Phys 4125 — Spring 2020
HW #6 Solutions

3.25 Know $\Omega(N,q) \simeq \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N$

(a) $S/k = \ln \Omega = q \ln\left(1+\frac{N}{q}\right) + N \ln\left(1+\frac{q}{N}\right)$

when we used Stirling, we dropped $\sqrt{N}, \sqrt{q}$ factors, so $\frac{1}{2}\ln N$ or $\frac{1}{2}\ln q$ are small compared to q & N.

(b) $q = \frac{U}{\epsilon} \Rightarrow S = k\frac{U}{\epsilon}\ln\left(1+\frac{N\epsilon}{U}\right) + kN\ln\left(1+\frac{U}{N\epsilon}\right)$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N \Rightarrow \frac{1}{kT} = \frac{1}{\epsilon}\ln\left(1+\frac{N\epsilon}{U}\right) - \cancel{\frac{U}{\epsilon}\frac{N\epsilon}{U^2}\frac{1}{1+N\epsilon/U}} + \cancel{kN\frac{1}{1+\frac{U}{N\epsilon}}\frac{1}{N\epsilon}}$$

$$\boxed{\frac{1}{kT} = \frac{1}{\epsilon}\ln\left(1+\frac{N\epsilon}{U}\right)} \Rightarrow$$

$$\Rightarrow \boxed{T = \frac{\epsilon}{k\ln(1+N\epsilon/U)}}$$

(c) $U \Rightarrow \quad 1+\frac{N\epsilon}{U} = e^{\epsilon/kT} \Rightarrow \boxed{U = \frac{N\epsilon}{e^{\epsilon/kT}-1}}$

$$\left[C = \left(\frac{\partial U}{\partial T}\right)_N = \frac{N\epsilon^2}{kT^2}\frac{e^{\epsilon/kT}}{(e^{\epsilon/kT}-1)^2} \right.$$

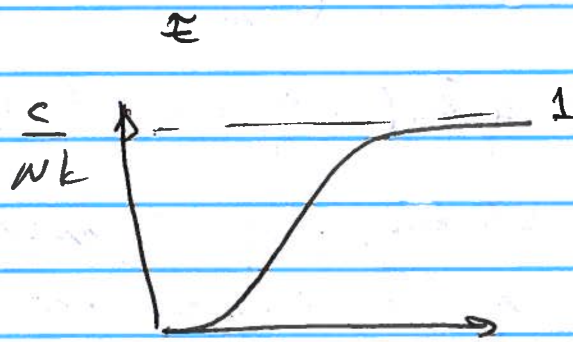
$$\left. = Nk\left(\frac{\epsilon}{kT}\right)^2 \frac{e^{\epsilon/kT}}{(e^{\epsilon/kT}-1)^2} \right]$$

(d) $T \to \infty \qquad e^{\epsilon/kT} \simeq 1 + \epsilon/kT$, then

$$c \simeq Nk\left[\left(\frac{\epsilon}{kT}\right)^2 \frac{1}{(\epsilon/kT)^2} + O\left(\left(\frac{\epsilon}{kT}\right)^2\right)\right] \simeq Nk$$

classical result

(e)



comparison : classical result $\simeq 3R$

Diamond only goes to about 40% of that value, so pick T^* where we have $c \simeq 0.4\,Nk$

$$c/Nk = \frac{x^2 e^x}{(e^x-1)^2} = 0.4$$: numerical solution $x^* = 3.46$

so we have $\frac{\epsilon}{kT^*} = 3.46 \Rightarrow \boxed{\epsilon = 3.46\,kT^*}$

for diamond $T^* \simeq 400\,\text{K} \simeq \epsilon/k \simeq 1{,}400\,\text{K}$

Al $T^* = 80\,\text{K} \Rightarrow \epsilon/k = 275\,\text{K}$

Pb $T^* = 15\,\text{K} \Rightarrow \epsilon/k \simeq 50\,\text{K}$

(f) $$\left[c/Nk \simeq \frac{x^2(1+x+x^2/2+x^3/6)}{x^2(1+x/2+x^2/6)^2} \simeq \frac{1+x+x^2/2}{1+x+\frac{7x^2}{12}} \simeq 1 - \frac{x^2}{12}\right]$$

so $$c \simeq Nk\left[1 - \frac{1}{12}\left(\frac{\epsilon}{kT}\right)^2\right]$$

#3.28

$P = \text{const} \Rightarrow C_P = \frac{7}{2} kN$

$V_f = 2V_i$, $P_i = P_f \Rightarrow PV = NkT \Rightarrow T_f = 2T_i$

so $\Delta S = \int_{T_i}^{T_f} \frac{7}{2} \frac{kN\,dT}{T} = \frac{7}{2} kN \ln 2$

we know $\sim 25\,\ell$ is 1 mole $\Rightarrow kN = R$ @ room T

so for $1\,\ell$ we have $kN = \frac{R}{25} \simeq \frac{1}{3}\,J/k$, so

$$\boxed{\Delta S = \frac{7}{2} \cdot \frac{1}{3} \cdot \ln 2 \simeq \frac{4.9}{6} \simeq 0.8\,J/k}$$

#3.32

(a) $W = F\,d\ell = 2 \cdot 10^3 N \cdot 10^{-3} m = 2J$

(b) sudden $\Rightarrow$ no heat added.

