

Wave packet spreading: Temperature and squeezing effects with applications to quantum measurement and decoherence

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A localized free particle is represented by a wave packet and its motion is discussed in most quantum mechanics textbooks. Implicit in these discussions is the assumption of zero temperature. We discuss how the effects of finite temperature and squeezing can be incorporated in an elementary manner. The results show how the introduction of simple tools and ideas can bring the reader into contact with topics at the frontiers of research in quantum mechanics. We discuss the standard quantum limit, which is of interest in the measurement of small forces, and decoherence of a mixed ("Schrödinger cat") state, which has implications for current research in quantum computation, entanglement, and the quantum-classical interface. © 2002 American Association of Physics Teachers. [DOI: 10.1119/1.1447540]

I. INTRODUCTION

The uncertainty principle in quantum mechanics implies that the position and momentum of a particle cannot be determined simultaneously with arbitrary precision. More explicitly

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \quad (1)$$

where $(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$ and $(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$ are the variance of the position and momentum of the particle, respectively. For a noninteracting particle the expected value at time t of an operator \mathcal{O} is given by

$$\langle \mathcal{O}(t) \rangle = \int_{-\infty}^{\infty} dx \psi^*(x, t) \mathcal{O} \psi(x, t), \quad (2)$$

where the wave function $\psi(x, t)$ is the solution of the time-dependent Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi, \quad (3)$$

with H the particle Hamiltonian. For the more general case, such as a particle interacting with a heat bath, the expected value would be given by¹

$$\langle \mathcal{O}(t) \rangle = \text{Tr}\{\rho(t)\mathcal{O}\}, \quad (4)$$

where the density matrix $\rho(t)$ is the solution of the von Neuman equation,

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho], \quad (5)$$

with H now the Hamiltonian for the entire system.

For a free particle, the stationary solutions of the Schrödinger equation are plane waves, for which the particle may be found with equal probability anywhere in space, that is, $\Delta x = \infty$, $\Delta p = 0$. However, one is often interested in describing a localized particle, which can be achieved by constructing a wave packet corresponding to a superposition of plane waves. Such a packet is necessarily not stationary and will spread (or shrink) in time. Wave packet spreading is of fundamental interest, appears in many contexts, and is discussed in introductory² and advanced^{3,4} quantum mechanics textbooks.

We begin our discussion in Sec. II, where we describe the motion of an arbitrary free-particle wave packet. This description is more or less standard, the main result being a general expression for the width at time t in term of the initial data. This expression does not, however, take into account the uncertainty principle; to do so one must evaluate the initial data using Eqs. (2) or (4). We do this in Sec. III, where we begin with a brief derivation of the uncertainty principle. There we introduce annihilation and creation operators that are analogous to those appearing in discussions of the harmonic oscillator,²⁻⁴ but which now apply to an arbitrary state. An immediate consequence is a simple construction of the most general minimum uncertainty wave packet. We conclude Sec. III with an expression for the spreading of an initial minimum uncertainty state.

In Sec. IV we consider the effect of finite temperature on wave packet spreading. An initial wave packet at finite temperature is in a mixed state: there is no corresponding wave function, and the state is described by a density matrix. Nevertheless, we can calculate the temperature effect by forming the observed quantities (the probability distribution or the expected values) with an initial wave function and then averaging over a Maxwell distribution of initial velocities. The result is an additional spreading that dominates when the thermal de Broglie wavelength is small compared to the initial width.

In Sec. V we consider spreading for squeezed states. Squeezing is generally discussed in the context of the harmonic oscillator, but here we discuss squeezing of a Gaussian wave packet. An interesting result is that, while for sufficiently long times the root-mean-square (rms) width of such a wave packet increases linearly with time, for short times it can even shrink. Finally, in Sec. VI we consider two topical applications: the standard quantum limit arising in connection with the measurement of small forces and decoherence. In particular, we give a new and simple demonstration of how to circumvent the standard quantum limit. This limit is of interest not only for gravitational wave detection, but for any application where the question of the accuracy of successive measurements arises. With regard to the decoherence problem, the temperature effect on wave packet spreading is an essential feature.

