A pendulum with a moving support point

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September 12, 2006

Consider a pendulum with mass m hanging from a rod of length l. The support point moves horizontally with a known function $\mathbf{R}(t) = X(t)\hat{\mathbf{i}} + Y(t)\hat{\mathbf{j}}$. We can use the angle θ between the vertical and the pendulum rod as a generalized coordinate, the only one needed to describe the system.

The position vector of the mass m is

$$\mathbf{r} = \mathbf{R} + l(\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$$

The velocity is

$$\mathbf{v} = \dot{\mathbf{R}} + l\dot{\theta}(\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}})$$

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(|\dot{\mathbf{R}}|^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta)$$

The potential energy is

$$V = -m\mathbf{g}\cdot\mathbf{r} = -mg\cos\theta - mgY$$

and although time dependent (through $Y(t)$, it is not dissipative, since it doesn't contain $\dot{\theta}$.

The Lagrangian is

$$\begin{aligned} L &= T - V \\ L(\theta,\dot{\theta},t) &= \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}(\dot{X}(t)\cos\theta - \dot{Y}(t)\sin\theta) + \frac{1}{2}m(\dot{X}(t)^2 + \dot{Y}(t)^2) + mgl\cos\theta + mgY(t) \end{aligned}$$

Lagrange's equation is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} \\ &= \frac{d}{dt}\left(ml^2\dot{\theta} + ml(\dot{X}\cos\theta - \dot{Y}\sin\theta)\right) - \left(ml\dot{\theta}(-\dot{X}\sin\theta - \dot{Y}\cos\theta) - mgl\sin\theta\right) \\ &= ml^2\ddot{\theta} + ml(\ddot{X}\cos\theta - \ddot{Y}\sin\theta) + mgl\sin\theta \\ &= ml^2\ddot{\theta} + ml(g - \ddot{Y})\sin\theta + ml\ddot{X}\cos\theta \end{aligned}$$

We see that the equation of motion is unchanged if $\ddot{\mathbf{R}} = 0$: this is because the support would be moving with constant velocity, and thus it is just like setting up the system in another inertial frame. We also notice that a vertical acceleration of the support point is like increasing or decreasing the local gravitational acceleration g: this can be recognized as the relativity principle (imagine Einstein in an elevator).

The energy function is

$$\begin{aligned}
h &= \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \\
&= \dot{\theta}\left(ml^2\dot{\theta} + ml(\dot{X}\cos\theta - \dot{Y}\sin\theta)\right) \\
&\quad - \left(\frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) + \frac{1}{2}mV^2 + mgl\cos\theta + mgY\right) \\
&= \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta - \frac{1}{2}mV^2 - mgY
\end{aligned}$$

The first two terms are the usual expression for the energy of a pendulum hanging from a fixed support (or the energy with respect to a frame moving with the support), but there are extra terms.

Since the Lagrangian is not homogeneous in the second order with respect to $\dot{\theta}$ (through the terms $\dot{\theta}\dot{X}, \dot{\theta}\dot{Y}$), the energy function is not equal to the mechanical energy, which we can calculate:

$$\begin{aligned}
E &= T + V \\
&= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mV^2 + ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) - mgl\cos\theta - mgY \\
&= h + ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) + mV^2
\end{aligned}$$

We can calculate the time derivative of the energy from this expression:

$$\begin{aligned}
E &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mV^2 + ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) - mgl\cos\theta - mgY \\
\frac{dE}{dt} &= ml^2\dot{\theta}\ddot{\theta} + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} + ml\ddot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) + ml\dot{\theta}(\ddot{X}\cos\theta - \ddot{Y}\sin\theta) \\
&\quad - ml\dot{\theta}^2(\dot{X}\sin\theta + \dot{Y}\cos\theta) + mgl\dot{\theta}\sin\theta - mg\dot{Y} \\
&= -ml\dot{\theta}(\ddot{X}\cos\theta - \ddot{Y}\sin\theta + g\sin\theta) + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} + ml\ddot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) + ml\dot{\theta}(\ddot{X}\cos\theta - \ddot{Y}\sin\theta) \\
&\quad - ml\dot{\theta}^2(\dot{X}\sin\theta + \dot{Y}\cos\theta) + mgl\dot{\theta}\sin\theta - mg\dot{Y} \\
&= m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} + ml\ddot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) - ml\dot{\theta}^2(\dot{X}\sin\theta + \dot{Y}\cos\theta) - mg\dot{Y}
\end{aligned}$$

where we have used Lagrange's equation $-l\ddot{\theta} = \ddot{X}\cos\theta - \ddot{Y}\sin\theta + g\sin\theta$ to obtain the final expression.

