \ldots)\end{aligned} \tag{123}$$

Hence

$$\Psi'(x) \cong \left(I - \frac{i}{2}\sigma_{\mu\nu}\omega^{\mu\nu} + x_\mu\omega^{\mu\nu}\partial_\nu + \ldots\right)\Psi(x) \tag{124}$$

From this expression we see that $J_{\mu\nu}$ is given by the operator

$$J_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} + i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{125}$$

We easily recognize the second term as the *orbital* angular momentum operator (we will come back to this issue shortly). The first term is then interpreted as

the *spin*. In fact, let us consider purely spacial rotations, whose infinitesimal generator are the space components of $J_{\mu\nu}$, *i.e.,*

$$J_{jk} = i(x_j\partial_k - x_k\partial_j) + \frac{1}{2}\sigma_{jk} \tag{126}$$

We can also define a three component *vector* J_ℓ as the 3-dimensional dual of J_{jk}

$$J_{jk} = \epsilon_{jkl} J_\ell \tag{127}$$

Thus, we get (after restoring the factors of $\hbar$)

$$\begin{aligned} J_\ell &= \frac{i\hbar}{2}\epsilon_{\ell jk}(x_j\partial_k - x_k\partial_j) + \frac{\hbar}{4}\epsilon_{\ell jk}\sigma_{jk} \\ &= i\,\hbar\,\epsilon_{\ell jk}\,x_j\partial_k + \frac{\hbar}{2}(\frac{1}{2}\epsilon_{\ell jk}\sigma_{jk}) \\ J_\ell &\equiv \left(\vec{x}\wedge\hat{\vec{p}}\right)_\ell + \frac{\hbar}{2}\sigma_\ell \end{aligned} \tag{128}$$

The first term is clearly the orbital angular momentum and the second term can be regarded as the spin. With this definition, it is straightforward to check that the spinors $\begin{pmatrix}\phi\\ \chi\end{pmatrix}$ which are solutions of the Dirac equation, carry spin one-half.

2.6 Transformation Properties of Field Bilinears in the Dirac Theory

We will now consider the transformation properties of a number of physical observables of the Dirac theory under Lorentz transformations. Let

$$x'^\mu = \Lambda^\mu_\nu\, x^\nu \tag{129}$$

be a general Lorentz transformation, and $S(\Lambda)$ be the induced transformation for the Dirac spinors $\psi_a(x)$ (with $a = 1,\ldots,4$):

$$\psi'_a(x') = S(\Lambda)_{ab}\,\psi_b(x) \tag{130}$$

Using the properties of the induced Lorentz transformation $S(\Lambda)$ and of the Dirac γ-matrices, is straightforward to verify that the following Dirac bilinears obey the following transformation laws:

1. $$\bar\psi'(x')\,\psi'(x') = \bar\psi(x)\,\psi(x) \tag{131}$$
   which transforms as a *scalar*.

2. $$\bar\psi'(x')\,\gamma_5\,\psi'(x') = \det\Lambda\bar\psi(x)\,\gamma_5\,\psi(x) \tag{132}$$
   which transforms as a *pseudo-scalar*.

3.
$$\bar{\psi}'(x')\,\gamma^\mu\,\psi'(x') = \Lambda^\mu_\nu\,\bar{\psi}(x)\,\gamma^\nu\,\psi(x) \tag{133}$$
which transforms as a *vector*.

4.
$$\bar{\psi}'(x')\,\gamma_5\gamma^\mu\,\psi'(x') = \det\Lambda\,\Lambda^\mu_\nu\,\bar{\psi}(x)\,\gamma_5\gamma^\nu\,\psi(x) \tag{134}$$
which transforms as a *pseudo-vector*.