II. FREE PARTICLE

We begin by reminding ourselves that from either Eq. (2) or (4), we can show that the rate of change of the expected value of an operator \mathcal{O} , with no explicit time dependence, is given by

$$i\hbar \frac{d\langle \mathcal{O} \rangle}{dt} = \langle [\mathcal{O}, H] \rangle. \quad (6)$$

For a free particle with $H = p^2/2m$, we can use the canonical commutation relation

$$[x, p] = i\hbar, \quad (7)$$

to show that

$$\frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m}, \quad \frac{d\langle p \rangle}{dt} = 0. \quad (8)$$

These are the classical equations of motion (Ehrenfest theorem). Therefore, we have

$$\langle x(t) \rangle = \langle x(0) \rangle + \frac{\langle p(0) \rangle}{m} t, \quad \langle p(t) \rangle = \langle p(0) \rangle. \quad (9)$$

In the same way, we can show that

$$\begin{aligned} \frac{d\langle x^2 \rangle}{dt} &= \frac{\langle xp + px \rangle}{m}, & \frac{d\langle xp + px \rangle}{dt} &= \frac{2\langle p^2 \rangle}{m}, \\ \frac{d\langle p^2 \rangle}{dt} &= 0, \end{aligned} \quad (10)$$

and therefore,

$$\langle x^2(t) \rangle = \langle x^2(0) \rangle + \frac{\langle x(0)p(0) + p(0)x(0) \rangle}{m} t + \frac{\langle p^2(0) \rangle}{m^2} t^2. \quad (11)$$

We can write the above results in terms of the variances as

$$\begin{aligned} \Delta x^2(t) &= \Delta x^2(0) + \frac{1}{m^2} \Delta p^2(0) t^2 \\ &+ \frac{\langle x(0)p(0) + p(0)x(0) \rangle - 2\langle x(0) \rangle \langle p(0) \rangle}{m} t. \end{aligned} \quad (12)$$

Hence, for sufficiently long times, $\Delta x(t)$, the rms width of the wave packet at time t , increases linearly with time. However, it is possible for a wave packet to shrink for a time, as we shall discuss in Sec. V.

In the formal solution of the equations of mean motion, it appears that the initial data, $\Delta x^2(0)$, $\Delta p^2(0)$, and $\langle x(0)p(0) + p(0)x(0) \rangle - 2\langle x(0) \rangle \langle p(0) \rangle$, could be given arbitrary values. We emphasize that this is not so, and the initial expectations must be obtained from the initial state by an expression of the form (2) or (4). In particular, the initial variances must satisfy the uncertainty principle (1).

III. MINIMAL (NONQUEEZED) GAUSSIAN WAVE PACKET

To make our discussion as simple as possible, we begin by restricting our discussion to states for which $\langle x \rangle = 0$ and $\langle p \rangle = 0$. We then introduce the operators

$$a = \frac{x}{2\sigma} + i \frac{\sigma p}{\hbar}, \quad a^\dagger = \frac{x}{2\sigma} - i \frac{\sigma p}{\hbar}, \quad (13)$$

where σ is a real parameter. These operators are formally identical with the annihilation and creation operators usually introduced in connection with the harmonic oscillator,²⁻⁴ but here they apply to an arbitrary state (pure or mixed) without reference to an external potential. Next, we form the necessarily positive quantity,

$$\langle a^\dagger a \rangle = \frac{\Delta x^2}{4\sigma^2} + \frac{\sigma^2 \Delta p^2}{\hbar^2} - \frac{1}{2} \geq 0, \quad (14)$$

where we have used the canonical commutation relation in Eq. (7). We seek the minimum of this quantity with respect to variations of σ^2 , which occurs when

$$\sigma^2 = \frac{\hbar \Delta x}{2\Delta p}. \quad (15)$$

With this value of σ^2 , we see that

$$\langle a^\dagger a \rangle = \frac{\Delta x \Delta p}{\hbar} - \frac{1}{2} \geq 0, \quad (16)$$

