Phys 4125 HW#5

Solutions

#2.42 a) need length out of G, M & fundamental constants. GM^2/r energy Mc^2 energy, so

$$\frac{GM}{rc^2} \text{ is dimensionless} \Rightarrow R \sim GM/c^2$$

$$R \simeq \frac{6.67\cdot10^{-11}\cdot2\cdot10^{30}}{9\cdot10^{16}} \simeq 1.5\cdot10^3\,m$$

b) 2.36 basically says, ignore the log $S \sim Nk$.
So suppose we made black hole out of N particles, it formed, i.e. $\Delta S \geq 0 \Rightarrow S \geq N$ # const
Now if we believe that only mass is relevant, we would have $Nm_0 \simeq M \Rightarrow S \sim \frac{M}{m_0}$, minimize m_0, maximize N

c) $\lambda_{max} \simeq \frac{GM}{c^2}$, so $\hbar\omega = \hbar\frac{2\pi c}{\lambda} = \frac{hc}{\lambda}$, so

$$N = \frac{Mc^2}{\hbar\omega} = \frac{Mc^2}{hc}\cdot\frac{GM}{c^2} = \frac{GM^2}{hc},$$

so we would expect $\boxed{S \sim \frac{GM^2}{\hbar c}k}$

d) The point is that

$$N \simeq \frac{GM^2}{hc} = \frac{6.7\cdot10^{-11}\cdot4\cdot10^{60}}{6.6\cdot10^{-34}\cdot3\cdot10^{8}} \simeq 10^{75}$$

so this is huge compared to our usual 10^{23} or so particles.

#3.5 $\Omega \simeq \left(\frac{eN}{q}\right)^q$ (had in previous HW)

Then $S = kq \ln\left(\frac{eN}{q}\right) = kq\left\{1 + \ln\frac{N}{q}\right\}$

Take $U = q\epsilon$, then $q = U/\epsilon$, so

$$S = \frac{kU}{\epsilon}\left(1 + \ln\frac{N\epsilon}{U}\right)$$

Then $\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N = \frac{k}{\epsilon}\left(1 + \ln\frac{N\epsilon}{U}\right) - \frac{kU}{\epsilon}\frac{1}{U} = \frac{k}{\epsilon}\ln\frac{N\epsilon}{U}$

so

$$\boxed{U = N\epsilon\, e^{-\epsilon/kT}}$$

#3.7 We had $S = \frac{8\pi^2 G M^2 k}{hc}$, energy $E = Mc^2$, so

$$S = AE^2, \text{ where } A = \frac{8\pi^2 G k}{hc^5}$$

Then $\frac{1}{T} = 2AE$, so

$$T = \frac{hc^5}{16\pi^2 Gk}\cdot\frac{1}{Mc^2} = \frac{hc^3}{16\pi^2 GMk}$$

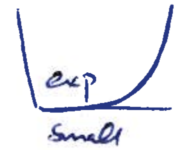
Then we have

$$T \simeq \frac{10^{-34}\cdot 10^{24}\cdot 27\cdot 6}{16\cdot 10\cdot 6\cdot 10^{-11}\cdot 2\cdot 10^{30}\cdot 1.4\cdot 10^{-23}} \simeq$$

$$\simeq 0.6\cdot\frac{10^{-10}}{10^{-4}} \simeq 10^{-7}\,K.$$

#3.8 $\frac{\partial U}{\partial T} = N\epsilon \left(\frac{\epsilon}{kT^2}\right) e^{-\epsilon/kT} = Nk\left(\frac{\epsilon}{kT}\right)^2 e^{-\epsilon/kT}$

we discussed this in class

C 

note this is only for $kT \ll \epsilon$

#3.12 Estimate heat loss from home

Monthly heating ~ \$100, so ~1500 kW·h,
Let's say 8% on a winter cold day ≃
≃ 110 kW·h per day ≃
$\simeq 110 \cdot 10^3 \cdot 3.6 \cdot 10^3 \simeq 4 \cdot 10^8$ J

Then ΔS outside $\frac{Q}{T_{out}}$ $\quad \Delta S_{in} = -\frac{Q}{T_{in}}$, so

$$\left[\Delta S = Q\left(\frac{1}{T_{out}} - \frac{1}{T_{in}}\right) \simeq Q\left(\frac{1}{273} - \frac{1}{300}\right) \simeq \right.$$
$$\simeq 4 \cdot 10^8 \frac{27}{300 \cdot 273} \simeq 4 \cdot 10^8 \frac{1}{3 \cdot 10^3} \simeq$$
$$\left. \simeq 1.3 \cdot 10^5 \text{ J/K} \right]$$

#3.14 If $C_v = aT + bT^3$, then

$$S(T) = \int_0^T \frac{C_v(T')dT'}{T'} = \int_0^T \left(\frac{aT' + bT'^3}{T'}\right) dT' =$$

$$\boxed{S(T) = aT + \frac{b}{3} T^3}$$ (assumed $S(0)=0$

Then $\cancel{S(1K)} =$ Note $\frac{3a}{b} \sim \frac{1.3 \cdot 10^{-3} \cdot 3}{2.5 \cdot 10^{-5}} \simeq 1.6 \cdot 10^2$, so
below $T^* \simeq 13$K $\quad (T^{*2} = \frac{3a}{b})$
almost all $S(T)$ is due to linear term