5.
$$\bar{\psi}'(x')\,\sigma^{\mu\nu}\,\psi'(x') = \Lambda^\mu_\alpha\,\Lambda^\nu_\beta\,\bar{\psi}(x)\,\sigma^{\alpha\beta}\,\psi(x) \tag{135}$$
which transforms as a *tensor*

Above we have denoted by Λ^μ_ν a Lorentz transformation and $\det\Lambda$ is its determinant. We have also used that

$$S^{-1}(\Lambda)\gamma_5 S(\Lambda) = \det\Lambda\gamma_5 \tag{136}$$

Thus, the Dirac algebra provides for a natural basis of the space of 4×4 matrices, which we will denote by

$$\Gamma^{\rm S} \equiv I, \quad \Gamma^{\rm V}_\mu \equiv \gamma_\mu, \quad \Gamma^{\rm T}_{\mu\nu} \equiv \gamma_{\mu\nu}, \quad \Gamma^A_\mu \equiv \gamma_5\gamma_\mu, \quad \Gamma^{\rm P} = \gamma_5 \tag{137}$$

where S, V, T, A and P stand for scalar, vector, tensor, axial vector (or pseudo-vector) and parity respectively. For future reference we will note here the following useful trace identities obeyed by products of Dirac γ-matrices

1.
$$\mathrm{tr}I = 4, \quad \mathrm{tr}\gamma_\mu = \mathrm{tr}\gamma_5 = 0, \quad \mathrm{tr}\gamma_\mu\gamma_\nu = 4g_{\mu\nu} \tag{138}$$

2. If we denote by a_μ and b_μ two arbitrary 4-vectors, then
$$\not{a}\not{b} = a_\mu b^\mu - i\sigma_{\mu\nu}\,a^\mu b^\nu, \qquad \text{and} \quad \mathrm{tr}\,\not{a}\not{b} = 4a\cdot b \tag{139}$$

2.7 Lagrangian for the Dirac Equation

We now seek a Lagrangian density $\mathcal{L}$ for the Dirac theory. It should be a local differentiable functional of the spinor field Ψ. Since the Dirac equation is first order in derivatives and it is Lorentz covariant, the Lagrangian should be Lorentz invariant and first order in derivatives. A simple choice is

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi \equiv \frac{1}{2}\bar{\Psi}i\overleftrightarrow{\not{\partial}}\Psi - m\bar{\Psi}\Psi \tag{140}$$

where $\bar{\Psi}\overleftrightarrow{\partial}\Psi \equiv \bar{\Psi}(\overrightarrow{\partial}\Psi) - (\partial_\mu\bar{\Psi})\gamma^\mu\Psi$. This choice satisfies all the requirements. The equations of motion are derived in the usual manner, *i.e.,* by demanding that the action $S = \int d^4x\,\mathcal{L}$ be stationary

$$\delta S = 0 = \int d^4x\,[\frac{\delta\mathcal{L}}{\delta\Psi_\alpha}\delta\Psi_\alpha + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Psi_\alpha}\delta\partial_\mu\Psi_\alpha + (\Psi\leftrightarrow\bar{\Psi})] \tag{141}$$

The equations of motion are

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\Psi_\alpha} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\Psi_\alpha} &= 0 \\ \frac{\delta\mathcal{L}}{\delta\bar{\Psi}_\alpha} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}_\alpha} &= 0 \end{aligned} \tag{142}$$

By direct substitution we find

$$(i\partial\!\!\!/ - m)\Psi = 0 \tag{143}$$

and

$$\bar{\Psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0 \tag{144}$$

which is the adjoint of the Dirac equation. Here, $\overleftarrow{\partial\!\!\!/}$ indicates that the derivatives are acting *on the left*.

Finally, we can also write down the Hamiltonian density that follows from the Lagrangian of Eq. 140. As usual we need to determine the canonical momentum conjugate to the field Ψ, *i.e.,*

$$\Pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\Psi(x)} = i\bar{\Psi}(x)\gamma^0 \equiv i\Psi^\dagger(x) \tag{145}$$

Thus the Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \Pi(x)\Psi(x) - \mathcal{L} = i\bar{\Psi}\gamma^0\partial_0\Psi - \mathcal{L} \\ &= \bar{\Psi}i\vec{\gamma}\cdot\vec{\bigtriangledown}\Psi + m\bar{\Psi}\Psi \\ &= \Psi^\dagger \underbrace{\left(i\vec{\alpha}\cdot\vec{\bigtriangledown} + m\beta\right)}_{H_{\rm Dirac}} \Psi \end{aligned} \tag{146}$$