which is just the uncertainty principle. The minimum uncertainty state, for which the inequality becomes an equality, must be a pure state that corresponds to a wave function ϕ satisfying $\langle a^\dagger a \rangle = \|a\phi\|^2 = 0$. That is, ϕ must satisfy

$$a\phi = \left(\frac{x}{2\sigma} + \sigma \frac{d}{dx} \right) \phi = 0, \quad (17)$$

where we have used the familiar realization of the momentum operator: $p = (\hbar/i)(d/dx)$. The solution of the first-order differential equation in Eq. (17) is

$$\phi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-x^2/4\sigma^2}, \quad (18)$$

where we have chosen the normalization so that $\int_{-\infty}^{\infty} dx \phi^*(x)\phi(x) = 1$.

To extend this result for nonvanishing $\langle x \rangle$ and $\langle p \rangle$, we need only make the replacements $x \rightarrow x - \langle x \rangle$ and $p \rightarrow p - \langle p \rangle$ in Eq. (13) and repeat the argument. The result is that the most general minimum uncertainty wave packet has the form:

$$\phi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left\{ -\frac{(x-x_0)^2}{4\sigma^2} + i \frac{mv_0 x}{\hbar} \right\}, \quad (19)$$

where σ , x_0 , and v_0 are real. Thus the minimum uncertainty wave packet is a Gaussian, centered at x_0 and moving with velocity v_0 .

Suppose we choose the initial state to be a minimum uncertainty state with $\psi(x,0)$ given by Eq. (19). Then we find,

$$\begin{aligned} \langle x(0) \rangle &= x_0, & \langle p(0) \rangle &= mv_0, \\ \langle x^2(0) \rangle &= x_0^2 + \sigma^2, & \langle p^2(0) \rangle &= m^2 v_0^2 + \frac{\hbar^2}{4\sigma^2}, \end{aligned} \quad (20)$$

$$\langle x(0)p(0) + p(0)x(0) \rangle = 2mx_0v_0.$$

With these expressions, Eq. (12) for the mean square width of the wave packet becomes

$$\Delta x^2(t) = \sigma^2 + \left(\frac{\hbar t}{2m\sigma} \right)^2. \quad (21)$$

The wave packet will expand so that the mean square width doubles in a time $t = 2m\sigma^2/\hbar$. During this time, the wave packet will have traveled a distance $\ell = v_0 t = 4\pi\sigma^2/\lambda$, where $\lambda = mv_0/2\pi\hbar$ is the de Broglie wavelength.

Although our derivation might appear to be similar to that found in some textbooks, there are important differences in that we have allowed from the beginning the possibility of a mixed state and we have shown that the minimum uncertainty state is a Gaussian. Furthermore, we have introduced the concept of creation and annihilation operators for arbitrary states and, concomitantly (as we shall see in Sec. V), this derivation enables us to consider squeezing of arbitrary states (as distinct from just harmonic oscillator states).

IV. EFFECT OF TEMPERATURE ON THE SPREADING OF A FREE PARTICLE WAVE PACKET

We proceed by first calculating the wave function for the particle at time t and then forming the probability distribution. This procedure is instructive, because it provides another method for calculating the result in Eq. (12) for $\Delta x^2(t)$. Next, we take into account the thermal distribution of initial velocities.

Consider the solution of the free-particle Schrödinger equation with a given initial state. The general solution is⁴

$$\psi(x,t) = \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} dx' \exp\left\{ -\frac{m(x-x')^2}{2i\hbar t} \right\} \psi(x',0). \quad (22)$$

We now apply this result to the case of an initial minimum uncertainty wave packet with $\psi(x,0)$ of the general form given by Eq. (19). We find,

$$\begin{aligned} \psi(x,t) = & \frac{1}{\left[2\pi \left(\sigma + \frac{i\hbar t}{2m\sigma} \right) \right]^{1/4}} \exp\left\{ -\frac{(x-x_0-v_0 t)^2}{4\sigma^2 + (2i\hbar t/m)} \right. \\ & \left. + i\frac{mv_0}{\hbar}x - i\frac{mv_0^2 t}{2\hbar} \right\}. \end{aligned} \quad (23)$$

The probability distribution is

$$\begin{aligned} P(x;t) = & |\psi(x,t)|^2 = [2\pi\Delta x^2(t)]^{-1/2} \\ & \times \exp\left\{ -\frac{(x-x_0-v_0 t)^2}{2\Delta x^2(t)} \right\}, \end{aligned} \quad (24)$$

which is a Gaussian centered at the mean position of the particle at time t with variance given by Eq. (21).

