Moment of Inertia and Strain

January 5, 2003

1 Introduction

Gravitational radiation is produced by changing gravitational fields. An object placed in this field will experience mechanical strains due to waves with a magnitude of the order of the dimensionless wave amplitude h (K.D. Kokkotas)[1]. In this report, the expressions for strain are derived from the second time derivatives of the trace free or reduced quadrupole moment tensor of different objects. In general, the trace free quadrupole moment tensor is given by

$$\Im_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} TrI \quad (1)$$

$$= I_{ij} - \frac{1}{3}\delta_{ij} \sum_{k=1}^{3} I_k^k \quad (2)$$

where I_{ij} is the component of the moment of inertia tensor. For discrete masses, the moment of inertia tensor can be found by using the expression (http://scienceworld.wolfram.com/physics/MomentofInertia.html)

$$I_{jk} = \Sigma_i M_i \Big(r_i^2 \delta_{jk} - x_{i,j} x_{i,k} \Big) \quad (3)$$

For continuous mass distribution

$$I_{jk} = \int_v \rho(r) \Big(r^2 \delta_{jk} - x_j x_k \Big) dv \quad (4)$$

where dv is the volume element. In the following sections, the moment of inertia for different objects is derived from the observer's point of view. In all the cases, initially, the moment of inertia in a rotating co-ordinate system is found by using the above formula and later Euler angle's are applied to obtain the I tensor in the observer's frame. Then the expression's for strain are written in terms of the second time derivative of the trace free quadrupole moment tensor components. Also, Energy radiated per unit time (Luminosity) is calculated for each object.

2 Strain and Luminosity

The gravitational waves are calculated by using the following strain formulae (Misner et al. 1973; see also New et al. 1995, and Schutz 1990, p. 229):

$$h_+ = \frac{G}{c^4} \frac{\ddot{\Im}_{xx} - \ddot{\Im}_{yy}}{r}, \quad (5)$$

$$h_\times = \frac{G}{c^4} \frac{2\ddot{\Im}_{xy}}{r}. \quad (6)$$

[1] K.D. Kokkotas. Department of physics, Aristotle University of Thessaloniki.

The luminosity is (cf., Schutz 1990, p. 240, eq. 9.123),

$$L = \dot{E}_{GR} = \frac{1}{5}\frac{G}{c^5}\left\langle \frac{d^3\Im_{ij}}{dt^3}\frac{d^3\Im^{ij}}{dt^3}\right\rangle . \tag{7}$$

3 For two different point masses rotating about z-axis

Consider two point objects of masses m_1 and m_2 rotating about z-axis. In the center of mass frame, we can get the moment of inertia tensor as follows. Let's assume that center of mass frame is denoted by S'. For a single mass,

$$x'_1 = r_1 \qquad y'_1 = 0 \qquad z'_1 = 0 \tag{8}$$

where r_1 is the distance of mass m_1 from the origin.
Now, we can derive the components of the moment of inertia tensor using equation 1.

$$I'_{xx1} = m_1[y'^2_1 + z'^2_1] = 0$$

$$I'_{yy1} = m_1[x'^2_1 + z'^2_1] = m_1 r_1^2$$

$$I'_{zz1} = m_1[x'^2_1 + y'^2_1] = m_1 r_1^2$$

All cross terms are zero

$$I'_{xy1} = m_1[-x'_1 y'_1] = 0$$

$$I'_{yz1} = m_1[-y'_1 z'_1] = 0$$

$$I'_{zx1} = m_1[-z'_1 x'_1] = 0$$

Similarly, for the other mass

$$I'_{xx2} = m_2[y'^2_2 + z'^2_2] = 0$$

$$I'_{yy2} = m_2[x'^2_2 + z'^2_2] = m_2 r_2^2$$

$$I'_{zz2} = m_2[x'^2_2 + y'^2_2] = m_2 r_2^2$$

$$I'_{xy2} = m_2[-x'_2 y'_2] = 0$$

$$I'_{yz2} = m_2[-y'_2 z'_2] = 0$$

$$I'_{zx2} = m_2[-z'_2 x'_2] = 0$$

Here r_2 is the distance of the second mass from the origin.

