Consider a system of (Fock) bosons with the single particle states $|\alpha_i\rangle = |\alpha\rangle$. We may form the Fock, many-particle states from these single particle states

$$|\alpha_1, \ldots, \alpha_n\rangle = \frac{1}{\sqrt{N!}} \sum_{\pi} (-1)^{\pi} |\alpha_{\pi(1)}\rangle \otimes \cdots \otimes |\alpha_{\pi(n)}\rangle$$

Fermions only

Let $|\varphi\rangle$ denote a general vector in Fock space

$$|\varphi\rangle = \sum_{n=0}^{\infty} \sum_{\alpha_1, \ldots, \alpha_n} \varphi_{\alpha_1, \ldots, \alpha_n} |\alpha_1, \ldots, \alpha_n\rangle$$

If we operate on $|\varphi\rangle$ by $a^+_\pi$, then the minimum number of particles increases by 1, so $|\varphi\rangle$ cannot be made an eigenstate of any $a^+_\pi$. However, since the number of particles is not bounded above, it is possible for $a_\alpha |\varphi\rangle = |\varphi\rangle$

Then, as

$$[a_\alpha, a^+_\pi]_e = \delta_{\alpha\pi} \quad [a_\alpha, a_\beta]_e = 0$$

so that

$$a_\alpha |\varphi\rangle = |\varphi\rangle$$

or

$$[\varphi_\alpha, \varphi_\beta]_e = 0$$

so that for Fermions, $\varphi_\alpha \varphi_\beta = - \varphi_\beta \varphi_\alpha$; i.e., Grassmann numbers.
Now let's specialize for Bosons, and expand $|\psi\rangle$ in the occupation basis

$$|\psi\rangle = \sum_{n_1, n_2, \ldots} |n_1, n_2, \ldots \rangle$$

where $|n_1, n_2, \ldots \rangle$ is a normalized symmetrized state with $n_i$ particles in single particle state $|\alpha_i\rangle$.

Then as

$$a_\alpha |\psi\rangle = \phi_\alpha |\psi\rangle$$

and since $a_\alpha$ is a lowering operator, i.e. $a(n) = \sqrt{n} a(n-1)$

$$\phi_{\alpha_1} \phi_{\alpha_2} \ldots \phi_{\alpha_{n-1}} = \sqrt{n_{\alpha_n}} \phi_{\alpha_{n-1}} \ldots \phi_{\alpha_1}$$

Then by induction if we continue to the coeff. of $|10\rangle$ (which we set to one), we obtain

$$\phi_{n_1, n_2, \ldots} = \frac{\phi_{n_1} \phi_{n_2} \ldots \phi_{n_1}}{n_{n_1} n_{n_2} \ldots n_{n_1}}$$

Then, since

$$|n_1, n_2, \ldots \rangle = \frac{(a_\alpha)^{n_1} (A_\alpha)^{n_2} \ldots (A_\alpha)^{n_{n_1}}}{\sqrt{n_{n_1} n_{n_2} \ldots n_{n_1}}} |10\rangle$$

If we substitute these two relations in $|\psi\rangle$, we get

$$|\psi\rangle = \sum_{n_1, n_2, \ldots} \frac{(\phi_{\alpha_1, \alpha_2} n_1)^{n_1}}{n_{n_1}} \frac{(\phi_{\alpha_2, \alpha_3} n_2)^{n_2}}{n_{n_2}} \ldots |10\rangle$$
or \( |\phi\rangle = e^{\sum_{\alpha} \phi^*_\alpha a_\alpha^+} |1\rangle \) and \( \langle \phi | = \langle \phi | e^{-\sum_{\alpha} \phi^*_\alpha a_\alpha} |0\rangle \).

We may now explore the effect of \( a_\alpha^+ \) on \( |\phi\rangle \):

\[
a_\alpha^+ |\phi\rangle = a_\alpha^+ e^{\sum_{\alpha} \phi^*_\alpha a_\alpha^+} |1\rangle = e^{\sum_{\alpha} \phi^*_\alpha \phi^*_\alpha} |1\rangle
\]

with adjoint \( \langle \phi | a_\alpha = \frac{\partial}{\partial \phi^*_\alpha} \langle \phi | \) and \( \langle \phi | a_\alpha^+ = \frac{\partial}{\partial \phi^*_\alpha} \langle \phi | \) as well.

The overlap of two coherent states is

\[
\langle \phi | \phi' \rangle = \sum_{n_\alpha} \frac{\phi^*_n}{n_\alpha!} \left( \frac{\phi^*_n'}{n_\alpha'}! \right)
\]

\[
\times \langle n_\alpha, n_\alpha', ..., n_\alpha' \rangle \langle n_\alpha, n_\alpha', ..., n_\alpha' \rangle
\]

\[
= e^{\sum_{\alpha} \phi^*_\alpha^* \phi^*_\alpha}
\]

The identity, which is essential for the evaluation of \( \Xi \), is

\[
\int \frac{d\phi^*_\alpha d\phi^*_\alpha}{2\pi i} e^{-\sum_{\alpha} \phi^*_\alpha^* \phi^*_\alpha} |1\rangle \otimes |\phi\rangle = 1
\]