Thus we discover that the "one-particle" Dirac Hamiltonian $H_{\rm Dirac}$ of Eq. 72 appears naturally in the field theory as well. Since this Hamiltonian is first order in derivatives (*i.e.,* in the "momentum), unlike its Klein-Gordon relative, it is not manifestly positive. Thus there is a question of the stability of this theory. We will see below that the proper quantization of this theory as a quantum field theory of *fermions* solves this problem. In other words, it will be necessary to *impose* the *Pauli Principle* for this theory to describe a stable system with an energy spectrum that is bounded from below. In this way we will see that there is *natural* connection between the *spin* of the field and the *statistics*. This connection is known as the *Spin-Statistics Theorem*.

2.8 Classical Electromagnetism as a field theory

We now turn to the problem of the electromagnetic field generated by a set of sources. Let $\rho(x)$ and $\vec{j}(x)$ represent the charge density and current at a point

x of space-time. Charge conservation requires that a continuity equation has to be obeyed.

$$\frac{\partial \rho}{\partial t} + \vec{\bigtriangledown} \cdot \vec{j} = 0 \tag{147}$$

Given an initial condition, *i.e.,* the values of the electric field $\vec{E}(x)$ and the magnetic field $\vec{B}(x)$ at some t_o in the past, the time evolution is governed by Maxwell's equations

$$\begin{array}{cc} \vec{\bigtriangledown} \cdot \vec{E} = \rho & \vec{\bigtriangledown} \cdot \vec{B} = 0 \\ \vec{\bigtriangledown} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \vec{j} & \vec{\bigtriangledown} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \end{array} \tag{148}$$

It is possible to recast these statements in a manner in which (a) the relativistic covariance is apparent and (b) the equations follow from a Least Action Principle. A convenient way to see the above is to consider the electromagnetic field tensor $F^{\mu\nu}$ which is the (contravariant) antisymmetric real tensor

$$F^{\mu\nu} = -F^{\nu\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{149}$$

In other words

$$\begin{aligned} F^{0i} &= -F^{i0} = -E^i \\ F^{ij} &= -F^{ji} = \epsilon^{ijk} B^k \end{aligned} \tag{150}$$

where ϵ^{ijk} is the Levi-Civita tensor:

$$\epsilon^{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123) \\ 0 & \text{otherwise} \end{cases} \tag{151}$$

The dual tensor $\tilde{F}$ is defined by

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{152}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the fourth rank Levi-Civita tensor. In particular

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^2 \\ B^3 & E^2 & -E^2 & 0 \end{pmatrix} \tag{153}$$

With these notations, we can rewrite Maxwell's equations in the manifestly covariant form

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu && \text{(Equation of Motion)} \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 && \text{(Bianchi Identity)} \\ \partial_\mu j^\mu &= 0 && \text{(Continuity Equation)} \end{aligned} \tag{154}$$

At this point it is convenient to introduce the vector potential A^μ whose contravariant components are

$$A^\mu(x) = (\frac{A^0}{c}, \vec{A}) \equiv (\frac{\Phi}{c}, \vec{A}) \tag{155}$$

where Φ is the scalar potential, and the current 4-vector $j^\mu(x)$

$$j^\mu(x) = (\rho c, \vec{j}) \equiv (j^0, \vec{j}) \tag{156}$$

The electric field strength $\vec{E}$ and the magnetic field $\vec{B}$ we defined to be

$$\begin{aligned} \vec{E} &= -\frac{1}{c}\vec{\bigtriangledown}A^0 - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\bigtriangledown} \times \vec{A} \end{aligned} \tag{157}$$

