ELLiptic InTEgrals & FCNs

EX 4. IN 1750S LEGENDRE POSED THE QUESTION: "HOW FAR DOES THE EARTH TRAVEL IN ONE YEAR?" IF THE EARTH'S ORBIT WERE EXACTLY CIRCULAR THEN

\[ \Phi = 2\pi R \quad \text{with} \quad R = 1 \text{ A.U.} \quad \text{is the answer.} \]

The distance between 01 Jan and 01 Apr is

\[ \Phi = \frac{1}{14} 2\pi R = \frac{2\pi}{14} R = \frac{1}{7} \pi R \]

In general, \( \Phi = \varphi R \) and the velocity is \( \frac{ds}{dt} = \dot{\varphi} R = \omega R \) where

\[ \omega = \frac{2\pi}{T} \quad \text{with} \quad T = 1 \text{ year}. \]

However, we know from Kepler the Earth and all other planets, comets, etc. move on ellipses! For the Earth the ellipse is approximately circular but for long period comets they are highly eccentric and the circular approximation fails badly.

The equation for an ellipse of major axis \( a > b \) = minor is

\[ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \]

which reduces to \( a = b = R \) for a circular orbit, since \( a \) and \( b \) have units of length \( x = a \sin \phi \), \( y = b \cos \phi \).
which gives \((\frac{a \sin \theta}{a})^2 + (\frac{b \cos \theta}{b})^2 = 1\)

Recall \(\frac{a^2 - b^2}{a^2} = e^2\) is the eccentricity.

\(e = 0 \implies a = b = R\) is circular.

To compute the arc length around the ellipse, we note

\[
\begin{align*}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 \\
&= \left[ a \cos \theta \, \mathrm{d}\theta \right]^2 + \left[ -b \sin \theta \, \mathrm{d}\theta \right]^2 \\
&= \left[ a^2 \cos^2 \theta + b^2 \sin^2 \theta \right] \mathrm{d}\theta \\
&= \left[ a^2 (1 - \sin^2 \theta) + b^2 \sin^2 \theta \right] \mathrm{d}\theta \\
&= \left[ a^2 - (a^2 - b^2) \sin^2 \theta \right] \mathrm{d}\theta \\
&= a \left[ 1 - e^2 \sin^2 \theta \right] \mathrm{d}\theta
\end{align*}
\]

Hence

\[
\mathrm{d}s = a \sqrt{1 - e^2 \sin^2 \theta} \, \mathrm{d}\theta
\]

Now we just integrate from \(\theta = 0\) to \(\theta = \varphi\) to get the arclength from \(0 \to \varphi\),

\[
S_{\varphi}^\theta = a \int_{0}^{\varphi} \sqrt{1 - e^2 \sin^2 \theta} \, \mathrm{d}\theta
\]

Only if \(e = 0\) does this reduce to

\[
S(e=0)_{\varphi}^\theta = a \int_{0}^{\varphi} \, \mathrm{d}\theta = a \varphi = R \varphi
\]

Some special cases may be worked out:

\(e = 1 \implies S_{\varphi}^\theta = a \int_{0}^{\varphi} \, \mathrm{d}\theta = \pi \sin \theta_{\varphi} - \pi \sin \theta_{0} = a \pi \sin \varphi_{\varphi} - a \pi \sin \theta_{0}
\]
For arbitrary \( \varepsilon \) there is no antiderivative! So the \( \log \), \( \text{erf} \), \( \text{erf}^{-1} \) we define elliptic integrals of the first and second kind as

\[
F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{1ST}
\]

\[
E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \ d\theta \quad \text{2ND}
\]

Hence the orbit problem reduces to solving an elliptic equation of the form

In one orbit, Earth travels

\[
S_\text{Earth} = a \int_0^{2\pi} \sqrt{1 - e^2 \sin^2 \theta} \ d\theta
\]

we may evaluate using

\[
S = a \times \text{EllipticE}[\phi, k]
\]

in Mathematica

\[
S_\text{Earth} = N[a \times \text{EllipticE}[\theta, e]] = 6.012 \text{ AU}
\]

