ASTR7741 (Fall, 2007)

1. Spherical Free-Fall Problem

Determine exactly how long it takes a (massless) particle to "free-fall" from rest onto a point mass of mass M if the particle is initially at a distance r_0 from the point mass. To answer this question, you need to integrate (twice) the following equation of motion (eq. 3.11, p. 120 in Vol. II of Padmanabhan):

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}\,. \tag{1}$$

More specifically, you should set $v_r = 0$ and $r = r_0$ at time $t = 0$, then determine a precise, analytical expression for the time τ_{ff} at which $r \rightarrow 0$.

- Hint #1: Multiply through by $2v$ (on the left-hand-side) and by $2dr/dt$ (on the right-hand-side), and integrate once to determine v^2 as a function of r, M, and r_0.
- Hint #2: To set up the second integral, replace v by dr/dt, and remember that the velocity must be negative.

ANSWER: Padmanabhan actually provides the answer to this problem, beginning with Eq. (3.11), p. 120 of Vol. II. First note that Eq. (1), above, can be rewritten in the form,

$$\frac{dv_r}{dt} = -\frac{GM}{r^2}\,. \tag{2}$$

Therefore,

$$\begin{aligned} 2v_r\frac{dv_r}{dt} &= -\frac{2GM}{r^2}\,\frac{dr}{dt} && (3)\\ \Rightarrow d(v_r^2) &= 2GM\ d(r^{-1}) && (4)\\ \Rightarrow \text{At any } r<r_0\,,\qquad v_r^2 &= 2GM\left(\frac{1}{r}-\frac{1}{r_0}\right) && (5)\end{aligned}$$

Since $v_r = dr/dt$, this also means,

$$\begin{aligned} \frac{dr}{dt} &= -\left[2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)\right]^{1/2} && (6)\\ \Rightarrow \left[2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)\right]^{-1/2} dr &= -dt && (7)\end{aligned}$$

To integrate this, it is convenient to adopt the substitution,

$$\cos^2\zeta \equiv \frac{r}{r_0}\,;\quad \text{hence}\quad dr = -r_0(2\cos\zeta\sin\zeta)\ d\zeta\,, \tag{8}$$

in which case $\zeta = 0$ when $t = 0$ ($r = r_0$) and when $r \to 0$, $\zeta \to \pi/2$. With this substitution, the equation to be integrated takes the form,

$$\begin{aligned} (2GM)^{1/2}\ dt &= \left[\frac{1}{r_0\cos^2\zeta}-\frac{1}{r_0}\right]^{-1/2} r_0(2\cos\zeta\sin\zeta)\ d\zeta && (9)\\ \Rightarrow \left[\frac{GM}{2r_0^3}\right]^{1/2} dt &= \left[\frac{1}{\cos^2\zeta}-1\right]^{-1/2}(\cos\zeta\sin\zeta)\ d\zeta && (10)\\ &= \cos^2\zeta\ d\zeta && (11)\\ \Rightarrow \left[\frac{GM}{2r_0^3}\right]^{1/2} t &= \int_0^\zeta \cos^2\zeta\ d\zeta && (12)\\ &= \left[\frac{1}{2}\zeta+\frac{1}{4}\sin(2\zeta)\right]_0^\zeta && (13)\\ &= \frac{1}{2}\zeta+\frac{1}{4}\sin(2\zeta) && (14)\end{aligned}$$

Hence, when $\zeta = \pi/2$,

$$t = \tau_{\rm ff} = \frac{\pi}{4}\left[\frac{2r_0^3}{GM}\right]^{1/2} = \left[\frac{\pi^2 r_0^3}{8GM}\right]^{1/2} \tag{15}$$

But also note that $\bar\rho_0 \equiv 3M/(4\pi r_0^3)$, hence, $(r_0^3/M) = 3/(4\pi\bar\rho_0)$, which means,

$$\tau_{\rm ff} = \left[\frac{3\pi}{32\ G\bar\rho_0}\right]^{1/2}. \tag{16}$$