In a more compact notation we write

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{158}$$

In terms of the vector potential A^μ, Maxwell's equations have the following additional symmetry

$$A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \Lambda(x) \tag{159}$$

(where $\Lambda(x)$ is an *arbitrary smooth* function of space-time). It is easy to check that, under the transformation of equation (42), the field strength remains unchanged *i.e.,*

$$F^{\mu\nu} \rightarrow F^{\mu\nu} \tag{160}$$

This property is called *Gauge Invariance* and it plays a *fundamental* role in modern physics. By directly substituting the definitions of $\vec{B}$ and $\vec{E}$ in terms of A^μ into Maxwell's equations, we get the wave equation. Indeed

$$\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu \tag{161}$$

which yields

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu \tag{162}$$

This is the *wave equation.* We can further use the gauge-invariance to further restrict A^μ (without these restrictions A^μ is not completely determined). These restrictions are known as the procedure of *fixing a gauge.* The choice

$$\partial_\mu A^\mu = 0 \tag{163}$$

known as the Lorentz gauge, yields the simpler wave equation

$$\Box A^\mu = j^\mu \tag{164}$$

Another "popular" choice is the radiation (or Coulomb) gauge

$$\vec{\bigtriangledown} \cdot \vec{A} = 0 \tag{165}$$

which yields (in units with $c = 1$)

$$\Box A^\nu - \partial^\nu(\partial_0 A^0) = j^\nu \tag{166}$$

In the absence of external sources ($j^\nu = 0$) we can further make the choice $A^0 = 0$. This choice reduces the set of three equations (one for each spacial component of $\vec{A}$) which satisfy

$$\begin{aligned} \Box \vec{A} &= 0 \\ \vec{\bigtriangledown} \cdot \vec{A} &= 0 \end{aligned} \tag{167}$$

The solutions are plane waves of the form

$$\vec{A}(x) = \vec{A} e^{i(p_0 x_0 - \vec{p}\cdot\vec{x})} \tag{168}$$

which are only consistent if $p_0^2 - \vec{p}^2 = 0$ and $\vec{p} \cdot \vec{A} = 0$. This choice is also known as the *transverse* gauge.

We can also regard the electromagnetic field as a dynamical system and construct a Lagrangian picture for it. Since Maxwell's equations are local and gauge invariant, we should demand that the Lagrangian density should have the same properties. A simple choice is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu + \text{gauge fixing terms} \tag{169}$$

This Lagrangian density is manifestly Lorentz invariant. Gauge invariance is satisfied if and only if j_μ is a conserved current ($\partial_\mu j^\mu = 0$) since under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x)$ the field strength does not change but the source term does.

$$\begin{aligned} \int d^4x \; j_\mu A^\mu \quad &\rightarrow \quad \int d^4x \; [j_\mu A^\mu + j_\mu \partial^\mu \Lambda] \\ &= \quad \int d^4x \; j_\mu A^\mu + \int d^4x \; \partial^\mu (j_\mu \Lambda) - \int d^4x \; \partial^\mu j_\mu \; \Lambda \end{aligned} \tag{170}$$

If the sources vanish at infinity, ($\lim_{|x|\rightarrow\infty} j_\mu = 0$), the surface term can be dropped. Thus the action $S = \int d^4x \mathcal{L}$ is invariant if and only if

$$\partial_\mu j^\mu = 0 \tag{171}$$

We can now derive the equations of motion by demanding that the action S be stationary, *i.e.,*

$$\delta S = \int d^4x \; [\frac{\delta \mathcal{L}}{\delta A^\mu} \delta A^\mu + \frac{\delta \mathcal{L}}{\delta \partial^\mu A^\mu} \delta \partial^\mu A^\mu] = 0 \tag{172}$$

Once again, we can integrate by parts to get

$$\delta S = \int d^4x\, \partial^\nu \left[\frac{\partial \mathcal{L}}{\delta \partial^\mu A^\mu} \delta A^\mu \right] + \int d^4x\, \delta A^\mu \left[\frac{\delta \mathcal{L}}{\delta A^\mu} - \partial^\nu \left(\frac{\delta \mathcal{L}}{\delta \partial^\nu A^\mu} \right) \right] \tag{173}$$

If we demand $\delta A^\mu \mid_{\text{surface}} = 0$, we get

$$\frac{\delta \mathcal{L}}{\delta A^\mu} = \partial^\nu \left(\frac{\delta \mathcal{L}}{\delta \partial^\nu A^\mu} \right) \tag{174}$$