Since the Lagrangian depends explicitly on time, we know that the energy function is not conserved:

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} = -ml\dot{\theta}(\ddot{X}\cos\theta + \ddot{Y}\sin\theta) - m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} - mg\dot{Y}$$

We can also use this result to calculate the rate of change in the mechanical energy:

$$\begin{aligned} E &= h + ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) + mV^2 \\ \frac{dE}{dt} &= \frac{dh}{dt} + \frac{d}{dt}\left(ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) + mV^2\right) \\ &= -mg\dot{Y} + ml\ddot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) - ml\dot{\theta}^2(\dot{X}\sin\theta + \dot{Y}\cos\theta) + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} \end{aligned}$$

which is of course the same expression we had obtained earlier.

Let us look at some particular cases:

- **Small angle approximation:** $\theta \ll 1$

  The Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) + \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) + mgl\cos\theta + mgY \\ &\approx \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2 + ml\dot{\theta}(\dot{X} - \dot{Y}\theta) + \frac{1}{2}mV^2 + mgl \end{aligned}$$

  The equation of motion is

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} &= ml^2\ddot{\theta} + ml(\ddot{X} - \ddot{Y}\theta) + mgl\theta \\ 0 &= l\ddot{\theta} + \ddot{X} + (g - \ddot{Y})\theta \end{aligned}$$

  The equation of motion does not necessarily with periodic solutions, unless $\ddot{X} = 0$ and $\ddot{Y}$ is constant, in which case the oscillations have a frequency $\omega^2 = (g - \ddot{Y})/l$. The frequency increases if the motion is accelerated down (in the same direction than gravity), or lower if the support is accelerated upwards.

  The energy is

$$\begin{aligned} E = &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mV^2 + ml\dot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) - mgl\cos\theta - mgY \\ &\approx \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2 + ml\dot{\theta}(\dot{X} - \dot{Y}\theta) + \frac{1}{2}mV^2 - mgY - mgl \end{aligned}$$

  The rate of change in energy is

$$\begin{aligned} \frac{dE}{dt} &= -mg\dot{Y} + ml\ddot{\theta}(\dot{X}\cos\theta - \dot{Y}\sin\theta) - ml\dot{\theta}^2(\dot{X}\sin\theta + \dot{Y}\cos\theta) + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} \\ &\approx l\ddot{\theta}\dot{X} - mg\dot{Y} + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} \end{aligned}$$

- **Uniform horizontal motion:** $\dot{X} = V, \ \dot{Y} = 0$

  This is like a pendulum inside a car moving with uniform velocity on a horizontal road. The equation of motion is not changed from that of a simple pendulum, but the energy is not constant. This is because there is a force of the vehicle on the pendulum, reacting to the motion of the pendulum itself. Small oscillations of the pendulum have the same frequency $\omega^2 = g/l$, and the energy change $dE/dt \approx lV\ddot{\theta} = -mgV\theta$ is also an oscillating function with the same frequency.

- **Uniform vertical motion:** $\dot{X} = 0, \ \dot{Y} = V$

  This is like a pendulum moving in an elevator moving with uniform velocity. The equations of motion are unchanged, but the energy is not constant. Small oscillations have the same frequency $\omega^2 = g/l$, but now $dE/dt \approx -mgV$: there is a constant increase or decrease of energy, depending on whether the support is going up (and energy is increasing), or down (and energy is decreasing). This can also be interpreted as the change in gravitational potential energy due to the vertical motion.