So, the total moment of inertia tensor in center of mass frame is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & m_1r_1^2 + m_2r_2^2 & 0 \\ 0 & 0 & m_1r_1^2 + m_2r_2^2 \end{pmatrix}$$

In the observers frame of reference, the components of moment of inertia tensor can be obtained by using Euler angle's.

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_1r_1^2 + m_2r_2^2 & 0 \\ 0 & 0 & m_1r_1^2 + m_2r_2^2 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (m_1r_1^2 + m_2r_2^2)\sin^2\phi & (m_1r_1^2 + m_2r_2^2)\sin\phi\cos\phi & 0 \\ (m_1r_1^2 + m_2r_2^2)\sin\phi\cos\phi & (m_1r_1^2 + m_2r_2^2)\cos^2\phi & 0 \\ 0 & 0 & (m_1r_1^2 + m_2r_2^2) \end{pmatrix}$$

Here ϕ is the angle between the rotating frame and the observer's frame and is defined as

$$\phi = \omega_1 t_1 = \omega_2 t_2$$

where ω_1 and ω_2 are the angular velocities of the two point masses.
Now, let $r_1 - r_2 = 2R$ and $M_1r_1 = -M_2r_2$. Therefore,

$$r_1 = \frac{2M_2R}{M_1 + M_2} \tag{9}$$

$$r_2 = \frac{-2M_1R}{M_1 + M_2} \tag{10}$$

Since, we already have expressions for I_{xx}, I_{yy} and I_{xy} from the above matrix, it is easy to calculate the trace free quadrupole moment tensor components and their double derivatives. If we take double time derivatives of $\Im_{ij}$ from equation (1) and substitute that in the strain expressions of (5) and (6), we see that the trace part of the quadrupole tensor cancels out for h_+ and vanishes for $h_\times$ becuase of δ_{ij}. So, the strain expressions can simply be written in terms of the moment of inertia tensor.

$$h_+ = \frac{G}{c^4}\frac{\ddot{I}_{xx} - \ddot{I}_{yy}}{r} \tag{11}$$

$$h_\times = \frac{G}{c^4}\frac{2\ddot{I}_{xy}}{r} \tag{12}$$

Now, Substituting

$$\omega_1^2 = \frac{GM_1}{r_1^3} \tag{13}$$

$$\omega_2^2 = \frac{GM_2}{r_2^3} \tag{14}$$

and writing in terms of R

$$h_+ = \frac{2M_1^2 M_2^2 (M_1 + M_2) G^2}{Rrc^4} \Big(\frac{\cos 2\omega_1 t_1}{M_2^3} + \frac{\cos 2\omega_2 t_2}{M_1^3} \Big) \tag{15}$$

$$h_\times = \frac{-2M_1^2 M_2^2 (M_1 + M_2) G^2}{Rrc^4} \Big(\frac{\sin 2\omega_1 t_1}{M_2^3} + \frac{\sin 2\omega_2 t_2}{M_1^3} \Big) \tag{16}$$

The luminosity can be calculated from equation (7). The triple time derivatives of the trace free quadrupole moment tensor is

$$\frac{d^3 \Im_{xx}}{dt^3} = -4\Big(m_1 r_1^2 \omega_1^3 \sin 2\omega_1 t_1 + m_2 r_2^2 \omega_2^3 \sin 2\omega_2 t_2 \Big) \tag{17}$$

$$\frac{d^3 \Im_{yy}}{dt^3} = 4\Big(m_1 r_1^2 \omega_1^3 \sin 2\omega_1 t_1 + m_2 r_2^2 \omega_2^3 \sin 2\omega_2 t_2 \Big) \tag{18}$$

$$\frac{d^3 \Im_{zz}}{dt^3} = 0 \tag{19}$$

$$\frac{d^3 \Im_{xy}}{dt^3} = -4\Big(m_1 r_1^2 \omega_1^3 \cos 2\omega_1 t_1 + m_2 r_2^3 \omega_2^3 \cos 2\omega_2 t_2 \Big) \tag{20}$$

$$\frac{d^3 \Im_{yx}}{dt^3} = \frac{d^3 \Im_{xy}}{dt^3} \tag{21}$$

Substituting for r_1, r_2, ω_1 and ω_2 and plugging in the triple time derivatives of $\Im_{ij}$, the luminosity is given by

$$L = \frac{G^4}{5} \Big(\frac{M_1 + M_2}{Rc} \Big)^5 \Big[\Big(\frac{M_1}{M_2} \Big)^5 + \Big(\frac{M_2}{M_1} \Big)^5 + 2\cos 2(\omega_1 t_1 - \omega_2 t_2) \Big] \tag{22}$$