Explicitly, we get

$$\frac{\delta \mathcal{L}}{\delta A^\mu} = -j_\mu \tag{175}$$

and

$$\frac{\delta \mathcal{L}}{\delta \partial^\nu A^\mu} = F^{\mu\nu} \tag{176}$$

Thus, we find

$$j^\mu = -\partial_\nu F^{\mu\nu} \tag{177}$$

or, equivalently

$$j^\nu = \partial_\mu F^{\mu\nu} \tag{178}$$

Therefore, the Least Action Principle *implies* Maxwell's equations.

2.9 The Landau Theory of Phase Transitions as a Field Theory

We now turn back to the problem of the statistical mechanics of a magnet which was introduced in the first lecture. In order to be a little more specific, we will consider the simplest *model* of a ferromagnet: the classical Ising model. In this model, one considers an array of atoms on some lattice (say cubic). Each atom is assumed to have a net spin magnetic moment $\vec{S}$. From elementary quantum mechanics we know that the simplest interaction among the spins is the Heisenberg exchange Hamiltonian

$$H = -\sum_{<ij>} J_{ij} \vec{S}(i) \cdot \vec{S}(j) \tag{179}$$

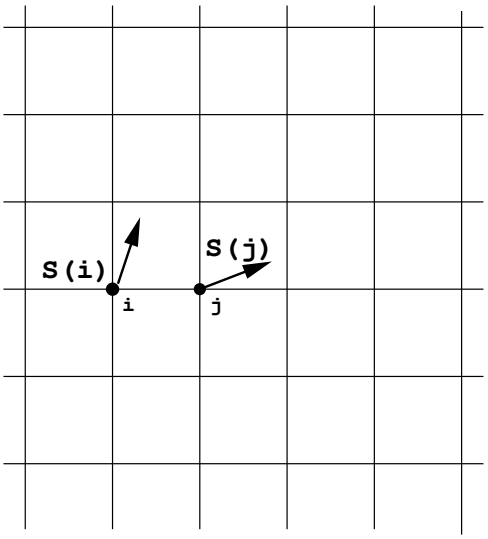
where $< ij >$ are nearest neighboring sites on the lattice. In many situations in which there is magnetic anisotropy, only the z-component of the spin operators play a role. The Hamiltonian now reduces to the Ising case H_I

$$H_I = -J \sum_{<ij>} \sigma(i)\sigma(j) \equiv E(\sigma) \tag{180}$$

where $\sigma(i)$ is the z-projection of the spin at site i.

The equilibrium properties of the system are determined by the partition function Z

$$Z = \sum_{\{\sigma\}} \exp\left(-\frac{E(\sigma)}{T} \right) \tag{181}$$



where T is the temperature and $\{\sigma\}$ is the set of all spin configurations.

In the 1950's, Landau developed a method (or rather a picture) to study these type of problems which in general, are very difficult. Landau first proposed to work not with the microscopic spins but with a set of *coarse-grained* configurations. One way to do this (this approach is actually due to Kadanoff and Wilson) is to subdivide a large system of size L into regions or *blocks* of linear size ℓ such that $a_0 << \ell << L$. Each one of these regions will be centered around a site, say $\vec{x}$. We will denote such a region by $\mathcal{A}(\vec{x})$. The idea is now to perform the sum, *i.e.,* the partition function Z, while keeping the total magnetization of each region $M(\vec{x})$ fixed

$$M(\vec{x}) = \frac{1}{N[\mathcal{A}]} \sum_{y \in \mathcal{A}(\vec{x})} \sigma(\vec{y}) \tag{182}$$

where $N[\mathcal{A}]$ is the number of sites in $\mathcal{A}(\vec{x})$. The restricted partition function is now a functional of $M(\vec{x})$.

$$Z[M] = \sum_{\{\sigma\}} \exp\{-\frac{E(\sigma)}{T}\} \prod_{\vec{x}} \delta \left(M(\vec{x}) - \frac{1}{N(\mathcal{A})} \sum_{\vec{y} \in \mathcal{A}(x)} \sigma(\vec{y}) \right) \tag{183}$$

The variables $M(\vec{x})$ have the property that, for $N(\mathcal{A})$ very large, they take values on the real numbers. Also, the configurations $\{M(\vec{x})\}$ are much more smooth than the configurations $\{\sigma\}$.