- **Horizontal periodic motion:** $X = X_0 \cos(\Omega t), \ Y = 0$

  The equation of motion for small oscillations is $l\ddot{\theta} + g\theta = \Omega^2 X_0 \cos(\Omega t)$. This is a driven oscillation, and after initial transients, the pendulum will oscillate with the same frequency Ω as the support point: $\theta = \theta_0 \cos(\Omega t)$, with amplitude θ_0 given by the equation of motion:

$$
\begin{aligned}
l\ddot{\theta} + g\theta &= \Omega^2 X_0 \cos(\Omega t) \\
(-l\Omega^2 + g)\theta_0 &= \Omega^2 X_0 \\
\theta_0 &= X_0 \frac{\Omega^2}{g - l\Omega^2} = \frac{X_0}{l} \frac{\Omega^2}{\omega_0^2 - \Omega^2}
\end{aligned}
$$

  with $\omega_0^2 = g/l$. The horizontal displacement of the pendulum with respect to the inertial frame is $x = X + l\theta$. The ratio of amplitude of the horizontal motion of the pendulum mass, x, to the amplitude of the horizontal motion of the top, X_0, is called the “transfer function” (a function of driving frequency) of the pendulum to support motion:

$$
F(\Omega) = \frac{x}{X_0} = X_0 + l\theta_0 = X_0 \left(1 + \frac{\Omega^2}{\omega_0^2 - \Omega^2}\right) = X_0 \frac{\omega_0^2}{\omega_0^2 - \Omega^2}
$$

  If driven at low frequencies ($\Omega^2 \ll g/l$), the pendulum will follow the support, with a small angle θ, ans $x \approx X$: the rod is almost vertical (although the maximum pendulum displacement is larger than the top's maximum displacement). At driving frequencies near the natural pendulum frequency, $\Omega \approx \omega_0$, the motion will grow very large: the system is near “resonance”. At driving frequencies higher than ω_0, the pendulum motion is out of phase with respect to the motion of the support, since the transfer function is negative, and the maximum pendulum displacement is now smaller than the top's maximum displacement. At very high driving frequencies ($\Omega \gg \omega_0$), the limit of the transfer function is zero, and the angular amplitude of the pendulum motion is X_0/l: the pendulum mass stays near the equilibrium point, while the top moves back and forth.

The energy (for small oscillations) is

$$\begin{aligned}
E &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2 + ml\dot{\theta}\dot{X} + \frac{1}{2}mV^2 \\
&= \frac{1}{2}ml(l\Omega^2 + g)\theta_0^2\cos^2(\Omega t) + ml\Omega^2\theta_0 X_0 \sin^2(\Omega t) + \frac{1}{2}m\Omega^2 X_0^2\cos^2(\Omega t) \\
&= \frac{1}{2}mX_0^2\left(\cos^2(\Omega t)\left((\Omega^2+\omega_0^2)\frac{\Omega^2}{\omega_0^2-\Omega^2} + \Omega^2\right) + \sin^2(\Omega^2 t)\Omega^2\frac{\Omega^2}{\omega_0^2-\Omega^2}\right)
\end{aligned}$$

and the power absorbed by the system is

$$\begin{aligned}
\frac{dE}{dt} &= mlV\ddot{\theta} + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} \\
&= mlX_0\theta_0\Omega^2\sin\Omega t\cos\Omega t + mX_0^2\Omega^2\sin\Omega t\cos\Omega t \\
&= \frac{1}{2}mX_0^2\Omega^2\left(\frac{\Omega^2}{\omega_0^2-\Omega^2}+1\right)\sin 2\Omega t \\
&= \frac{1}{2}mX_0^2\frac{\Omega^2\omega_0^2}{\omega_0^2-\Omega^2}\sin 2\Omega t
\end{aligned}$$

Since dE/dt is periodic with frequency 2Ω, the system absorbs energy during half of the cycle of the top motion, and "returns" the energy in the other half. Which half is which depends on the driving frequency being larger or smaller than the pendulum energy, i.e., the pendulum moving in phase or out of phase with the support. The peak power is proportional to a factor $X_0^2\omega_0^2\Omega^2/(\omega_0^2-\Omega^2)$: this is very large near resonance; small and proportional to $X_0\Omega^2$ at low driving frequencies; and independent of driving frequency at large driving frequencies, $\propto X_0^2\omega_0^2$.