But $\omega_1 t_1 = \omega_2 t_2 = \phi$. Therefore, the luminosity is

$$L = \frac{G^4}{5} \Big(\frac{M_1 + M_2}{Rc} \Big)^5 \Big[\Big(\frac{M_1}{M_2} \Big)^5 + \Big(\frac{M_2}{M_1} \Big)^5 + 2 \Big] \tag{23}$$

4 For equal mass objects rotating about z-axis

In this case $m_1 = m_2 = M$. Hence the expression for strain reduces to

$$h_+ = \frac{2G^2M^2}{Rrc^4}\cos 2\omega t \tag{24}$$

$$h_\times = \frac{-2G^2M^2}{Rrc^4}\sin 2\omega t \tag{25}$$

And the luminosity is

$$L = \frac{128G^4}{5}\Big(\frac{M}{Rc}\Big)^5 \tag{26}$$

5 For Uniform density ellipsoid rotating about z-axis

Consider a non-axisymmetric ellipsoid with semi-major and semi-minor axes a,b,c. Let's assume that the ellipsoid is rotating about the 'c-axis'. Now we fix our rotating co-ordinate system in such a way that the z-axis is along the rotating axis. First we find the time derivatives of the moment of inertia inorder to derive the strain equations. The expression for moment of inertia is for continuous mass distribution is given in eq(4).

[**Correction:** I have corrected the expression's for moment of inertia in all the places. Actually, i have the correct expressions in my notes. Also, during our discussion meetings we have used the correct expressions, including the factor$\frac{1}{5}$. It was my mistake not to write it.]

For rotating coordinate system, let x,y and z axes be along the a,b and c axes of the ellipsoid. So $x' = a, y' = b, z' = c$. The moment of Inertia in the rotating frame is

$$I'_{xx} = \frac{1}{5}\rho(b^2 + c^2)\frac{4}{3}\pi abc \tag{27}$$

But $\rho = \frac{M}{\frac{4}{3}\pi abc}$. Therefore,

$I'_{xx} = \frac{1}{5}M(b^2 + c^2), \qquad I'_{yy} = \frac{1}{5}M(a^2 + c^2), \qquad I'_{zz} = \frac{1}{5}M(a^2 + b^2)$

All the cross terms are zero[2]

$$I'_{xy} = \rho \int -x'y'dv \qquad = 0 \tag{28}$$

So the Moment of inertia matrix in the rotating frame is

$$\begin{pmatrix} \frac{1}{5}M(b^2+c^2) & 0 & 0 \\ 0 & \frac{1}{5}M(a^2+c^2) & 0 \\ 0 & 0 & \frac{1}{5}M(a^2+b^2) \end{pmatrix}$$

To get the Inertia matrix in observer's frame we apply Euler angle's similar to the one used in the previous section. Here, the angle ϕ is the angle between long axis 'a' and x-axis. Then, the components of the moment of inertia tensor are

$$I_{xx} = \frac{1}{5}M[(a^2+c^2)\sin^2\phi + (b^2+c^2)\cos^2\phi]\,, \tag{29}$$

$$I_{yy} = \frac{1}{5}M[(a^2+c^2)\cos^2\phi + M(b^2+c^2)\sin^2\phi]\,, \tag{30}$$

$$I_{zz} = \frac{1}{5}M(a^2+b^2)\,, \tag{31}$$

$$I_{xy} = I_{yx} = \frac{1}{5}M(a^2-b^2)\sin\phi\cos\phi\,. \tag{32}$$

Now we define ellipticity e_b and e_c as (Goldstein)