At very high temperatures the average magnetization $< M >$ is zero since the system is paramagnetic. On the other hand, if the temperature is low, the average magnetization may be non-zero since the system may now be ferromagnetic. Thus, at high temperatures the partition function Z is dominated by configurations which have $< M >= 0$ while at very low temperatures, the most frequent configurations have $< M > \neq 0$. Landau proceeded to write down an approximate form for the partition function in terms of sums over *smooth*, *continuous*, configurations $M(\vec{x})$ which can be represented in the form

$$Z \approx \int \mathcal{D}M(\vec{x}) \ \exp[-\frac{E\,(M(\vec{x}),T)}{T}] \tag{184}$$

where $\mathcal{D}M(\vec{x})$ is a measure which means "sum over configurations." If the relevant configurations are *smooth* and *small*, the energy functional $E(M)$ can be written as an expansion in powers of $M(\vec{x})$ and of its space derivatives. This is the Landau-Ginzburg (LG) form of the energy and it is given by

$$\begin{aligned} F(M) &\equiv \frac{E_{LG}(M)}{T} \\ &= \int d^d x \ \left[\frac{1}{2}K(T)|\vec{\bigtriangledown}M(\vec{x})|^2 + \frac{a(T)}{2}M^2(\vec{x}) + \frac{b(T)}{4!}M^4(\vec{x}) + \ldots\right] \end{aligned} \tag{185}$$

Thermodynamic stability requires that the stiffness $K(T)$ and the non-linearity coefficient $b(T)$ be positive. The second term has a coefficient $a(T)$ with can have either sign. A simple choice of parameters is

$$\begin{aligned} K(T) &\approx K_0 \\ b(T) &\approx b_0 \\ a(T) &\approx \bar{a}(T - T_c) \end{aligned} \tag{186}$$

where T_c is an approximation to the critical temperature.

The *free energy* $F(M)$ defines a *Classical* (or *Euclidean*) Field Theory. In fact, by rescaling the field $M(x)$ in the form

$$\Phi(x) = \sqrt{K}M(x) \tag{187}$$

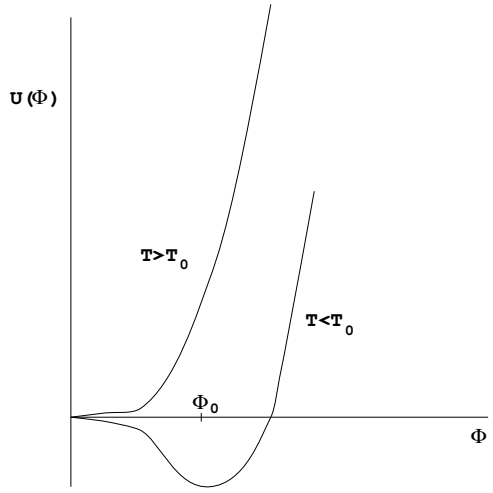
we can write the free energy as

$$F(\Phi) = \int d^d x \ \left\{\frac{1}{2}(\vec{\bigtriangledown}\Phi)^2 + U(\Phi)\right\} \tag{188}$$

where the potential $U(\Phi)$ is

$$U(\Phi) = \frac{\bar{m}^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 + \ldots \tag{189}$$

where $\bar{m}^2 = \frac{a(T)}{K}$ and $\lambda = \frac{b}{K^2}$. Except for the absence of the term involving the canonical momentum $\Pi^2(x)$, $F(\Phi)$ has a striking resemblance to the *Hamiltonian* of a scalar field in Minkowski space! We will see below that this is not an accident.



