Phys 4125 – 2020

HW # 4 Solutions

$\boxed{\# 2.8}$

a) Macrostate : distribution of energy.

fix q_A, that fixes $q - q_A = q_B$ too. So

$q_A = 0, 1 \dots 20 \Rightarrow \boxed{21}$ states

b) We have formula

$$\Omega(20,20) = \frac{39!}{20!\,19!} \simeq 7\cdot 10^{10}$$

c) $q_A = 20 \quad q_B = 0 \Rightarrow \Omega_B = 1$

$$\Omega_A = \Omega(10,20) = \frac{29!}{20!\,9!} \simeq 10^7, \text{ so}$$

$$P(q_A = 20) = \frac{\Omega_A \Omega_B}{\Omega} = \frac{10^7}{7\cdot 10^{10}} \simeq 1.4\cdot 10^{-4}$$

d) $$\Omega_A = \Omega_B = [\Omega(10,10)]^2 = \left[\frac{19!}{10!\,9!}\right]^2 =$$

$$= \left[\frac{19\cdot \cancel{18}\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot \boxed{12}\cdot 11}{1\cdot 2\cdot \cancel{3}\cdot \boxed{4}\cdot 5\cdot \cancel{6}\cdot 7\cdot 8\cdot 9}\right]^2 =$$

$$= \left[\frac{19\cdot 17\cdot \cancel{2}\cdot \cancel{3}\cdot 2\cdot 13\cdot \cancel{3}\cdot 11}{\cancel{2}\cdot \cancel{5}\cdot \cancel{3}\cdot \cancel{9}}\right]^2 =$$

$$= [19\cdot 17\cdot 13\cdot 11\cdot 2]^2 = 8.5\cdot 10^9$$

So

$$P = \frac{8.5\cdot 10^9}{7\cdot 10^{10}} \simeq 1.2\cdot 10^{-1} \simeq 12\%$$

(e)

(e) Strictly speaking → never.
However, if we start with max multiplicity = max probability = state in (d) we would not be able to go back without decreasing entropy.

$\boxed{\#2.13}$

a) $e^{a \ln b} = [e^{\ln b}]^a = b^a$

b) $\ln(a+b) = \ln[a(1+\frac{b}{a})] =$

$= \ln a + \ln(1+\frac{b}{a}) \simeq \ln a + \frac{b}{a}$

$\boxed{\#2.15}$

$N = 50$ $\qquad N! \simeq 3.04141 \cdot 10^{64}$
according to my calculator

$\ln N! = 148.47777$

Stirling: $\ln N! = N \ln N - N =$
$= N(\ln N - 1) = 145.6012$

If we include next

$\ln N! \simeq N \ln N - N + \frac{1}{2} \ln(2\pi N)$

$\simeq N \ln N - N + 2.87 \simeq 148.47$

so this becomes almost exact.

#2.17 Consider $\Omega(N,q) \simeq \frac{(q+N)!}{q!\,N!}$ (only requires $q, N \gg 1$)

Then $\Omega(N,q) \simeq (N+q)\ln(N+q) - N\ln N - q\ln q$,

but now expand in $\frac{q}{N}$, not $\frac{N}{q}$ $\Rightarrow$

$$(N+q)\ln(N+q) \simeq (N+q)\left[\ln N + \frac{q}{N}\right] =$$

$$= (N+q)\ln N + q + \frac{q^2}{N}, \text{ so}$$

$$\ln\Omega \simeq q\ln\frac{N}{q} + q \Rightarrow \boxed{\Omega \simeq \left(\frac{eN}{q}\right)^q}$$

#2.23 $N = 10^{23}$ ⓐ $2^{10^{23}}$ total, but our states

are $\Omega_{1/2} = \binom{N}{N/2} = \frac{N!}{\left(\frac{N}{2}!\right)^2} \simeq \frac{N^N e^{-N}\sqrt{2\pi N}}{N^N 2^{-N} e^{-N}\pi N} =$

$$\Rightarrow \Omega \simeq 2^N\sqrt{\frac{2}{\pi N}}$$

so it is smaller than <u>total</u> # of states only by a factor of $\sim\frac{1}{\sqrt{N}}$.