$$e_b = \sqrt{1-\frac{b^2}{a^2}} \tag{33}$$

$$e_c = \sqrt{1-\frac{c^2}{a^2}} \tag{34}$$

Substituting these expressions in the above matrix and taking second time derivatives of I_{xx}, I_{yy} and I_{xy} gives

$$\ddot{I}_{xx} = \frac{-2}{5}Ma^2\omega^2e_b^2\cos 2\omega t \tag{35}$$

$$\ddot{I}_{yy} = \frac{2}{5}Ma^2\omega^2e_b^2\cos 2\omega t \tag{36}$$

$$\ddot{I}_{xy} = \frac{-2}{5}Ma^2e_b^2\omega^2\sin 2\omega t \tag{37}$$

[2]If we change the ellipsoid into unit sphere by writing dimensionless units $x = \frac{x'}{b}$,$y = \frac{y'}{a}$, $z = \frac{z'}{c}$, then the integral becomes,

$I'_{xy} = -\rho a^2b^2c\int r\cos\theta\sin\phi\; r\sin\theta\sin\phi\; r^2\sin\theta dr d\theta d\phi = 0$
The integration of the angle θ gives zero and hence the integral is zero.

We saw in the last section that when calculating the strain from trace free quadrupole tensor, taking double derivatives of inertia tensor is equivalent to taking double derivatives of $\Im_{ij}$. Hence

$$h_+ = \frac{-4GMa^2e_b^2\omega^2}{5rc^4}\cos 2\omega t \tag{38}$$

$$h_\times = \frac{-4GMa^2e_b^2\omega^2}{5rc^4}\sin 2\omega t \tag{39}$$

The above expressions of strain are derived basing on the definition of the ellipticity as shown in equations (31) and (32). We can also write the strain equations in terms of eccentricity ϵ. It is defined as (New.K ,Tohline.J, 1995 ApJ),

$$\epsilon_b = \frac{a-b}{\sqrt{ab}} \tag{40}$$

We can derive a relation between e_b and ϵ_b. Equation (38) can be re-arranged to get a quadratic equation.

$$\frac{b^2}{a^2} - \frac{b}{a}(2+\epsilon^2) + 1 = 0 \tag{41}$$

Defining $\eta = \frac{b}{a}$, the quadratic equation reduces to

$$\eta^2 - (2+\epsilon^2)\eta + 1 = 0 \tag{42}$$

and has the roots

$$\eta = \frac{(2+\epsilon^2) + \sqrt{(2+\epsilon^2)^2 - 4}}{2} \tag{43}$$

$$= \tfrac{(2+\epsilon^2)}{2}\left[1 \pm \sqrt{1 - \tfrac{4}{(2+\epsilon^2)^2}}\right]$$

Assuming $b < a$,

$$\eta = \frac{(2+\epsilon^2)}{2}\left[1 \pm \sqrt{1 - \frac{1}{(1+\frac{1}{2}\epsilon^2)^2}}\right] = \frac{(2+\epsilon^2)}{2}[1 \pm \epsilon] \tag{44}$$

Therefore,

$$\eta = 1 \pm \epsilon \tag{45}$$

But $\eta^2 = \frac{b^2}{a^2}$. Then,

$$\eta^2 = \frac{b^2}{a^2} \approx (1 \pm \epsilon)^2 \approx 1 \pm 2\epsilon \tag{46}$$

So,

$$e_b^2 \approx 2\epsilon \tag{47}$$

As we now know the relation between, e_b and ϵ_b, we can write the strain equations in terms of ϵ as follows.

$$h_+ = \frac{-8GMa^2\omega^2}{5rc^4}\epsilon\cos 2\omega t \tag{48}$$

$$h_\times = \frac{-8GMa^2\omega^2}{5rc^4}\epsilon\sin 2\omega t \tag{49}$$

Also, the luminosity can be calculated using equation (7). Therefore

$$L_{ell} = -\frac{128\ GM^2a^4\omega^6\epsilon^2}{125c^5} \tag{50}$$

The above strain equations and luminosity agree with the relations given in New et al.(ApJ 1995), if we define $I_3 = \frac{2}{5}Ma^2$. Then

