The Structure, Stability, and Dynamics of Self-Gravitating Systems

Origin of the Poisson Equation

The Poisson Equation [I.D.1] is derived straightforwardly from Isaac Newton’s inverse-square law of gravitation. In presenting this derivation, we follow closely the presentation in Chapter 2, §1 of Binney & Tremaine (1987).¹

According to Isaac Newton’s inverse-square law of gravitation, the acceleration $a(x)$ felt at any point in space $x$ due to the gravitational attraction of a distribution of mass $\rho(x')$ is obtained by integrating over the accelerations exerted by each small mass element, $\rho(x') \, d^3x'$, as follows:

$$a(x) = \nabla_x \int |x' - x|^{-1} G \rho(x') \, d^3x'.$$

[Equation I.H.1] = BT87, Chapter 2, Eq. (2-2)

First, if we adopt the following

**Definition of the Gravitational Potential**

$$\Phi(x) = -\int |x' - x|^{-1} G \rho(x') \, d^3x'.$$

[Equation I.H.2] = BT87, Chapter 2, Eq. (2-3)

EFE, Chapter 2, §10, Eq. (11)²

Tassoul ’78, Chapter 4, Eq. (12)

and notice that the gradient of the function $|x' - x|^{-1}$ with respect to $x$ is

$$\nabla_x [ |x' - x|^{-1}] = [ (x' - x) \, |x' - x|^{-3} ],$$

[Equation I.H.3] = BT87, Chapter 2, Eq. (2-4)

we find that we may write the gravitational acceleration as

$$a(x) = \nabla_x \int |x' - x|^{-1} G \rho(x') \, d^3x' = -\nabla_x \Phi.$$  

[Equation I.H.4] = BT87, Chapter 2, Eq. (2-5)
Next, realize that the divergence of the gravitational acceleration takes the form,

\[ \nabla \cdot \mathbf{a}(\mathbf{x}) = \nabla \cdot \int \left[ \frac{\mathbf{x}' - \mathbf{x}}{r^3} \right] G \rho(\mathbf{x}') \, d^3 \mathbf{x}' \]

\[ = \int G \rho(\mathbf{x}') \{ \nabla \cdot \left[ \frac{\mathbf{x}' - \mathbf{x}}{r^3} \right] \} \, d^3 \mathbf{x}'. \]

[Equation I.H.5]

BT87, Chapter 2, Eq. (2-6)

Now

\[ \nabla \cdot \left[ \frac{\mathbf{x}' - \mathbf{x}}{r^3} \right] = -3 \left| \mathbf{x}' - \mathbf{x} \right|^{-3} + 3 \left[ (\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x}) \right] \left| \mathbf{x}' - \mathbf{x} \right|^{-5}. \]

[Equation I.H.6]

BT87, Chapter 2, Eq. (2-7)

When \( \mathbf{x}' - \mathbf{x} \neq 0 \) we may cancel the factor \( \left| \mathbf{x}' - \mathbf{x} \right|^2 \) from top and bottom of the last term in this equation to conclude that

\[ \nabla \cdot \left[ \frac{\mathbf{x}' - \mathbf{x}}{r^3} \right] = 0. \]

[Equation I.H.7]

BT87, Chapter 2, Eq. (2-8)

Therefore, any contribution to the integral (on the right-hand-side of Eq. [I.H.5]) must come from the point \( \mathbf{x}' = \mathbf{x} \), and we may restrict the volume of integration to a small sphere ... centered on this point. Since, for a sufficiently small sphere, the density will be almost constant through this volume, we can take \( \rho(\mathbf{x}') = \rho(\mathbf{x}) \) out of the integral. Via the divergence theorem (see BT87 for details), the remaining volume integral may be converted into a surface integral over the small volume centered on the point \( \mathbf{x}' = \mathbf{x} \) and, in turn, this surface integral may be written in terms of an integral over the solid angle \( d^2 \Omega \) to give:

\[ \nabla \cdot \mathbf{a}(\mathbf{x}) = -G \rho(\mathbf{x}) \int d^2 \Omega \]

\[ = -4 \pi G \rho(\mathbf{x}). \]

[Equation I.H.8]

BT87, Chapter 2, Eq. (2-9b)

Finally, by combining expressions [I.H.4] and [I.H.8], we derive the
which serves as one of the principal governing equations in our examination of the structure, stability, and dynamics of self-gravitating systems.

Footnotes

1 Text in green is taken verbatim from Chapter 2, §1 of Binney & Tremaine (1987).

2 Note that throughout his book entitled, "Ellipsoidal Figures of Equilibrium," Chandrasekhar adopts a sign convention for the scalar gravitational potential that is opposite to the sign convention used herein.

3 Note that there is a typographical error in Eq. (2-7) of BT87. As printed, the first term on the right-hand-side of the equation is \(-3 |x' - x|^1\) whereas it should be \(-3 |x' - x|^3\) as written here in [I.H.6].