Let us now ask the following question: is there a configuration $\Phi_c(\vec{x})$ which gives the *dominant* contribution to the partition function Z? If so, we should be able to approximate

$$Z = \int \mathcal{D}\Phi \ \exp\{-F(\Phi)\} \approx \exp\{-F(\Phi_c)\}\{1 + \cdots\} \tag{190}$$

This statement is usually called the *Mean Field Approximation*. Since the integrand is an exponential, the dominant configuration Φ_c must be such that F has a (local) minimum at Φ_c. Thus, configurations Φ_c which leave $F(\Phi)$ *stationary* are good candidates (we actually need local minima!). The problem of finding extrema is simply the condition $\delta F = 0$. This is the same problem we solved for classical Field Theory in Minkowski space. Notice that in the derivation of F we have invoked essentially the same type of arguments: (a) invariance and (b) differentiability. The Euler-Lagrange equations can be derived by using the same arguments that we employed in the context of a scalar field theory. In the case at hand they are

$$-\frac{\delta F}{\delta \Phi(\vec{x})} + \bigtriangledown_j \frac{\delta F}{\delta \bigtriangledown_j \Phi(\vec{x})} = 0 \tag{191}$$

For the case of the Landau theory we get

$$0 = - \bigtriangledown^2 \Phi_c(\vec{x}) + \bar{m}^2 \Phi_c(\vec{x}) + \frac{\lambda}{3!}\Phi_c^3(\vec{x}) \tag{192}$$

The solutions $\Phi_c(x)$ that *minimize* the energy are uniform in space and thus have $\bigtriangledown_j \Phi_c = 0$. Hence, Φ_c is the solution of the very simple equation

$$\bar{m}^2 \Phi_c + \frac{\lambda}{3!}\Phi_c^3 = 0 \tag{193}$$

Since λ is positive and $\bar{m}^2$ may have either sign, depending on whether $T > T_c$ or $T < T_c$, we have to explore both cases.

For $T > T_c, \bar{m}^2$ is also positive and the only real solution is $\Phi_c = 0$. This is the paramagnetic state. But, for $T < T_c, \bar{m}^2$ is negative and *two* new solutions are available, namely

$$\Phi_c = \pm \sqrt{\frac{6 \mid \bar{m}^2 \mid}{\lambda}} \tag{194}$$

These are the solutions with lowest energy and they are *degenerate*. They both represent the *magnetized* state.

We now must ask if this procedure is correct, or rather when can we expect this approximation to work. It is correct $T = 0$ and it will also turn out to be correct at very high temperatures. The answer to this question is the central problem of the theory of Critical Phenomena which describes the behavior of statistical systems in the vicinity of a continuous (or second order) phase transitions. It turns out that this problem is also connected with a central problem of Quantum Field theory, namely when and how is it possible to remove the singular behavior of perturbation theory , and in the process remove all dependence on the short distance (or high energy) cutoff from physical observables. In QFT this procedure amounts to a definition of the continuum limit. The answer to these questions motivated the development of the Renormalization Group which solved both problems simultaneously. It will be a central theme in Physics 583.

2.10 Analytic Continuation: Classical Field Theory and Classical Statistical Mechanics.

We are now going to discuss a mathematical "trick" which will allow us to connect field theory with classical statistical mechanics. Let us go back to the action for a real scalar field $\Phi(x)$ in $D = d + 1$ space-time dimensions

$$S = \int d^D x \; \mathcal{L}(\Phi, \partial_\mu \Psi) \tag{195}$$

where $d^D x$ is

$$d^D x \equiv dx_0 d^d x \tag{196}$$

Let us *formally* carry out the *analytic continuation* of the *time* component x_0 of x_μ from *real* to *imaginary time* x_D

$$x_0 \to -ix_D \tag{197}$$

under which

$$\Phi(x_0, \vec{x}) \to \Phi(\vec{x}, x_D) \equiv \Phi(x) \tag{198}$$

where $x = (\vec{x}, x_D)$. Under this transformation, the action (or rather i times the action) becomes

$$iS \equiv i \int dx_0 \; d^d x \; \mathcal{L}(\Phi, \partial_0 \Psi, \bigtriangledown_j \Phi) \to \int d^D x \; \mathcal{L}(\Phi, -i\partial_D \Phi, \bigtriangledown_j \Phi) \tag{199}$$