Compare to circular approx \( 2\pi R \approx 6.283 \text{ AU} \) which is off by \( 0.271 \text{ AU} \) about \( \frac{1}{3} \) the distance to sun!

\[
S_\text{Halley} = 75.48 \text{ AU} \quad \text{compare} \quad 2\pi \cdot R_\text{Halley} \approx 113.66
\]

off by \( 38 \times \) times Earth to Sun.
**LEGENDRE FORM ELLIPTIC INTEGRALS**

**Eq 12.1**

**1st Kind**

\[ F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad 0 \leq k \leq 1 \]

**2nd Kind**

\[ E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \quad 0 \leq k \leq 1 \]

These are also called **incomplete** E.I.

**Complete** E.I. correspond to \( \frac{1}{4} \) Ellipse

**E(k) = E(\pi/2, k)**

\[ H(k) = F(\pi/2, k) \]

**Jacobi Form**

\[ t = \tan \theta \]

\[ F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-t^2 \sqrt{1-k^2 t^2}}} \]

\[ E(\phi, k) = \int_0^\phi \sqrt{1-k^2 t^2} \, d\theta \]

**Proof**

\[ t = \tan \theta \quad dt = \sec^2 \theta \, d\theta = \sqrt{1+t^2} \, d\theta \]

\[ \theta : \theta \rightarrow \phi \]

\[ \phi \rightarrow \pi \rightarrow \pi \phi = \pi \]

\[ t : \tan 0 \rightarrow \tan \pi = \pi \]

**COMPLETE JACOBI**

\[ K(k) = F(1, k) \]

\[ E(k) = E(1, k) \]
Ex 1

Evaluate
\[ I = \int_0^{\pi/3} \sqrt{1 - \left(\frac{L}{2L}\right)^2 \sin^2 \theta} \, d\theta \]

Compare to Eq 12.1
\[ k^2 = \frac{L}{2L} \quad \lambda = \frac{L}{2L} = \frac{\sqrt{2}}{2} \quad \omega = \frac{\pi}{3} \]

\[ I = E[\pi/3, \sqrt{2}/2] \]

\[ = N \left[ \text{Elliptic E} \left[ \frac{\pi}{3}, \sqrt{2}/2 \right] \right] \approx 0.96 \]

Ex 5

We now can solve complete pendulum problem.

Recall Eq. 6.5
\[ \frac{1}{2} \Omega^2 = \gamma \cos \theta + \text{const} \]

Previous we solved initial amplitude \( \theta(t=0) = 90^\circ \)

Only with Bessel. Now want to solve for \( \theta(t=0) = \alpha \) general.

At \( t=0 \), \( \dot{\theta} = 0 \) and so

\[ t=0 \]
\[ \gamma^2 \cos \alpha = -\text{const} \]

\[ \Rightarrow \text{const} = -\gamma^2 \cos \alpha \]

\[ \Rightarrow \frac{1}{2} \dot{\theta}^2 = \gamma^2 \cos \theta - \gamma^2 \cos \alpha \]

\[ = \gamma^2 \left[ \cos \theta - \cos \alpha \right] \]

\[ \frac{d\theta}{dt} = \sqrt{2\gamma^2 \left[ \cos \theta - \cos \alpha \right]} \]
\[
\frac{d\theta}{dt} = \frac{1}{\sqrt{\cos \theta - \cos \chi}}
\]

\[
\int_{0}^{\chi} \frac{d\theta}{\sqrt{\cos \theta - \cos \chi}} = \frac{\sqrt{2\pi} \, t_x}{4}
\]

But \( t_x = \frac{1}{4} T_x \) is the period.

\[
T_x = \frac{4}{\sqrt{2}} \, y^{-1} \int_{0}^{\chi} \frac{d\theta}{\sqrt{\cos \theta - \cos \chi}}
\]

\[
= \int_{0}^{\pi/2} \frac{1}{\sqrt{2}} \int_{0}^{\chi} \frac{d\theta}{\sqrt{\cos \theta - \cos \chi}}
\]