These results are standard quantum mechanics. Next we consider an ensemble of particles in thermal equilibrium, but so weakly coupled to a heat bath that we can neglect dissipation in the equation of motion. Each particle has a wave function of the form (19), with a Maxwell distribution of initial velocities. (Note that the wave functions differ only by the phase factor $\exp[imv_0 x/\hbar]$ and that the distribution in initial velocities implies a corresponding distribution of the phase.) We obtain the corresponding probability distribution by averaging the distribution (24) over a thermal distribution of initial velocities. The result is

$$\begin{aligned} P_T(x;t) = & \sqrt{\frac{m}{2\pi kT}} \int_{-\infty}^{\infty} dv_0 \exp\left\{ -\frac{mv_0^2}{2kT} \right\} P(x;t) \\ = & \frac{1}{\sqrt{2\pi\Delta x_T^2(t)}} \exp\left\{ -\frac{(x-x_0)^2}{2\Delta x_T^2(t)} \right\}, \end{aligned} \quad (25)$$

in which

$$\Delta x_T^2(t) = \Delta x^2(t) + \frac{kT}{m} t^2 = \sigma^2 + \left(\frac{\hbar^2}{4m^2\sigma^2} + \frac{kT}{m} \right) t^2. \quad (26)$$

Here we have introduced the subscript T to emphasize that the probability distribution corresponds to finite temperature. Thus, there is an additional spreading that is due to the Maxwell distribution of particle velocities. As we shall see, in Sec. VI B, it is this extra term in the spreading of the wave packet that is the origin of decoherence.

We could just as well obtain Eq. (26) by averaging the expressions in Eq. (20) for the initial moments. The result is

$$\begin{aligned} \langle x(0) \rangle_T = & x_0, \quad \langle p(0) \rangle_T = 0, \\ \langle x^2(0) \rangle_T = & x_0^2 + \sigma^2, \quad \langle p^2(0) \rangle_T = mkT + \frac{\hbar^2}{4\sigma^2}, \\ \langle x(0)p(0) + p(0)x(0) \rangle_T = & 0. \end{aligned} \quad (27)$$

If we substitute these results into Eq. (12) for Δx , we find the result (26).

The center of the distribution $P_T(x,t)$ in Eq. (25) does not move because the mean initial velocity is zero. The variance in Eq. (26) is the sum of three terms: the initial variance σ^2 , the uncertainty principle spreading $(\hbar t/2m\sigma)^2$, and the thermal spreading kTt^2/m . The ratio of the last two is

$$\frac{mkT\sigma^2}{4\hbar^2} = \frac{\sigma^2}{4\bar{\lambda}^2}, \quad (28)$$

where $\bar{\lambda} = \hbar/\sqrt{mkT}$ is the mean de Broglie wavelength. Therefore, the thermal spreading will dominate when the mean de Broglie wavelength is small compared to the initial width of the packet.

Note that the initial thermal state we have described is what is called a mixed state. For such a state there is no single wave function with which one can form observables such as the probability distribution in Eq. (24); rather the state is described by a density matrix.¹ We have avoided introducing the density matrix to keep the discussion simple, but for those who would like to see it, the density matrix is given by

$$\begin{aligned} \langle x|\rho(0)|x' \rangle = & \overline{\phi(x)\phi^*(x')} \\ = & \sqrt{\frac{m}{2\pi kT}} \int_{-\infty}^{\infty} dv_0 \exp\left\{ -\frac{mv_0^2}{2kT} \right\} \phi(x)\phi^*(x') \\ = & \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-x_0)^2 + (x'-x_0)^2}{4\sigma^2} \right. \\ & \left. - \frac{mkT(x-x')^2}{2\hbar^2} \right\}, \end{aligned} \quad (29)$$