(c) $\Delta U = W = 2J$

(d) Once we reach equilibrium, it does not matter how we got there (entropy is a state function), so we take a slow equivalent process, then $\Delta S = \frac{dQ}{T}$, so

$$\Delta S = \frac{\Delta U + P\Delta V}{T} = \frac{2J + 10^5 Pa \cdot 10^{-2} m^2 \cdot 10^{-3} m}{3 \cdot 10^2 K} =$$

$$= 3.3 \cdot 10^{-3}\,J/k$$ note $\Delta S > 0$

#3.34

(a) This is a coin toss (Heads - left, tails - right)

so we have

$$\frac{S}{k} = \ln \frac{N!}{N_R!\,N_L!} = N\ln N - N_R \ln N_R - N_L \ln N_L$$

(b)

$$\boxed{\frac{S}{k} = N\ln N - N_R \ln N_R - (N - N_R) \ln (N - N_R)}$$

(b) $L = \ell(N_R - N_L)$ by construction

$L = \ell(2N_R - N)$, $N_R = \frac{1}{2}\left[\frac{L}{\ell} + N\right]$

$N_L = \frac{1}{2}\left[N - \frac{L}{\ell}\right]$

(c) We had $dU = TdS + PdV$, and now replace

$$\boxed{dU = \underset{\substack{\uparrow \\ \text{heat}}}{TdS} + \underset{\substack{\uparrow \\ \text{work to stretch}}}{FdL}} \quad (*)$$

(d) $F = -T\left(\frac{\partial S}{\partial L}\right)_U$ from $(*)$

Note $\frac{\partial S}{\partial L} = \frac{1}{2\ell}\frac{\partial S}{\partial N_R} = \frac{k}{2\ell}\frac{\partial}{\partial N_R}\left\{-N_R \ln N_R - (N-N_R)\ln(N-N_R)\right\}$

$= -\frac{k}{2\ell}\left\{\ln N_R + 1 - \ln(N-N_R) - 1\right\} =$

$= +\frac{k}{2\ell}\ln\left\{\frac{N-N_R}{N_R}\right\}$, so that

$$\boxed{F = +\frac{kT}{2\ell}\ln\frac{N_R}{N_L} = \frac{kT}{2\ell}\ln\frac{1+L/N\ell}{1-L/N\ell}}$$

(e) $L \ll N\ell \Rightarrow$ $F \simeq \frac{kT}{2\ell}\cdot\frac{2L}{N\ell} \simeq \frac{kTL}{N\ell^2} \sim L$

$\ln(1 \pm x) \simeq \pm x$

(f) If T increases, band contracts $\rightarrow$ complete analogy with magnet, at higher T number of up and down spins similar.

$F \sim T$, so increases @ higher T

(a) Sudden = adiabatic $\Rightarrow \Delta S = 0$.
But ΔS_{config} decreases: more right than left, fewer microstates. So some other entropy should increase, e.g. vibrational, i.e. increase in T.

#3.36 We saw $S = kq \ln\left(1+\frac{N}{q}\right) + kN \ln\left(1+\frac{q}{N}\right)$
(see problem 3.25a) above.

Now $\mu = -T\left(\frac{\partial S}{\partial N}\right)_U$, $U = $ const means $q = $ const.

so

$$\boxed{\mu = -Tk\left\{ q \frac{1}{1+\frac{N}{q}} \frac{1}{q} + \ln\left(1+\frac{q}{N}\right) - \frac{Nq}{N^2}\frac{1}{1+q/N} \right\} = -kT \ln\left(1+\frac{q}{N}\right)}$$

(b) $N \gg q$ $\quad \mu \simeq -kT \frac{q}{N}$ $\quad |\mu| \ll kT$

small change in entropy if we already have more oscillators than available energy

$N \ll q$ $\quad \mu = -kT \ln \frac{q}{N}$, so this is $O(kT)$

and adding one particle increases multiplicity significantly

$\boxed{\#3.37}$ $\tilde{U} = U + Nmgz$, where U is the ideal gas expression

since this is additive

$$\left[\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V} = \mu_{gas} + mgz = \right.$$

$$\left. = -kT \ln\left[\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right] + mgz\right]$$

