The Inverse Square Law

The Inverse Square Law is a principle you are already very familiar with, even if you don't recognize the name. We all know that a light viewed from a distance looks fainter than it does when it is close (think of the example of oncoming car headlights on a highway). For the same reason, a distant star will appear fainter than it would if the star were nearby. We can use this principle to help us determine the distance to stars.

The simple concept that a star will appear fainter if it is farther away is very familiar, but it's not really useful until we quantify it. How much fainter does it appear, and just how far away is it? To answer these questions we have to use a little math. All we need is basic geometry plus some multiplication and division. Not only will this enable us to determine the distance to stars, it will also show us why a light appears fainter from a distance.

If we are going to quantify our description of brightness, we have to define exactly what we mean by "brightness". What creates the sensation of brightness when we see a light? Our perception of brightness is determined by the amount of electromagnetic energy hitting our eye. What we call brightness then depends not merely on how much power a light source produces, but also on how much of it is concentrated into the area of our eye. Since we want to describe apparent brightness by how much power goes into a given area (the area of our eyes), what we are really saying is that we want to know the power per unit area. In other words, the brightness is just the power of the light source divided by the area covered by the light. To write this statement as an equation, let \( L \) represent the power radiated by the light bulb (usually called the luminosity in astronomy) and let \( A \) represent the area over which that power is distributed. Then the brightness, \( B \) is defined as

\[
B = \frac{L}{A}
\]

As an example, let's assume that we are looking at a 100 Watt light bulb. The 100 Watt rating tells us the luminosity of the bulb. This is what determines the true brightness of the bulb. We know a 100 Watt bulb is intrinsically brighter than a 40 Watt bulb. However, we also know that the 40 Watt bulb may appear brighter if it's right in our face while the 100 Watt bulb is across the street. To describe this difference in apparent brightness, we have to consider how the energy of the light bulb is spread out over an area. Imagine the 100 Watt bulb is one meter away. If we assume the light spreads out equally in all directions from the bulb (not a very good assumption for real light bulbs, but fine for stars), then the 100 Watts produced by the light bulb not only goes into our eyes, it also goes into all of the space around us. Only a tiny fraction of that 100 Watts actually gets to our eyes. How much? If the bulb is located one meter away from us, the 100 Watts produced by the bulb must spread out over a sphere one meter in radius by the time the light reaches us (See Fig. 1). That sphere has a surface area of

\[
A = 4\pi r^2
\]

Plugging in \( r = 1 \) m tells us that the area of the sphere is \( A = 12.6 \) m\(^2\). Every square meter of area on the surface of this sphere will receive the same fraction of the 100 Watts produced by the light bulb. To express this mathematically, we use equation (1) with \( L = 100 \) Watts and \( A = 12.6 \) m\(^2\).

\[
B = \frac{L}{A} = \frac{100W}{12.6m^2}
\]
\[ B = 7.94 \text{W/m}^2 \]

So this 100 Watt bulb has an apparent brightness of 7.94W/m² when seen from a distance of one meter. If the bulb is farther away, equation (2) tells us how much greater an area the light must cover before it reaches us. Dividing the same luminosity by a greater area results in a smaller value for the brightness, \( B \). To see the general behavior, substitute the expression for area given by equation (2) in place of the \( A \) in equation (1). We get

\[ B = \frac{L}{4\pi r^2} \]  

Equation (3) is the Inverse Square Law. It tells us that brightness \( B \) depends upon the inverse of the square of distance \( r \). We have seen that the Inverse Square Law is simply a consequence of geometry and the fact that energy is spread evenly over that geometry.

The Inverse Square Law tells us that if we measure the apparent brightness of a star (this is the science of photometry) and we know the star's distance (from parallax measurements), we can calculate the luminosity, \( L \), of the star from equation (3). This is very important because once we know the true luminosity of some stars, we can begin to notice how different properties (like color, temperature, etc...) depend upon a star's luminosity. We'll discuss some of these properties in future labs, but for now just remember that we can discover certain “signs” that are unique to intrinsically luminous stars and other “signs” that are unique to intrinsically dim stars. Having discovered these “signs”, we turn the process around. When we see a star that has the same “signs” as the most luminous stars, but is too far from us to measure its parallax, we can assume that star has the same luminosity that we have calculated for a nearby star with the same “signs”. We use that luminosity in equation (3), along with the brightness that we measure for the distant star, and solve the equation for the distance, \( r \)

**The Magnitude Scale**

Now that we understand how brightness varies with distance, we will cover one additional topic: the magnitude scale which is used to measure brightness. Magnitudes are simply the units astronomers normally use to measure brightness. Thousands of years ago, people didn't think of brightness in terms of energy; they just knew that some stars looked brighter than others. The ancient Greeks divided all the stars they could see into six groups. They called the brightest stars "first magnitude" and the faintest stars "sixth magnitude". Notice that the magnitude scale is inversely proportional to brightness; brighter stars are denoted by smaller magnitude numbers. Using this scale, they tried to estimate by eye how bright each star appeared and assigned each star a magnitude from one to six. This scale is still used today with only a few minor changes.