ⓑ 10^9 times/sec 10^{10} years $\simeq 3\cdot10^{17}$ seconds (see class notes)

Then we explore $3\cdot10^{26}$ states

In contrast $2^{10^{23}} \simeq 2^{10\cdot10^{22}} \simeq 10^{3\cdot10^{22}}$

so we explore a fraction of

$$10^{-3\cdot10^{22}+26} \simeq 10^{-3\cdot10^{22}} \text{ th}$$

(c) Therefore accessible means that we can explore one of equivalent states and don't know a priori which one.

$\boxed{\#2.25}$ Random walk

(a) Obviously you are most likely to return to the same place: equal number of steps forward & back. This somewhat depends on whether N is even or odd, but for large N, when you traveled N steps, and find yourself 0 or ± 1 steps from where you started, this is not too relevant.

b) We saw that this is another version of coin toss that near maximum we have $e^{-x^2/2N}$ steps

To compute distance we ~~would~~ don't care in which direction we go, so only how large typical $|x|$ is. We expect $x \sim \sqrt{2N}$
Technically we would write

not required but bonus points

$$P(x) \simeq A e^{-x^2/2N}$$

$$\langle x \rangle \simeq \int_0^\infty A x e^{-x^2/2N} dx = A \int_0^\infty dy\, e^{-y/N} = AN$$

and
$$A \int_0^\infty e^{-x^2/2N} dx = 1 \quad \text{which gives}$$

$$A \sqrt{\frac{\pi}{2} 2N} = 1 \quad A = \sqrt{\frac{2}{\pi N}} \quad \boxed{\langle x \rangle = \sqrt{\frac{2N}{\pi}}}$$

(c) We saw (p. 42) $\ell \simeq 150\,nm$ (above Eq. 1.63)

Number of steps $\simeq \frac{1}{\Delta t}$ where Δt is time between collisions $N \sim \frac{1}{3\cdot 10^{-10}} \simeq 3\cdot 10^{9}$, so

we expect $\langle x \rangle \sim \sqrt{2N}\cdot \ell \simeq \sqrt{30}\cdot 10^{4}\cdot 1.5\cdot 10^{-7}\sqrt{2}$

$$\simeq \overset{8}{\cancel{5.5}}\cdot 1.5\cdot 10^{-3} \simeq 12\,mm$$

- Grows with time as $\sqrt{t} \Rightarrow N \sim t$, $x \sim \sqrt{N}$
- ℓ depends on T as $\ell \sim \frac{V}{N} \sim \frac{kT}{p}$

but $\bar{v} \sim \sqrt{T}$, so $N \sim \frac{1}{\Delta t} \sim \frac{\bar{v}}{\ell} \sim \frac{\sqrt{T}}{T} \sim \frac{1}{\sqrt{T}}$

so we have

$$\boxed{\langle x \rangle \sim \sqrt{N}\,\ell \sim T\cdot T^{-1/4} \sim T^{3/4}}$$

consistency with diffusion

$D \sim \ell \bar{v}$ and blob $\exp(-x^2/Dt)$

so we need now $\Delta t = \frac{\ell}{v}$, so

$$x^2 \sim Dt \sim \ell v t \sim \ell^2 \frac{t}{\Delta t}$$

which is sensible result for us

#2.26 Need $\Omega_1 \sim A \cdot A_p / h^2$ $\quad A_p \sim \pi p_{max}^2 \sim$
$\sim \pi \cdot 2mU$

Then we have

$$\Omega \approx \frac{A^N (2\pi m U)^N}{h^{2N} \cdot [N!]^2}$$

#2.27 We need

$$P_1 \sim (0.99) \quad \text{for each, so}$$

$$P_N \sim (0.99)^N = (0.99)^{10^{23}}$$

$$\ln P_N \sim 10^{23} \ln 0.99 \sim -10^{21}$$

$$\left[P_N \sim e^{-10^{21}} \simeq 10^{-4 \cdot 10^{20}} \right]$$

#2.30 (a) we solved this in class, (problem 2.22) too
instructions to the grader are to
be harsh on that problem.

