Lecture 23
Fourier methods

- Theoretical preliminaries
- Discrete data: Nyquist interval and aliasing.

Fourier methods have a long tradition and literature both in math circles and engineering circles. They have taken a while to be incorporated into "numerical analysis" texts.

We have seen instances (ie, PDEs) where Fourier methods were used to tackle problems in numerical analysis. But their applicability ranges further. In many occasions the quantity of physical interest IS the Fourier transform of something.

Although I'll assume you have all encountered Fourier transforms and series in your life, we'll start with a review of basic formulae. We will not prove theorems but remind ourselves of various results that are useful when one uses Fourier transforms.

A physical process can be described in the "time domain" $h(t)$ or in the "frequency domain" via a (generally complex) function $H(f)$ with f ranging from minus to plus infinity.

The two functions are related via a Fourier transform,

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt$$
$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t} df$$

If t is measured in seconds, then f is measured in Hertz (cycles per second). Of course one can use other units, if the process is a function of distance, not time, the transform is a function of inverse wavelength (cycles per meter).

In physics, angular frequency is more common; that is, radians per second. The relation is,

$$\omega \equiv 2\pi f \qquad H(\omega) \equiv [H(f)]_{f=\omega/2\pi}$$

Which leads to the pesky factors,

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{i\omega t} dt$$
$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega$$

The transform is obviously a linear operation. If the function h(t) has symmetries, the function H(f) will have symmetries too. These can be important at the time of increasing computational efficiency,

If...	then...
$h(t)$ is real	$H(-f) = [H(f)]^*$
$h(t)$ is imaginary	$H(-f) = -[H(f)]^*$
$h(t)$ is even	$H(-f) = H(f)$ [i.e., $H(f)$ is even]
$h(t)$ is odd	$H(-f) = -H(f)$ [i.e., $H(f)$ is odd]
$h(t)$ is real and even	$H(f)$ is real and even
$h(t)$ is real and odd	$H(f)$ is imaginary and odd
$h(t)$ is imaginary and even	$H(f)$ is imaginary and even
$h(t)$ is imaginary and odd	$H(f)$ is real and odd

Here are some other reasonably straightforward properties,

$$h(at) \Longleftrightarrow \frac{1}{|a|} H(\frac{f}{a}) \qquad \text{“time scaling”}$$

$$\frac{1}{|b|} h(\frac{t}{b}) \Longleftrightarrow H(bf) \qquad \text{“frequency scaling”}$$

$$h(t - t_0) \Longleftrightarrow H(f)\, e^{2\pi i f t_0} \qquad \text{“time shifting”}$$

$$h(t)\, e^{-2\pi i f_0 t} \Longleftrightarrow H(f - f_0) \qquad \text{“frequency shifting”}$$

Given two functions f(t), g(t) and their Fourier transforms F(t),G(t), we can perform two combinations of interest:

The **convolution** of two functions in the time domain is a function in the time domain. The operation is symmetric in f,g and is defined by,

$$g * h \equiv \int_{-\infty}^{\infty} g(\tau)h(t-\tau)\, d\tau$$

The relevance of this operation is that it is the "Fourier transform of the product",

$$g * h \Longleftrightarrow G(f)H(f) \qquad \text{"Convolution Theorem"}$$

The **correlation** (or "lag") is also a function in the time domain,

$$\mathrm{Corr}(g,h) \equiv \int_{-\infty}^{\infty} g(\tau+t)h(\tau)\, d\tau$$

$$\mathrm{Corr}(g,h) \Longleftrightarrow G(f)H^*(f) \qquad \text{"Correlation Theorem"}$$

(for real functions, otherwise it is G(f)H(-f).

$$\mathrm{Corr}(g,g) \Longleftrightarrow |G(f)|^2$$

(Wiener-Khinchin)

The total power (integral of the amplitude square) of a signal should be the same computed in the time or frequency domain, and this is the content of Parseval's theorem,

$$\text{Total Power} \equiv \int_{-\infty}^{\infty} |h(t)|^2 \, dt = \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

On many occasions it is of interest to know how much power is available "per frequency bin", that is, between f and f+df. In these cases one does not distinguish between positive and negative frequencies and defines the "one sided power spectral density",

$$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2 \qquad 0 \leq f < \infty$$

So the total power is given by the integral of $P_h(f)$. If the function is real the two terms are equal. Many books and articles are vague as to if they are computing the "one sided" or "two sided" spectral density...

Fourier transform of discretely sampled data:

In most practical situations the function h is sampled at regular intervals of time Δ so the sequence of sampled values is,

$$h_n = h(n\Delta) \qquad n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

The reciprocal of the time interval Δ is the sampling rate.

Sampling theorem and aliasing:

We are reasonably accustomed to the effect of discretely sampling a function in the "time domain": getting derivatives is hard, etc. More surprising effects happen in the frequency domain. To see this we first define the Nyquist critical frequency, $f_c=1/(2\Delta)$.

If a sine wave of the Nyquist frequency is sampled at its positive peak, the next sampling happens at the negative peak, that is, it is sampled twice per cycle.