Follow hints in Prob 11.12.17

\text{TRIG} \quad \cos \theta = 1 - 2 \cos^2 \left( \theta/2 \right) \quad \cos \chi = 1 - 2 \cos^2 \left( \chi/2 \right)

\[
T_x = \frac{2 \sqrt{2}}{y} \int_{0}^{\chi} \frac{d\theta}{\sqrt{[1-2 \cos^2 \theta/2] - [1-2 \cos^2 \chi/2]}}
\]

\[
= \frac{2 \sqrt{2}}{y} \int_{0}^{\chi} \frac{d\theta}{\sqrt{2 [\cos^2 \theta/2 - \cos^2 \chi/2]}}
\]

\[
= \frac{2 \sqrt{2}}{y} \int_{0}^{\chi} \frac{d\theta}{\sqrt{1 - \left( \frac{2 \cos \theta/2}{\cos \chi/2} \right)^2}}
\]

Let \( \chi = \frac{2 \cos \theta/2}{\cos \chi/2} \quad c \chi = \frac{1}{2} \cos \theta/2 \quad d\theta\)
\[ d\theta = \frac{2\sin \theta/2}{\cos \theta/2} \, d\theta/2 \]
\[ = \frac{2 \sin \theta/2}{\sqrt{1 - \sin^2 \theta/2}} \, d\theta 
= \frac{2 \sin \theta/2}{\sqrt{1 - \sin^2 \theta/2} \, \cos \theta/2} \, d\theta 
= \frac{-2 \cos \theta/2 \, d\theta}{\sqrt{1 - \sin^2 \theta/2} \, \cos \theta/2} \]
\[ k = \sin \theta/2 \]

\[ T_x = \frac{2}{\sin \theta} \int_0^1 \frac{2k \, dk}{\sqrt{1 - k^2 x^2} \, \sqrt{1 - x^2}} \]
\[ = \frac{4}{\sin \theta} \int_0^1 \frac{dx}{\sqrt{1 - x^2} \, \sqrt{1 - k^2 x^2}} \]

This is Eq. 12.3 the complete \( E.I. \) of 1st kind in Jacobi Form \( K(k) = F(\pi/2, k) \)

Hence
\[ T_x = \frac{4}{\sqrt{9}} \, K \left[ \sin \theta/2 \right] \]

To approximate this let's expand about \( k = \sin \theta/2 \), which we can do in mathematics taking result from Schaun's Math Handbook.
$K(k) = \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right) k^4 + \cdots \right]$

If $k << 1$ then $k = \frac{\pi}{2} \ll \frac{\pi}{2} << 1$

and we get

$T \approx 4 \sqrt{\frac{k}{\rho}} \frac{\pi}{2} \left[ 1 + \frac{1}{4} \left( \frac{k^2}{2} \right) \right]$

$= 2 \pi \sqrt{\frac{k}{\rho}} \left[ 1 + \frac{k^2}{16} \right]$

\text{phys 101 simple harmonic oscillation solution}
Elliptic Functions

In the pendulum problem we expressed
\[ t = t(\theta) \] but we really want \( \theta = \theta(t) \) as the solution to \( \dot{\theta}^2 = \omega^2 \cos \theta + C \)

To do this we need the inverse of the elliptic integrals.

Recall from Calculus
\[
\arcsin^{-1}(x) = \frac{\pi}{2} - \arcsin(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}
\]

which is just the integral form of
\[
\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \tag{Schaum's Eq. 45.20}
\]

In a similar way we define the elliptic function inverse in terms of Eq. 12.2 1st Kind
\[
\begin{align*}
\text{sn}^{-1}(x) & = \int_0^x \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}} = F(\phi, k) \\
\phi & = a \sin \phi \\
\text{sn}^{-1}(x) & = \int_0^{a \sin \phi} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = F(\phi, k)
\end{align*}
\]