Then $S(1K) \simeq a = 1.4 \cdot 10^{-3}\ J/K$

For 10 K two are more comparable

$$S(10K) \simeq 1.3 \cdot 10^{-2} + \frac{1}{3} 2.5 \cdot 10^{-5} \cdot 10^{3} \simeq$$

$$\simeq 1.3 \cdot 10^{-2} + 1.25 \cdot 10^{-2} \simeq 2.5 \cdot 10^{-2}\ J/K$$

The two contributions are about equal

$\boxed{\#3.16}$ a) $1\,GB = 2^{10}\,MB = 2^{10}2^{10}\,KB = 2^{30}B = 2^{33}\,bits$

So we started with a specified combination, ended up in any of 2^{33}, so

$$S = k \cdot \ln 2^{2^{33}} = k \cdot 2^{33} \ln 2 \simeq \cancel{k \cdot 10^{11} \ln 2} \simeq$$

$$\simeq k (2^{10})^{3.3} \ln 2 \simeq k\, 10^{10} \ln 2 \simeq$$

$$\simeq 0.6 \cdot 10^{10} \cdot k = 8.4 \cdot 10^{-14}\ J/K$$

b) $Q = T \Delta S = 3 \cdot 10^{2} K \cdot 8.4 \cdot 10^{-14}\ J/K \simeq 2.5 \cdot 10^{-11}\ J$
small

$\boxed{\#3.20}$ We need $x = \frac{\mu B}{kT} = \frac{9 \cdot 10^{-24} \cdot 2}{1.38 \cdot 10^{-23} \cdot 2.2} \simeq 6 \cdot 10^{-1} = 0.6$

Then $\frac{U}{N\mu B} = -\tanh x \simeq -0.56 \qquad \frac{M}{N\mu} \simeq 0.56$

We saw in class

$$\frac{S}{k} = N \ln N - N_p \ln N_p - (N - N_p) \ln (N - N_p)$$

$$= N \ln N - N_p \ln \frac{N_p}{N} - (N - N_p) \ln \frac{N - N_p}{N} - N \ln N$$

$$= -N \left\{ \frac{N_p}{N} \ln \frac{N_p}{N} + \frac{N - N_p}{N} \ln \frac{N - N_p}{N} \right\}$$

and

$$\cancel{\frac{N_\uparrow}{N} = \frac{N}{N_\mu}} \qquad U = \mu B (N - 2N_\uparrow), \text{ so}$$

$$\frac{U}{\mu B N} = 1 - 2\frac{N_\uparrow}{N}$$

$$\frac{N_\uparrow}{N} = \left(1 - \frac{U}{\mu B N}\right)\frac{1}{2}$$

(recall $U<0$)

Then

$$\frac{N - N_\uparrow}{N} = \left(1 + \frac{U}{\mu B N}\right)\frac{1}{2}$$

Now

$$\frac{S}{Nk} = -\frac{1}{2}\left(1 - \frac{U}{\mu B N}\right)\ln\left(1 - \frac{U}{\mu B N}\right) - $$

$$-\frac{1}{2}\left(1 + \frac{U}{\mu B N}\right)\ln\left(1 + \frac{U}{\mu B N}\right)$$

Max entropy : disordered $S = k \ln 2^N = kN\ln 2$

So

$$\frac{S}{S_{max}} = -\frac{1}{2\ln 2}\left\{\left(1 - \frac{U}{\mu B N}\right)\ln\left(1 - \frac{U}{\mu B N}\right) - 2\ln 2\right.$$

$$\left. + \left(1 + \frac{U}{\mu B N}\right)\ln\left(1 + \frac{U}{\mu B N}\right)\right\} \simeq 0.76$$

To get $\tanh x = 0.99 \Rightarrow x \simeq 2.7$

So we need to quadruple field or lower T, to $1/4$ of the value . . .

$\boxed{3.23}$ If we take

$$\frac{1}{Nk}S = -\frac{1}{2}\left\{\left(1-\frac{U}{\mu BN}\right)\ln\left(1-\frac{U}{\mu BN}\right) - 2\ln 2 + \right.$$

$$\left. + \left(1+\frac{U}{\mu BN}\right)\ln\left(1+\frac{U}{\mu BN}\right)\right\}$$

$$= -\frac{1}{2}\left\{\ln\left(1-\left(\frac{U}{\mu BN}\right)^2\right) + \frac{U}{\mu BN}\ln\frac{1+U/\mu BN}{1-U/\mu BN} - 2\ln 2\right\}$$

$$= -\frac{1}{2}\left\{\ln\left(1-\tanh^2 x\right) + \tanh x\,\ln\frac{1+\tanh x}{1-\tanh x} - 2\ln 2\right\}$$

$$= -\frac{1}{2}\left\{\ln\frac{1}{\cosh^2 x} + \tanh x\,\ln\left[\frac{1+\frac{e^x-e^{-x}}{e^x+e^{-x}}}{1-\frac{e^x-e^{-x}}{e^x+e^{-x}}}\right]\right\}$$

$$= \frac{1}{2}\left\{2\ln\cosh x + \tanh x\,\ln\left\{\frac{2e^x}{2e^{-x}}\right\}\right\}$$

$$= \frac{1}{2}\left\{2\ln\cosh x + 2x\tanh x + 2\ln 2\right\}$$

$$= \ln\cosh x + x\tanh x + \ln 2 =$$

$$= \boxed{\ln(2\cosh x) + x\tanh x}$$