$$h_+ = \frac{-4G\omega^2\epsilon}{rc^4}I_3\cos 2\omega t \tag{51}$$

$$h_\times = \frac{-4G\omega^2\epsilon}{rc^4}I_3\sin 2\omega t \tag{52}$$

$$L_{ell} = -\frac{32G\omega^6\epsilon^2}{5c^5}I_3^2 \tag{53}$$

6 Collapsing, Axisymmetric Spheroid

From the moment of inertia matrix that follows eq. (26), for a uniform-density, oblate spheroid ($a = b$),

$$I_{xx} = \frac{1}{5}M(b^2+c^2) = \frac{1}{5}M(a^2+c^2)\,, \tag{54}$$

$$I_{yy} = \frac{1}{5}M(a^2+c^2)\,, \tag{55}$$

$$I_{zz} = \frac{1}{5}M(a^2+b^2) = \frac{2}{5}Ma^2\,, \tag{56}$$

and all off-diagonal components are zero. Hence,

$$TrI = \frac{2}{5}M(2a^2+c^2)\,, \tag{57}$$

and the nonzero components of the reduced moment of inertia tensor are,

$$\Im_{xx} = \frac{1}{15}M(c^2-a^2)\,, \tag{58}$$

$$\Im_{yy} = \Im_{xx}\,, \tag{59}$$

$$\Im_{zz} = \frac{2}{15}M(a^2-c^2)\,. \tag{60}$$

So, if we "square" this matrix – that is, evaluate $\Im_{ij}\Im^{ij}$ – we derive,

$$\begin{aligned}\Im_{ij}\Im^{ij} &= \Im_{xx}^2 + \Im_{yy}^2 + \Im_{zz}^2 &(61)\\ &= \frac{2}{75}M^2(a^2-c^2)^2\,. &(62)\end{aligned}$$

Finally, then, the luminosity is,

$$\begin{aligned}L = \dot{E}_{GR} &\approx \frac{1}{5}\frac{G}{c^5}\frac{d^3}{dt^3}[\Im_{ij}\Im^{ij}] &(63)\\ &= \frac{2}{375}\frac{GM^2}{c^5}\frac{d^3}{dt^3}(a^2-c^2)^2\,. &(64)\end{aligned}$$

The two polarizations of the strain, as viewed by an observer looking down the x-axis:

$$\begin{aligned}h_+ &= \frac{G}{c^4}\frac{1}{r}[\ddot{\Im}_{yy} - \ddot{\Im}_{zz}] &(65)\\ &= \frac{3}{15}\frac{GM}{c^4}\frac{1}{r}[\ddot{c^2} - \ddot{a^2}] &(66)\\ h_\times &= \frac{G}{c^4}\frac{1}{r}\ddot{\Im}_{yz} &(67)\\ &= 0\,. &(68)\end{aligned}$$

7 Summary

In the previous sections the expressions for strain are derived for various objects. Let's summarize the results now.

Strain. (h_+)

For two different point masses	:	$\frac{2M_1^2M_2^2(M_1+M_2)G^2}{Rrc^4}\left(\frac{\cos 2\omega_1 t_1}{M_2^3} + \frac{\cos 2\omega_2 t_2}{M_1^3}\right)$
For equal mass	:	$\frac{2M^2G^2}{Rrc^4}\cos 2\omega t$
For rotating ellipsoid	:	$\frac{-8Ma^2G\omega^2}{5rc^4}\epsilon\cos 2\omega t$
For collapsing, axisymmetric spheroid.	:	$\frac{3}{15}\frac{GM}{c^4}\frac{1}{r}[\ddot{c^2} - \ddot{a^2}]$

Luminosity.

For two different point masses	:	$\frac{G^4}{5}\left(\frac{M_1+M_2}{Rc}\right)^5\left[\left(\frac{M_1}{M_2}\right)^5 + \left(\frac{M_2}{M_1}\right)^5 + 2\right]$
For equal mass	:	$\frac{128G^4}{5}\left(\frac{M}{Rc}\right)^5$

For rotating ellipsoid	:	$-\frac{32G\omega^6\epsilon^2}{5c^5}I_3^2$
For collapsing, axisymmetric spheroid.	:	$\frac{2}{375}\frac{GM^2}{c^5}\frac{d^3}{dt^3}(a^2 - c^2)^2$