where ϕ is given by Eq. (19). We also note that we have neglected dissipation during the time development. Of course, in order to come to thermal equilibrium, a particle

must be coupled to a heat bath and there must be a corresponding dissipation. The strength of this coupling would be measured by a typical decay rate γ . If the coupling is weak, we must wait a long time of order γ^{-1} for the system to come to equilibrium, but the equilibrium state will be independent of dissipation. The situation is like that for an ideal gas: collisions are necessary to bring about an approach to equilibrium, but do not appear in the equation of state nor in the velocity distribution. On the other hand, the effect of dissipation on the time development of the initial state can be neglected only for times short compared to γ^{-1} . Our simple expression (26) for wave packet spreading is therefore valid only for short times ($\gamma t \ll 1$) such that the motion is that of a free particle; this short-time behavior is exactly what is relevant for the calculation of decoherence times, as we shall discuss in detail in Sec. VIB.

V. EFFECTS DUE TO SQUEEZING

In general, a squeezed state is defined as one in which the uncertainty of one variable is reduced at the expense of an increase in its conjugate variable.⁵ If the original state is a minimum uncertainty state, then the squeezed state may also be a minimum uncertainty state (the so-called ideal squeezed state, such as a coherent state), but, more generally it is not. We start with the simple case of a minimum uncertainty state which is squeezed so that the uncertainty in x remains unchanged but the uncertainty in p increases. The corresponding squeezed state is also Gaussian,

$$\phi_C(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-[(1-iC)x^2/4\sigma^2]}. \quad (30)$$

This squeezed state can be represented as the result of a unitary operation on the minimum uncertainty state (18),

$$\phi_C(x) = e^{1/4 iC(a+a^\dagger)^2} \phi(x). \quad (31)$$

If we use the general formula (2) with $\psi(x,0) = \phi_C(x)$, we find

$$\begin{aligned} \langle x(0) \rangle &= 0, & \langle p(0) \rangle &= 0, \\ \langle x^2(0) \rangle &= \sigma^2, & \langle p^2(0) \rangle &= \frac{\hbar^2(1+C^2)}{4\sigma^2}, \\ \langle x(0)p(0) + p(0)x(0) \rangle &= \hbar C. \end{aligned} \quad (32)$$

The squeezed state is therefore not a minimum uncertainty state, because $\Delta x \Delta p = (\hbar/2) \sqrt{1+C^2} > (\hbar/2)$.

With these expressions, Eq. (12) for the variance of the wave packet becomes

$$\Delta x^2(t) = \sigma^2 \left(1 + \frac{C\hbar t}{2\sigma^2 m} \right)^2 + \left(\frac{\hbar t}{2m\sigma} \right)^2. \quad (33)$$

If $C < 0$, the wave packet first contracts, then expands; for very long times the wave packet always expands.

Finally, it is clear from the analysis given in Sec. IV that the thermal contribution to the spreading is the same for both the squeezed and unsqueezed states.

VI. APPLICATIONS

A. Standard quantum limit

Accurate measurements of the position of a free mass is a subject of much current interest, in particular in the context

of gravitational-wave detection where questions have arisen as to whether there are fundamental quantum mechanical limits on detection sensitivity. The point is that the act of measurement introduces spreading that affects subsequent measurements. In this context, Braginsky and Vorontsov⁶ have argued that in two successive measurements of the position x of a free mass m made at a fixed time interval t , there is an uncertainty $\Delta x(t)$ in the result of the second measurement satisfying

$$\Delta x(t) \geq \sqrt{\frac{\hbar t}{m}}. \quad (34)$$

The inequality (34) is called the standard quantum limit. On the other hand, Yuen,⁷ while agreeing that this is the correct result for free masses prepared in coherent states (that is, minimum uncertainty Gaussian states), has argued that the inequality can be violated for squeezed states. We now give a simple derivation of the standard quantum limit and Yuen's result.