If $\mathcal{L}$ has the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - V(\Phi) \equiv \frac{1}{2}(\partial_0 \Phi)^2 - \frac{1}{2}(\vec{\bigtriangledown}\Phi)^2 - V(\Phi) \tag{200}$$

then the analytic continuation yields

$$\mathcal{L}(\Phi, -i\partial_D \Psi, \bigtriangledown \Phi) = -\frac{1}{2}(\partial_D \Phi)^2 - \frac{1}{2}(\vec{\bigtriangledown}\Phi)^2 - V(\Phi) \tag{201}$$

Then we can write

$$iS(\Phi, \partial_\mu \Phi) \xrightarrow[x_0 \to -ix_D]{} - \int d^D x \left[\frac{1}{2} \; (\partial_D \Phi)^2 + \frac{1}{2}(\vec{\bigtriangledown}\Phi)^2 + V(\Phi) \right] \tag{202}$$

This expression has the same form as (minus) the potential energy $E(\Phi)$ for a classical field Φ in $D = d + 1$ *space* dimensions. However it is also the same as the energy for a classical statistical mechanics problem in the same number of dimensions *i.e.,* the Landau-Ginzburg free energy of the last section.

In Classical Statistical Mechanics, the equilibrium properties of a system are determined by the *partition function*. For the case of the Landau theory of phase transitions the partition function is

$$Z = \int \mathcal{D}\Phi \; e^{-E(\Phi)/T} \tag{203}$$

where the symbol "$\int \mathcal{D}\Phi$" means *sum over all configurations* (we will discuss the definition of the "measure" $\mathcal{D}\Phi$ later on). If we choose for energy functional $E(\Phi)$ the expression

$$E(\Phi) = \int d^D x \; \left[\frac{1}{2}(\partial \Phi)^2 + V(\Phi) \right] \tag{204}$$

where

$$(\partial \Phi)^2 \equiv (\partial_D \Phi)^2 + (\vec{\bigtriangledown}\Phi)^2 \tag{205}$$

we see that the partition function Z is formally the analytic continuation of

$$\mathcal{Z} = \int \mathcal{D}\Phi \; e^{iS(\Phi, \partial_\mu \Phi)/\hbar} \tag{206}$$

where we have used $\hbar$ which has units of action (instead of the temperature).

What is the physical meaning of $\mathcal{Z}$? This expression suggests that $\mathcal{Z}$ should have the interpretation of a sum of all possible functions $\Phi(\vec{x}, t)$ (*i.e.,* the *histories* of the configurations of the field Φ) weighed by the phase factor $\exp\{\frac{i}{\hbar} S(\Phi, \partial_\mu \Phi)\}$. We will discover later on that if T is identified with the Planck constant $\hbar$ (which is dimensionally consistent), then $\mathcal{Z}$ represents the *path-integral quantization* of the field theory! Notice that the *semiclassical* limit $\hbar \to 0$ is formally equivalent to the *low temperature* limit of the statistical mechanical system.

The analytic continuation procedure that we just discussed is known under the name of a *Wick rotation*. It amounts to a passage from $D = d + 1$-dimensional Minkowski space to a D-dimensional Euclidean space. We will find that this analytic continuation is a very powerful tool. As we will see below, a number of difficulties will arise when the theory is defined directly in Minkowski space. Primarily, the problem is the presence of ill-defined integrals which are given precise meaning by a deformation of the integration contours from the real time (or frequency) axis to the imaginary time (or frequency) axis. The deformation of the contour amounts to a definition of the theory in Euclidean rather than in Minkowski space. It is an underlying *assumption* that the analytic continuation can be carried out without difficulty. Namely, the assumption is that the result of this procedure is unique and that, whatever singularities may be present in the complex plane, they do not affect the result. It is important to stress that the success of this procedure is not guaranteed. However, in almost all the theories that we know of, this assumption seems to hold. The only case in which problems are known to exist is the theory of Quantum Gravity (which we will not discuss here).