Consider one \( \alpha \) and that the corresponding coefficient is \( \phi^*_n / n_\alpha! \), on

\[
|\phi\rangle = \sum_n \frac{\phi^*_n}{n_\alpha!} |n\rangle
\]
So that

\[ \int \frac{d^4 \Phi}{(2\pi)^4} e^{-\Phi \cdot \Phi} \sum_{nm} \frac{\Phi_n^{*} \Phi_m}{m!} \langle m | \Phi_n^{*} \Phi_m | n \rangle \]

\[ \Phi_n = r e^{i\phi} \]

\[ = \int_0^\infty \frac{dr \, d\phi}{2\pi} \, e^{-r^2} \sum_{nm} \frac{(r e^{i\phi})^n}{n!} \langle m | \Phi_n^{*} \Phi_m | n \rangle \]

\[ = \sum_{n} \left\{ \begin{array}{ll} 0 & n \neq m \\ 2 & n = m \end{array} \right. \]

\[ = \int_0^\infty dr \, dr \, e^{-r^2} r^{2n} = \frac{n!}{2} \text{e} \pi n^2 \]

\[ = \sum_n \ln | n | = \ln \]

This completeness relation leads to a form for the trace

\[ \text{Tr} A = \sum_n \langle n | A | 1 \rangle = \sum_n \langle n | e^{-\sum \Phi^* \Phi} \sum \langle 0 | \Phi^* \Phi | n \rangle \langle n | A | 1 \rangle \]

\[ = \sum_n \langle n | e^{-\sum \Phi^* \Phi} \langle 0 | A | \sum \langle n | \Phi^* \Phi | 1 \rangle \]

\[ = \sum_n \langle n | e^{-\sum \Phi^* \Phi} \langle 0 | A | n \rangle \]

The term \( \langle \phi | A | \phi \rangle \), or more generally \( \langle \phi | A | \phi \rangle \), may be evaluated immediately if \( A \) is normal ordered.
Since \( a_x |\phi\rangle = \phi_x |\phi\rangle \) and \( \langle \phi | a_x^+ = \langle \phi | \phi^*_x \)

and \( \langle \phi | \phi' \rangle = \exp \Sigma \phi^*_x \phi^*_x \), it immediatly follows that for any normal ordered \( A \)

\[
\langle \phi | A(a_x a_x^*) |\phi\rangle = A(\phi^*_x, \phi_x) e^{\Sigma \phi^*_x \phi^*_x}
\]

All of this formulas are the same, in form, as their Bosonic counter parts, so they yield the same form for \( \mathcal{Z} \)

\[
\mathcal{Z} = \mathcal{Z} \int \mathcal{Z} e^{-\mathcal{Z} H} = \int d\phi^*_x d\phi_x e^{-\phi^*_x \phi_x} \langle \phi^*_x | e^{-\mathcal{Z} H} |\phi_x\rangle
\]

where \( \phi_x = \phi_x^* \phi^*_x = \phi_x^* \) in

Let \( e^{-\mathcal{Z} H} = (e^{-\mathcal{Z} H})^N + \text{insert } N-1 \text{ identities} \)

\[
\int d\phi^*_x d\phi_x / 2 \pi i e^{-\phi^*_x \phi_x} 1 \phi_x X \phi^*_x
\]

evaluate each

\[
\langle \phi, e^{-\mathcal{Z} H} |\phi_{x-1}\rangle = e^{\phi^*_x \phi_{x-1}} e^{-\mathcal{Z} H(\phi^*_x \phi_{x-1})}
\]

collect term and identify \( \mathcal{Z} = \lim_{\mathcal{Z} \to \infty} \int d\phi^*_x \mathcal{Z} e^{-\mathcal{Z} H} \)

\[
\mathcal{Z} = \sum_{N=1}^{\infty} 2 \phi^*_x (\frac{\phi_x - \phi_{x-1}}{\Delta t}) + H(\phi^*_x \phi_{x-1})
\]
Gaussian Integral for Bosons

For Hermitian \( A \), one may show that

\[
\int_{\mathbb{R}^n} \exp \left( -z^* A z + u^* z + u z^* \right) = (\det A)^{-\frac{n}{2}} \exp \left( u^* A^{-1} u \right)
\]

First complete the square on the lhs, so that

\[
\int_{\mathbb{R}^n} \exp \left( -z^* A z + u^* A^{-1} u \right)
\]

\[
= e^{u^* A^{-1} u} \int_{\mathbb{R}^n} \exp \left( -x^* A x \right) \quad \text{with} \quad x = z - A^{-1} u
\]

Now diagonalize \( A \) with an orthogonal transform \( P \),

The Jacobian of this transformation is \( 1 \)

\[
x = P y \quad x^* = y^* P \quad y = P x \quad y^* = x^* P^* \quad P^* = P^{-1}
\]

so that

\[
-x^* A x = -y^* P A P y = -y^* D y \quad \text{diagonal}
\]

\[
e^{u^* A^{-1} u} \int y^* e^{-y^* D y} \, dy = e^{u^* A^{-1} u} \left( \det D \right)^{-\frac{n}{2}} = e^{u^* A^{-1} u} (\det A)^{-\frac{n}{2}}
\]

Apart from the factor of \( |A|^{-\frac{n}{2}} \) out front, this is

the same as we found for fermions. Thus all

the correlators, including Wick's theorem

apply to Bosons as they do for Fermions.