(a) $\Omega(N,q) \simeq \Omega(2N, 2N) = \dfrac{\cancel{(4N)!}}{} = \dfrac{2^{4N}}{\sqrt{8\pi N}}$

because

$$\Omega(N,q) \simeq \frac{(q+N)^{q+N} e^{-(q+N)}}{q e^{-q} N e^{-N}} \sqrt{\frac{2\pi(q+N)}{2\pi N \, 2\pi q}} \quad \frac{N}{q+N}$$

$$\simeq \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N \sqrt{\frac{N}{2\pi q (q+N)}}$$

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so $$\frac{S}{k} = 4N \ln 2 - \frac{1}{2}\ln(8\pi N) =$$
$$= 4\cdot 10^{23} \ln 2 - \frac{1}{2}\ln(4\pi) - \frac{1}{2}23 \ln 10$$
$$\simeq 2.7\cdot 10^{23} - 28$$

(b) $$\Omega(N,N) = \frac{2^{2N}}{\sqrt{4\pi N}} \Rightarrow \Omega_0 = \frac{2^{4N}}{4\pi N},$$

(c) so $$\frac{S}{k} - \frac{S_0}{k} = \ln N - \frac{1}{2}\ln N$$
$$= \frac{1}{2}\ln N = \boxed{28}$$

fractional difference $\left[\frac{\Delta S}{S} \simeq \frac{28}{2.7\cdot 10^{23}} = 10^{-22}\right]$

(d) Violation of 10^{-22} is not measurable.

II 2.31 Take Eq. (2.40) $$\Omega = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{(3N/2)!}\left(\sqrt{2mU}\right)^{3N}$$

and log

$$\ln\Omega = \underline{N\ln V} + \frac{3N}{2}\ln(2mU) + \frac{3N}{2}\ln\pi - 3N\ln h$$
$$- \underline{N\ln N} + N - \underline{\frac{3N}{2}\ln\frac{3N}{2}} + \frac{3N}{2} =$$
$$= N\left\{\ln\left[\frac{V}{N}\left(\frac{2mU\pi}{h^2}\right)^{3/2}\left(\frac{2}{3N}\right)^{3/2}\right] + \frac{5}{2}\right\}$$
$$= N\left\{\ln\left[\frac{V}{N}\left(\frac{4\pi mU}{3h^2N}\right)^{3/2}\right] + \frac{5}{2}\right\}$$

$\boxed{\#2.32}$ We use result of 2.26

$$\ln \Omega = N \ln A + N \ln(2\pi m U/h^2) - 2(N \ln N - N) =$$

$$= N\left\{\ln\left[\frac{A}{N} \frac{2\pi m U}{N h^2}\right] + 2\right\}$$

$\boxed{\#2.33}$

$$\left[S = kN\left\{\ln\left[\frac{kT}{P}\left(\frac{4\pi m}{3h^2}\right)^{3/2} \frac{3}{2}kT\right)^{3/2}\right] + \frac{5}{2}\right\}$$

$$= R\left[\ln\left[\frac{kT}{P}\left(\frac{2\pi m kT}{h^2}\right)^{3/2}\right] + \frac{5}{2}\right] \simeq 160\ \mathrm{J/K}$$

$\boxed{\#2.39}$ Distinguishable atoms: no $N!$ for permutations. See eq. (2.57)

$\cancel{\Omega} =$

$$S_{dist} = Nk\left[\ln\left(V\left(\frac{4\pi m U}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right]$$

$$S = Nk\left[\ln\left(\frac{V}{N}\left(\frac{4\pi m U}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right]$$

$$\Delta S = Nk\{\ln N - 1\} \simeq R \cdot \{23 \ln 6 - 1\}$$

$$\simeq R \cdot 41.2 \simeq 360\ \mathrm{J/K}$$

Recall $S \simeq 126\ \mathrm{J/K}$, (see. (2.50))

So $\boxed{S_{dist} \simeq 4S}$