Consider the squeezed state (30) for which $\Delta x(0) = \sigma$ and $\Delta x^2(t)$ is given by Eq. (33). What is the value of σ^2 for which $\Delta x^2(t)$ is a minimum for a given t ? By calculating the derivative with respect to σ^2 , we find that the minimum occurs when

$$\sigma^2 = \sqrt{1+C^2} \frac{\hbar t}{2m}. \quad (35)$$

At the minimum, $\Delta x^2(t)$ has the value

$$\Delta x^2(t)_{\min} = (\sqrt{1+C^2} + C) \frac{\hbar t}{m}. \quad (36)$$

We therefore have in place of (34) the general inequality

$$\Delta x(t) \geq \sqrt{(\sqrt{1+C^2} + C) \frac{\hbar t}{m}}. \quad (37)$$

For a minimum uncertainty state, where $C=0$, Eq. (37) corresponds to the standard quantum limit (34). On the other hand, for C large and negative, the right-hand side of (37) can be as small as one likes, as noted by Yuen.⁷

While our explicit result (37) supports Yuen's general conclusion that more sensitive detection is possible when the initial measurement results in a squeezed state, a more careful examination of the result makes it clear that it may be difficult to achieve in practice, because the condition (35) places a restriction on σ (the width of the initial wave packet). In fact, a large negative C implies that σ be large, so the initial measurement must have large uncertainty. Moreover, we see from Eqs. (32) and (35) that, when the minimum value of $\Delta x(t)$ is achieved,

$$\langle E \rangle = \left\langle \frac{p^2}{2m} \right\rangle = \frac{m\sigma^2}{2t^2} = \frac{\hbar}{4t} \sqrt{1+C^2}. \quad (38)$$

Thus the energy needed to produce the state increases with increasing C .

B. Decoherence

Decoherence refers to the destruction of a quantum interference pattern and is relevant to the many experiments that depend on achieving and maintaining entangled states. Examples of such efforts are in the areas of quantum teleportation,⁸ quantum information and computation,^{9,10} en-

tangled states,¹¹ Schrödinger cats,¹² and the quantum-classical interface.¹³ For an overview of many of the interesting experiments involving decoherence, we refer to Refs. 11 and 14.

Much of the discussion of decoherence^{15–18} has been in terms of a particle moving in one dimension that is placed in an initial superposition state (a Schrödinger “cat” state) corresponding to two widely separated wave packets, each of the form (19) but having $x_0 = \pm d/2$ so that the packets are separated by a distance d . Thus, in an obvious notation we write the wave function of the two-Gaussian state as

$$\psi(x, t) = N[\psi_1(x, t) + \psi_2(x, t)], \quad (39)$$

where ψ_1 and ψ_2 are each given by the right-hand side of Eq. (23), but with x_0 replaced by $d/2$ and $-d/2$, respectively, and the normalization constant N is

$$N = \frac{1}{\sqrt{2(1 + e^{-d^2/8\sigma^2})}}. \quad (40)$$

Hence

$$P(x, t) = N^2(|\psi_1|^2 + |\psi_2|^2 + 2 \operatorname{Re}\{\psi_1^* \psi_2\}). \quad (41)$$

Thus, the probability distribution consists of three contributions, two of which correspond to the separate packets, whereas the third is an interference term. For a free particle at rest at zero temperature [that is, in Eq. (23) we take $v_0 = 0$], an elementary calculation leads to the result

$$P(x, t) = \frac{N^2}{\sqrt{2\pi\Delta x^2(t)}} \left(\exp\left\{-\frac{\left(x - \frac{d}{2}\right)^2}{2\Delta x^2(t)}\right\} + \exp\left\{-\frac{\left(x + \frac{d}{2}\right)^2}{2\Delta x^2(t)}\right\} + 2 \exp\left\{-\frac{x^2}{2\Delta x^2(t)} - \frac{d^2}{8\Delta x^2(t)}\right\} \cos \frac{\hbar t dx}{4m\sigma^2\Delta x^2(t)} \right). \quad (42)$$

The first two terms are of the single-Gaussian form given by (24) while the interference term is characterized by the cosine factor. The key point to be made here is that this interference term persists for all time. More generally, when either temperature or dissipative effects are present, one measures the disappearance of the interference term, that is, the loss of coherence (decoherence), by defining an attenuation coefficient $a(t)$, which is the ratio of the factor multiplying the cosine to twice the geometric mean of the first two terms. From an examination of Eq. (42), we see that $a(t) = 1$, corresponding to the absence of decoherence.