Hence
\[
\begin{align*}
\text{IF} \quad u_k^{-1}(x) & = F(\phi, k) \\
\text{Then} \quad u_k^{-1}(x) & = \text{sn}^{-1}(x) \\
\Rightarrow \quad \text{sn}(x) & = x = a \sin \phi
\end{align*}
\]
In this form we define elliptic cosine

$$\text{cn}(u) = \cos \varphi = \frac{1}{\sqrt{1 - \kappa^2 u}}$$

The elliptic derivative $\text{dn}$ is

$$\text{dn}(u) = \frac{d\varphi}{du} = \frac{1}{\frac{dF(\varphi, u)}{d\varphi}} = \left[ \frac{1}{\sqrt{1 - \kappa^2 u}} \right]^{-1}$$

$$= \sqrt{1 - \kappa^2 u^2}$$
$$= \sqrt{1 - \kappa^2 \sin^2 \varphi}$$
$$= \sqrt{1 - \kappa^2 x^2}$$

Just like sine and cosine
$\text{sn}$ and $\text{cn}$ have lots of identities:
see Ch 35 Schaun's

$$\frac{d}{du} \text{cn}(u) = \frac{d}{du} \sin \varphi = \cos \varphi \frac{d\varphi}{du} = \kappa u \text{dn}(u)$$
Back to pendulum!

Recall

$$T_c = \int_0^L \frac{d\theta}{\sqrt{\cos \theta - \cos \alpha}} = \frac{1}{4} T_x$$

we may rewrite this

$$T \phi = \frac{1}{\sqrt{2m}} \int_0^\phi \frac{d\theta}{\sqrt{\cos \theta - \cos \alpha}}$$

For any time $t \phi$ that is not related to $T_x$, Inverting gives $\phi = \phi(t)$

Following our previous derivation we get by inspection

$$T \phi = 4 \ T \phi = \frac{16}{\phi} \ \frac{1}{2\pi} \ \frac{\phi(t)}{\phi} \ \int_0^{\phi(t)} \frac{d\theta}{\sqrt{1 - \left(\frac{\sin \theta}{2\pi \phi} \right)^2}}$$

where $\alpha = \phi(t = 0)$ which we can rewrite as

$$T \phi = \frac{16}{\phi} \ \int_0^{k\sin \phi(t)} \frac{dx}{\sqrt{1 - x^2} \ \sqrt{1 - \frac{k^2 x^2}{2\pi^2}}}$$

where $k = \frac{2\pi \phi}{\alpha}$

Compare to Eq. 12.12

$$u = \sqrt{T \phi} = \int_0^{k\sin \phi(t)} \frac{dx}{\sqrt{1 - x^2} \ \sqrt{1 - \frac{k^2 x^2}{2\pi^2}}} = 2\pi^{-1} \left[ \arcsin \left( \frac{\sin \phi(t)}{2\pi \phi} \right) \right]$$

Note $\sqrt{T \phi} = \sqrt{\frac{3}{2}} \ \frac{t}{s} = \sqrt{\frac{m}{s^2 \ \frac{1}{m^2}}}$, $s = 1$ is dimensionless
Hence

\[ \sin(\phi(t)) = k \sin \left[ k \frac{\gamma t}{16} \right] \]

\[ \phi(t) = \arcsin \left[ k \sin \left( \frac{\gamma t}{16} \right) \right] \]

where \( \gamma \) is dimensionless time \( \gamma = \frac{\gamma t}{16} \).

Recall \( k = \alpha \sin \beta \) where \( \alpha \) is \( \phi(t=0) \).

Hence we can plot \( \phi_k(t) \) for different starting angles \( \alpha \).

Notice \( \frac{\gamma t}{16} = \omega x \) has units of \( \text{radian} \times \text{time} \).

\[ \phi(t) = \arcsin \left[ k \sin \left( -\omega t \right) \right] \]

Recall for small angle \( \alpha \ll 1 \) this reduces to \( \phi(t) \approx \omega \sin \left( -\omega t \right) \) simple harmonic.

\( \phi(t) \approx \omega \sin \left( \omega t \right) \)
However as the initial $l$ becomes large, we get "hang time"

\[ \phi_{\text{m}}(\Phi) = \text{ArcSin} \sum l \text{ JacobsSin}[u, l]\]

$\Phi = \pi$

$\phi_{c-x}$

See 11.12.4b or 11.12.4b, p.d.f.