(b) Diffusive equilibrium means $\mu_1 = \mu_2$, or,

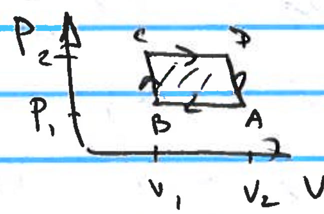
if we assume T to be the same, let's rewrite this as density $n = \frac{N}{V}$, then

$$kT \ln n(z) + mgz = kT \ln(n(z=0))$$

so

$$\boxed{n(z) = n(z=0)\, e^{-mgz/kT}}$$

$\boxed{\#4.1}$ (a)



$$W = (P_2 - P_1)(V_2 - V_1) =$$
$$= 2P_1V_1$$

heat absorbed $Q = C_V(T_C - T_B) + C_p(T_D - T_C)$

$$Q = \frac{5}{2} V_1(P_2 - P_1) + \frac{7}{2} P_2(V_2 - V_1) =$$

$$= \frac{5}{2} P_1V_1 + \frac{7}{2} \cdot 2P_1 \cdot 2V_1 = \frac{33}{2} P_1V_1 \text{, so}$$

$$\left[\epsilon = \frac{2}{33/2} = \frac{4}{33} \approx 12\%\right]$$

(b) $PV = NkT$ so $P_2 V_2 = 6 P_1 V_1 \Rightarrow T_h = 6 T_c$

$\left[\text{so } \epsilon_{ideal} = 1 - \frac{1}{6} = \frac{5}{6} = 83\% \right]$

#4.2

(a) $\epsilon_{max} = 1 - \frac{293}{773} \simeq 62\%$

(b) $\tilde{\epsilon}_{max} = 1 - \frac{293}{873} \simeq 66\%$

about 4% diff, so we get, instead of

1GW $\quad \frac{66}{62} \cdot 1GW \simeq \frac{33}{31} = 1.065\,GW$

So we get extra $6.5 \cdot 10^{-2} GW = 65\,MW$

so in a year $(365 \cdot 24 = 8{,}760 \text{ hours})$, we make

$$65 \cdot 10^3 \cdot 8{,}760 \cdot 5 \cdot 10^{-2} \simeq 20\,M\$$$

#4.6

(a) we have still $\frac{Q_c}{T_{cw}} = \frac{Q_h}{T_{hw}}$ to get entropy balance, so

$$\boxed{\frac{Q_c}{Q_h} = \frac{T_{cw} - T_c}{T_h - T_{hw}} = \frac{T_{cw}}{T_{hw}}}$$

(b) $\text{power} = \frac{\text{Work}}{\text{cycle time}} = \frac{W}{2\Delta t} = \frac{Q_h - Q_c}{2\Delta t}$

where

$$\Delta t = \frac{Q_c}{K(T_{cw} - T_c)} = \frac{Q_h}{K(T_h - T_{hw})}$$

$$\left[\text{power} = K(T_{cw} - T_c)\,\frac{1}{2}\left[\frac{Q_h}{Q_c} - 1\right] 2 \right.$$

$$= K(T_{cw} - T_c)\cdot\frac{1}{2}\left[\frac{T_{hw}}{T_{cw}} - 1\right]$$

$$\left. = \frac{K}{2} T_c \left(\frac{T_{cw}}{T_c} - 1\right)\left(\frac{T_{hw}}{T_{cw}} - 1\right) \right]$$

we want to eliminate T_{cw}; so rewrite

$$\text{power} = \frac{K(T_h - T_{hw})}{2}\left[1 - \frac{T_{cw}}{T_{hw}}\right]$$

with $T_{cw} - T_c = T_{cw}\left(\frac{T_h}{T_{hw}} - 1\right)$ so

$$T_{cw} = \frac{T_c}{2 - T_h/T_{hw}}, \quad \text{so}$$

$$\left[\text{power} = \frac{K(T_h - T_{hw})}{2}\left[1 - \frac{T_c}{2T_{hw} - T_h}\right] \right]$$

(c) $$\frac{2}{K}\frac{\partial P}{\partial T_{hw}} = -\left[1 - \frac{T_c}{2T_{hw} - T_h}\right] + \frac{2T_c(T_h - T_{hw})}{(2T_{hw} - T_h)^2} = 0$$

this gives $$2T_c(T_h - T_{hw}) - (2T_{hw} - T_h)^2 +$$

$$+\, T_c(2T_{hw} - T_h) = 0$$

$$\cancel{4T_{hw}^2 + (2T_c - 4T_h}$$

$$(2T_{hw} - T_h)^2 - T_c T_h = 0$$

$$T_{hw} = \frac{1}{2}\left(T_h \pm \sqrt{T_c T_h}\right)$$

only + sign makes sense. ~~since T_{hw} should approach~~ $T_{hw} > T_c$, so

$$\boxed{T_{hw} = \frac{1}{2}\left(T_h + \sqrt{T_c T_h}\right)}$$

and then

$$T_{cw} = \frac{T_c}{2 - T_h/T_{hw}} = \frac{1}{2}\left(T_c + \sqrt{T_c T_h}\right)$$

(d) Now $\epsilon = 1 - \frac{T_{cw}}{T_{hw}} = 1 - \frac{T_c + \sqrt{T_c T_h}}{T_h + \sqrt{T_c T_h}} =$

$$= 1 - \sqrt{\frac{T_c}{T_h}}$$

For $T_c = 300\ \text{K}$ $T_h = 873\ \text{K}$

$$\left[\epsilon \simeq 1 - \sqrt{\frac{300}{873}} \simeq 40\%\right]$$