To take into account the effect of finite temperature, we first form the probability distribution (41) with ψ_1 and ψ_2 given by the form (23) with $x_0 = \pm d/2$, but keeping the terms with v_0 . It is a simple matter to see that the resulting probability distribution can be obtained from (42) with the replacement $x \rightarrow x - v_0 t$. We then average this probability distribution over a thermal distribution of initial velocities, as in (25), to obtain the probability distribution corresponding to a finite temperature T . After a bit of algebra, the result can be written in the form

$$P_T(x; t) = \frac{N^2}{\sqrt{2\pi\Delta x_T^2(t)}} \left(\exp\left\{-\frac{\left(x - \frac{d}{2}\right)^2}{2\Delta x_T^2(t)}\right\} + \exp\left\{-\frac{\left(x + \frac{d}{2}\right)^2}{2\Delta x_T^2(t)}\right\} + 2a(t) \exp\left\{-\frac{x^2}{2\Delta x_T^2(t)} - \frac{d^2}{8\Delta x_T^2(t)}\right\} \cos \frac{\hbar t dx}{4m\sigma^2\Delta x_T^2(t)} \right), \quad (43)$$

where the attenuation coefficient $a(t)$ is given by

$$a(t) = \exp\left\{-\frac{\frac{kT}{m} t^2 d^2}{8\sigma^2\Delta x_T^2(t)}\right\} = \exp\left\{-\frac{\frac{kT}{m} t^2 d^2}{8\sigma^4 + 8\sigma^2 \frac{kT}{m} t^2 + \frac{2\hbar^2 t^2}{m^2}}\right\}. \quad (44)$$

Once again, we see that $a(t) = 1$ for $T = 0$. However, for nonzero T and short times (characteristic of decoherence time scales), whereas the t dependent terms in the denominator are negligible, the t dependent terms in the numerator remain, and thus we obtain

$$a(t) \cong e^{-t^2/\tau_d^2}, \quad (45)$$

where the decoherence time is

$$\tau_d = \frac{\sqrt{8}\sigma^2}{\bar{v}d}, \quad (46)$$

and $\bar{v} = \sqrt{kT/m}$ is the mean thermal velocity. This is consistent with the results obtained in Refs. 16–18, where we found that the dominant contribution to decoherence at high temperatures ($kT \ll \hbar\gamma$, where γ is typical dissipative decay rate), is independent of dissipation. However, for very low temperatures T , dissipation plays an important role, in which case one must use sophisticated techniques from nonequilibrium quantum statistical mechanics.^{16,18}

To see why decoherence is a short-time phenomenon, consider as an example an electron at room temperature (300 K), $\bar{v} = 6.8 \times 10^6$ cm/s. Hence, if we take $d = 1$ cm and $\sigma = 0.4$ Å, then using Eq. (46), we obtain $\tau_d = 6.9 \times 10^{-24}$ s, which is orders of magnitude smaller than typical γ^{-1} values. For this reason it is permissible to take $\gamma t \ll 1$ for calculations involving the calculation of decoherence times and this is why the simple derivation outlined above (which is solely within the framework of elementary quantum mechanics and equilibrium statistical mechanics) works.

VII. CONCLUSIONS

Wave packet spreading is of fundamental interest in quantum mechanics. By extending some of the usual methods, we have been led to many interesting phenomena: temperature and squeezing effects on wave packet spreading with applications to topological phenomena such as how the standard quantum limit may be circumvented by squeezing and how the

temperature dependence of the rate of decoherence may be calculated in a simple manner. Concomitantly, we have introduced simple tools and ideas that are at the frontiers of cutting-edge research in quantum mechanics, such as Schrödinger cat states, decoherence, entanglement, and the classical-quantum interface.

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