(Here we solve the partial differential equation for steady state temperature distribution in the semi-infinite slab with the Laplace equation $\nabla^2 T[x,y] = 0$ with the Boundary Conditions (BC) $T(x,0)=T_0=100$, $T[x,\infty]=0$, $T[0,y]=0$, $T[l,y]=0$ where $L=10$. Assuming a solution of the form $T[x,y] = X(x)Y(y)$ and using the 2D Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ which gives $X''[x]Y[y] + X[x]Y''[y] = 0$. Dividing by $X[x]Y[y] \neq 0$ we get $X''[x]/X[x] + Y''[y]/Y[y] = 0$, which can only be satisfied if both terms are the same constant or $X''[x]/X[x] = -Y''[y]/Y[y] = -k^2$, where $k$ is the separation constant. We can solve these two Diffy-Q's using DSolve. *)

DSolve[$X''[x] + k^2 X[x] = 0$, $X[x], x$]

DSolve[$Y''[y] - k^2 Y[y] = 0$, $Y[y], y$]

(* That is $X[x] = A x \sin[k x] + B x \cos[k x] + C \exp[k x] + D \exp[-k x]$, as in the notes. Demanding the BC's $X[0]=0$, $X[L]=0$ gives $B=0$ and $k= n \pi / L$ (the "quantization condition") respectively, where $L=10$. Demanding $Y[\infty]=0$ gives $C=0$. That gives us a final solution of $T[x,y]=X[x]Y[y]=A \exp[k x] Y[y]+D \exp[-k x] Y[y]$, where $b_n=A \exp[-k x] Y[y]$ solves the partial diffy-q but no single one satisfies the BC, so we use superposition and linearity to construct a more general solution:

$T[x,y]=X[x]Y[y]=\sum_{n=0}^{\infty} b_n \exp[-k_n x] Y[y]$, with the hope that the sum of the individual solutions solves the Diffy-Q (by default) AND the final BC.

The final BC for this new solution reads:

$T[x,0]=X[x]Y[0]=\sum_{n=0}^{\infty} b_n \exp[-k_n x] Y[y] = \sum_{n=0}^{\infty} b_n \sin[k_n x] = T_0=100$.

This is the Fourier Sine Series for the function:

$f[x]=T_0=\sum_{n=0}^{\infty} b_n \sin[k_n x]$

Note the functions $\sin[k_n x]=\sin[n \pi x / L]$, for $n=0,2,3,...$ are orthogonal on the interval $x \in [0,L]$.*

Integrate[$\sin[n \pi x / L] \sin[m \pi x / L]$, $\{x, 0, L\}$, 
Assumptions -> $\{\text{Element}[n,\text{Integers}], \text{Element}[m,\text{Integers}], L > 0\}$]

\[ L \frac{n \cos[n \pi] - L m \cos[m \pi] \sin[n \pi]}{m^2 \pi - n^2 \pi} \]

(* Which is zero if $m \neq n$. For $m=n$ we have: *)
\begin{equation}
\text{Integrate}\left[\sin[n \cdot \pi \cdot x / L] \cdot \sin[n \cdot \pi \cdot x / L], \{x, 0, L\}\right], \text{Assumptions} \rightarrow \{\text{Element}[n, \text{Integers}], \text{Element}[m, \text{Integers}], L > 0\}
\end{equation}

\begin{align*}
\frac{1}{L} \cdot \text{Log} \left( \frac{2 - \sin[2 \cdot n \cdot \pi]}{n \cdot \pi} \right)
\end{align*}

(*Which is L/2 if n is an integer. Hence the functions \(|n\rangle = \text{Sqrt}[2/L] \cdot \sin[n \cdot \pi \cdot x / L]\) form an orthonormal set, such that \(<n|m> \cdot \delta_{nm}. We can use these to find \(b_n\). Rewriting the BC in terms of the orthonormal Sine functions we get:

\begin{equation}
f[x] = T_0 \cdot \sum_{n=0}^{\infty} \langle \text{Sqrt}[2/L] \rangle \cdot b_n \cdot \langle \text{Sqrt}[2/L] \rangle \cdot \sin[k_n \cdot x]) \cdot \sum_{n=0}^{\infty} c_n \cdot |n\rangle,
\end{equation}

where \(c_n = \text{Sqrt}[L/2] \cdot \delta_{nm} \cdot c_n\). Hence we can then use:

\begin{equation}
\langle n | f \rangle = \sum_{n=0}^{\infty} c_n \cdot \langle n | m \rangle = \sum_{m=0}^{\infty} c_n \cdot \langle n | m \rangle = \sum_{m=0}^{\infty} c_n \cdot \delta_{nm} = c_n.
\end{equation}

Hence we can now compute \(c_n\) using \(f=T_0\) to get,

\begin{equation}
c_n = \text{FullSimplify}[\text{Integrate}\left[\text{Sqrt}[2/L] \cdot \sin[n \cdot \pi \cdot x / L] \cdot T_0, \{x, 0, L\}\right], \{n > 0, L > 0\}]
\end{equation}

\begin{align*}
\frac{\sqrt{2} \cdot \sqrt{L} \cdot (-1 + \cos[n \cdot \pi]) \cdot T_0}{n \cdot \pi}
\end{align*}

\(b_n = \text{FullSimplify}[\text{Sqrt}[2/L] \cdot c_n, \{n > 0, L > 0\}]
\end{equation}

\begin{align*}
\frac{2 \cdot (-1 + \cos[n \cdot \pi]) \cdot T_0}{n \cdot \pi}
\end{align*}

(*Note the conditional assumption

\(|n>0,L>0\) forces n and L to be treated at positive real numbers,
simplifying the result. This is the result in the notes. Notice that this is zero if n-
even, and \(b_n=4 \ast T_0/(n \pi)\) if n-odd so we replace n->2m+1 to get: \(b_{2m+1}=2 \ast T_0 / ((2 \ast m + 1) \ast \pi)\)

which gives us \(T[x,y] = (4 \ast T_0 / \pi) \ast \sum_{m=0}^{\infty} \exp[-(2 \ast m + 1) \ast \pi/10 \ast x] \ast \sin((2 \ast m + 1) \ast \pi/10 \ast x)] / (2 \ast m + 1).\) We can sum the first few terms by recalling \(T_0=100\) and defining a Fourier series partial sum.\)

\(s[M_x, x_{-}, y_{-}]:=
\end{equation}

\begin{align*}
\frac{400}{\pi} \ast \sum_{m=0}^{\infty} \exp[-(2 \ast m + 1) \ast \pi / 10 \ast y] \ast \sin((2 \ast m + 1) \ast \pi / 10 \ast x) / (2 \ast m + 1), \{m, 0, M\}
\end{align*}

(*We can plot the first 100 terms as a 3D plot.\)
Plot3D[s[100, x, y], {x, 0, 10}, {y, 0, 5}, PlotRange -> {0, 100}, AxesLabel -> {x, y, T[x, y]}]

(*Click on the plot and you can rotate it with your mouse. The plot in the x direction shows the oscillations die out quickly with more terms but also that the function dives to zero at x=0 and x=10 and clearly it is a step function at y=0 where T=100 and then drops off exponentially with increasing y. There are no real oscillations in the Laplace equation \( \nabla^2 T[x,y]=0 \) since there is no term proportional to T. For oscillations we would solve the wave equation instead \( \nabla^2 T[x,y]+K^2 T[x,y]=0 \).*)