- **Vertical periodic motion:** $Y = Y_0\cos(\Omega t),\ X = 0$**.**

  The equation of motion is $-l\ddot{\theta} = l(g-\ddot{Y})\sin\theta = l(g+\Omega^2 Y_0\cos\Omega t)\sin\theta$. The motion of the support adds an oscillating component to the gravitational acceleration. This is not a driven oscillator like the previous case, because the oscillatory driving force is vertical but the natural oscillatory pendulum motion is horizontal. If $Y_0\Omega^2 \ll g$, we can use a perturbative approach to find a solution starting with the pendulum solution; if $Y_0\Omega^2 \gg g$, we can also use a perturbative approach to find a solution starting with the driven pendulum solution.

The energy is

$$\begin{aligned} E &= \frac{1}{2}ml^2\dot\theta^2 + \frac{1}{2}mV^2 + ml\dot\theta(\dot X\cos\theta - \dot Y\sin\theta) - mgl\cos\theta - mgY \\ &= \frac{1}{2}ml^2\dot\theta^2 - mgl\cos\theta + \frac{1}{2}mY_0^2\Omega^2\cos^2\Omega t + mlY_0\Omega\dot\theta\sin\theta\cos\Omega t - mgY_0\cos\Omega t \end{aligned}$$

and the power absorbed is

$$\begin{aligned} \frac{dE}{dt} &= -mg\dot Y + ml\ddot\theta(\dot X\cos\theta - \dot Y\sin\theta) - ml\dot\theta^2(\dot X\sin\theta + \dot Y\cos\theta) + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} \\ &= mgY_0\Omega\sin\Omega t + mlY_0\Omega\ddot\theta\sin\Omega t\sin\theta + mlY_0\Omega\dot\theta^2\sin\Omega t\cos\theta + mY_0^2\Omega^2\sin\Omega t\cos\Omega t \\ &= mgY_0\Omega\sin\Omega t + \frac{1}{2}mY_0^2\Omega^2\sin 2\Omega t + mlY_0\Omega\sin\Omega t(\ddot\theta\sin\theta + \dot\theta^2\cos\theta) \end{aligned}$$

- **Support point in uniform circular motion**

  The support will move with $X(t) = R\sin\Omega t,\ Y(t) = R\cos\Omega t$.

$$\begin{aligned} L &= \frac{1}{2}ml^2\dot\theta^2 + ml\dot\theta(\dot X(t)\cos\theta - \dot Y(t)\sin\theta) + \frac{1}{2}m(\dot X(t)^2 + \dot Y(t)^2) + mgl\cos\theta + mgY(t) \\ &= \frac{1}{2}ml^2\dot\theta^2 + mgl\cos\theta + mlR\Omega\dot\theta\cos(\Omega t - \theta) + \frac{1}{2}mR^2\Omega^2 + mgR\cos\Omega t \end{aligned}$$

Lagrange's equation is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} \\ &= ml^2\ddot\theta + mgl\sin\theta - mlR\Omega^2\sin(\Omega t - \theta) \\ &= ml^2\ddot\theta + ml(g + R\Omega^2\cos\Omega t)\sin\theta - mlR\Omega^2\cos\theta\sin\Omega t \end{aligned}$$

For small oscillations, we see that there is an oscillating contribution to the restoring gravitational force, and an oscillatory driving term.

The energy is

$$E = \frac{1}{2}ml^2\dot\theta^2 - mgl\cos\theta + mlR\dot\theta\sin(\Omega t - \theta) + \frac{1}{2}mR^2\Omega^2 - mgR\cos\Omega t$$

and the power absorbed by the system is

$$\begin{aligned} \frac{dE}{dt} &= -mg\dot Y + ml\ddot\theta(\dot X\cos\theta - \dot Y\sin\theta) - ml\dot\theta^2(\dot X\sin\theta + \dot Y\cos\theta) + m\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}} \\ &= mgR\Omega\sin\Omega t + mlR\ddot\theta\Omega\cos(\Omega t - \theta) + mlR\dot\theta^2\Omega\sin(\Omega t - \theta) \end{aligned}$$