Parisi Trick for Monte Carlo

What I find amazing is that we may combine what we learn here for Monte Carlo, with what we learned about mean field theory, sort of.

In describing Monte Carlo we did not describe the sequence of proposed moves that define the Markov chain. Some practitioners simply scan through the lattice sequentially. First proposing to flip spin 1, then 2, then 3

\[ S_1 \rightarrow S_2 \rightarrow S_3 \]

Some choose a random spin each time and propose a flip

\[ S_i \rightarrow S_j \rightarrow S_k \quad i, j, k \text{ random} \]

Suppose we instead propose a large number of local spin flips

\[ \bigcirc \rightarrow \bigcirc \rightarrow \bigcirc \quad \text{many} \]

Clearly this would not be very efficient, unless we may perform the many local moves analytically.

In fact let many \( \rightarrow \infty \) and consider 2 spins which are well separated \( i \) and \( j \), and measure

\[ \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle = X_{ij}. \]

Of course the vacuum term vanishes, leaving \( X_{ij} = \langle S_i S_j \rangle \) and now \( \langle \rangle \) refers to a MC average. Now if we perform the local MC average analytically we must replace the value of the spin with its average obtained by \( \infty \) many local MC hits. If we know the local configuration of \( S_i \& \delta \text{ surrounding} i \).
\[ r_i = \sum_{s_{i+\delta}} \text{then the result of the accumulated local hits is to replace } S_i \text{ in the estimator by} \]

\[ S_i \rightarrow \frac{e^{\beta r_i} - e^{-\beta r_i}}{e^{\beta r_i} + e^{-\beta r_i}} = \tanh \beta r_i \]

\[ \langle S_i S_j \rangle \rightarrow \langle \tanh \beta r_i \tanh \beta r_j \rangle \]

This replacement is exact provided that \( r_i \) and \( S_j \) or \( r_j \) and \( S_i \) have no spins in common. Suppose that \( i = j \), then \( S_i^2 = 1 \). If \( i \) and \( j \) are min. min. then the formula above may be used. The problem arises only when \( i \) and \( j \) are min. In this case you must replace \( S_i S_j \) by its average given the six environment spins shown below. I will leave this exercise for your homework.

\[ \begin{array}{c}
\text{9-1}
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Why do this? By averaging over the local spin configurations in \( \langle S_i S_j \rangle \) the variance, and hence the error bar, in the Markov process is reduced by an order of magnitude \((\Rightarrow 100 \text{ fold speedup!})\).