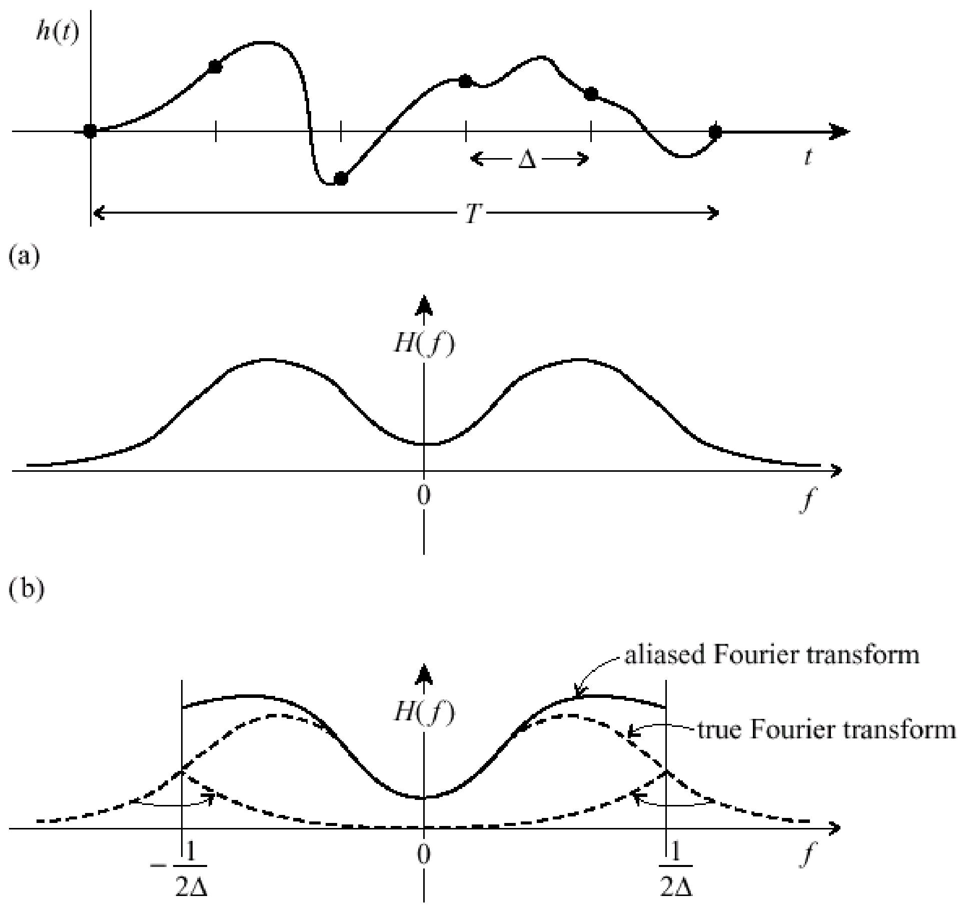
If one samples at intervals Δ a continuous function f(t) that is bandwidth limited to frequencies less than f_c (that is, its transform H(f) vanishes for $f>f_c$) then its Fourier transform is **completely determined** by the samples,

$$h(t) = \Delta \sum_{n=-\infty}^{+\infty} h_n \frac{\sin[2\pi f_c(t-n\Delta)]}{\pi(t-n\Delta)}$$

This might sound great, after all, in reality one is frequently dealing with signals that are bandwidth limited (for instance the output of almost all amplifiers is).

But in reality it also points out an obvious limitation: *we are not able to resolve properly frequencies that are higher than the sampling frequency*. This is the analog in Fourier of what we encountered when we solved PDE's: **the grid ought to be smaller than the features we want to resolve**.

What this means in practice is that all the power spectrum living outside an interval determined by the Nyquist frequency is shifted inside that interval. This is called **aliasing**.



There is little to be done about aliasing once it happened. Either you filter your waveform before you sample or you sample with a rate such that *at least two samplings happen per cycle for the highest frequency Fourier component.*

Another way to view things is to assume that a sampled function has zero Fourier components outside its Nyquist band. One can check how well this assumption holds in reality by computing the Fourier component and see if it is going to zero or not as one approaches the limits of the Nyquist band. If it is not, then aliasing is occurring.

Discrete Fourier transform:

Suppose we have N sampled values of a function,

$$h_k \equiv h(t_k), \qquad t_k \equiv k\Delta, \qquad k = 0, 1, 2, \ldots, N-1$$

It is obvious that as a result we cannot compute (without interpolation) more than N points of its Fourier transform. Instead of trying to compute the transform for any value of the frequency, let us compute it for the N discrete values spanning the Nyquist interval,

$$f_n \equiv \frac{n}{N\Delta}, \qquad n = -\frac{N}{2}, \ldots, \frac{N}{2}$$

(we assume at the moment that N is even and that the function is either zero or periodic outside the time domain specified).

(it also turns out we are really looking at N+1 points, it will turn out that the value of the transform at both ends is equal).

We now proceed to discretize the integral in the Fourier transform,

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t} dt \approx \sum_{k=0}^{N-1} h_k \, e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k \, e^{2\pi i k n/N}$$

We denote,

$$H_n \equiv \sum_{k=0}^{N-1} h_k \, e^{2\pi i k n/N} \qquad\qquad H(f_n) \approx \Delta H_n$$

Since the function is periodic with period N, we can let n vary between 0 and N-1. With this convention n=0 is zero frequency, positive frequencies go from 1 to N/2-1 and negative frequencies from N/2+1 to N-1. The value at N/2 corresponds to both f_c and $-f_c$.

It is easy to check that all the properties we listed in the table at the beginning of the class for $H(f_n)$ also hold for H_n.

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n \, e^{-2\pi i k n/N}$$

The inverse transform is very similar to the transform so the same routine with small modifications can handle both.

Summary

- Review of basic concepts of Fourier transform. Convolution. Parseval theorem.
- Aliasing is the analogue in Fourier transforms of the limitations of working with a finite grid in finite differences.