By the nineteenth century, the nature of light as a form of electromagnetic energy was understood and scientists could relate the magnitude scale for brightness to the definition in terms of energy per unit area which we described above. The human eye perceives brightness logarithmically. Since the magnitude scale is based upon visual observations, this means that equal increments of the magnitude scale correspond to equal ratios of brightness. Today we know that a first magnitude star is about 100 times as bright as a sixth magnitude star. So a difference of five magnitudes \((5 = 6 - 1)\) corresponds to a brightness ratio of 100. Writing this statement mathematically gives us the modern definition of the magnitude scale.

Suppose we have one star named \( \alpha \) and another named \( \beta \). Let \( m_\alpha \) represent the magnitude of star \( \alpha \) and let \( B_\alpha \) represent the brightness of star \( \alpha \) (say in Watts/m²). Let \( m_\beta \) and \( B_\beta \) represent the same quantities for star \( \beta \). Then we define the difference in magnitudes of the two stars by

\[ m_\beta - m_\alpha = 2.5 \log(B_\alpha/B_\beta) \]
If you aren't comfortable with this equation (maybe you wonder where the 2.5 came from?), consider the example where $m_\alpha = 1$ and $m_\beta = 6$. Since we know $B_\alpha = 100B_\beta$ in this case, we can plug these values into equation (4) and immediately see

$$6 - 1 = 2.5 \log(100)$$

$$6 - 1 = 2.5 \times 2$$

$$6 - 1 = 5$$

which should reassure you.

Notice that equation (4) only defines differences in magnitudes. We set the zero point of the scale by defining certain stars as reference points for the magnitude scale. Modern measurements are a good deal more accurate than the ancient Greek's visual estimates, so not all the stars they called "first magnitude" have a magnitude of exactly 1.0. The star Altair, in the Summer Triangle, does have a magnitude of 1.0. Some stars are even brighter and thus have even lower magnitudes. For example, Sirius has a magnitude of -1.5.

The important thing to remember is that the magnitude scale is just that - a scale for measuring brightness. The apparent brightness of a star depends upon the nature of the star itself (its luminosity) and its distance from us (via the Inverse Square Law). In principle, that apparent brightness may be measured in any units we wish; in Watts/m², or in magnitudes, or in some other units. Astronomers have traditionally used magnitudes to describe the brightness of stars, so that is what we will use in this class. Finally, remember that since we have been discussing the apparent brightness of stars, the magnitudes that we use to measure that apparent brightness are therefore apparent magnitudes. We'll talk about how to apply the magnitude scale to the true brightness of a star in a future lab.
QUESTIONS

(Answer on a separate sheet of paper and remember to show your work)

1. Assume Mars is twice as far from the Sun as the Earth. If you travel to Mars, how bright will the Sun appear, compared to how it appeared from Earth? (I just want to know the answer as a fraction of the brightness seen from Earth, not an actual value in Watts/m²).

2. Alpha Centauri is the closest star (other than the Sun) to Earth. Parallax measurements show it is about 280,000 times as far away as the Sun. It appears $1.3 \times 10^{-11}$ times as bright as the Sun. Calculate the luminosity of Alpha Centauri relative to the Sun using the Inverse Square Law (i.e. just express the luminosity of Alpha Centauri as a multiple of the solar luminosity - don't worry about the numerical value of the solar luminosity).

3. Both the Sun and Alpha Centauri are classified as "G-type" stars. Observations of the apparent brightness and parallax-determined distances of other nearby G-type stars indicate that ALL G-type stars have about the same luminosity as the Sun. Suppose we observe a G-type star which is too far away to determine its distance via parallax. If we note that the star appears 1/100 as bright as Alpha Centauri, how many times farther away is this star than Alpha Centauri?

4. Alpha Centauri has an apparent magnitude of 0. Based on the information given above, what is the apparent magnitude of the more distant star described in problem 3?

5. (Next page.)