The Structure, Stability, and Dynamics of Self-Gravitating Systems

**Euler's Equation**
(The Vector Equation of Motion)

Among the principal governing equations, we have included the

**Standard Lagrangian Representation**
of Euler's Equation,

\[ \mathbf{D} \mathbf{v} = - \frac{1}{\rho} \nabla P - \nabla \Phi. \]

[Equation I.A.1]
BT87, Appendix 1.E, Eq. (1E-6)

Replacing the Lagrangian (or total) time-derivative on the left-hand-side of this equation [I.A.1] by the Eulerian (or partial) time-derivative via the relationship [VI.M.14] defined in an accompanying appendix, we readily derive the

**Standard Eulerian Representation**
of Euler's Equation,

\[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{1}{\rho} \nabla P - \nabla \Phi. \]

[Equation I.A.2]
Landau & Lifshitz ’75, Chapter I, Eq. (2.4)
BT87, Appendix 1.E, Eq. (1E-8)

Multiplying the standard Lagrangian representation of Euler's Equation [I.A.1] through by the mass density \( \rho \) produces the relation,

\[ \rho \mathbf{D} \mathbf{v} = - \nabla P - \rho \nabla \Phi, \]

[Equation I.A.3]

which may be rewritten as,

\[ \mathbf{D}(\rho \mathbf{v}) - \mathbf{v} \mathbf{D} \rho = - \nabla P - \rho \nabla \Phi. \]

[Equation I.A.4]

Combining this with the standard Lagrangian representation of the Continuity Equation [I.B.1], we derive,

\[ \mathbf{D}(\rho \mathbf{v}) + (\rho \mathbf{v}) \nabla \cdot \mathbf{v} = - \nabla P - \rho \nabla \Phi. \]

[Equation I.A.5]

Note that if, at any location in space, the gradient in the gas pressure \( P \) and the gradient in the gravitational potential \( \Phi \) conspire to make the right-hand-side of this expression equal to zero, this form of Euler's equation may be cast into a generic Lagrangian Conservative Form [I.B.5],

\[ \mathbf{D} \left[ \ln (\rho \mathbf{v}) \right] = - \nabla \cdot \mathbf{v}, \]

[Equation I.A.6]
in which the momentum of the gas is identified as a conserved quantity. This is simply a mathematical way of expressing the physical
concept that any particle of gas that happens to be passing through that location in space will instantaneously experience no acceleration and, hence, its (linear) momentum will remain unchanged as it passes through that location.

For any two vectors \( \mathbf{A} \) and \( \mathbf{B} \), we can utilize the following

\[
\nabla ( \mathbf{A} \cdot \mathbf{B} ) = ( \mathbf{A} \cdot \nabla ) \mathbf{B} + \mathbf{A} \times ( \nabla \times \mathbf{B} ) + ( \mathbf{B} \cdot \nabla ) \mathbf{A} + \mathbf{B} \times ( \nabla \times \mathbf{A} ) .
\]

Letting \( \mathbf{A} = \mathbf{B} = \mathbf{v} \), this identity implies,

\[
( \mathbf{v} \cdot \nabla ) \mathbf{v} = (1/2) \nabla ( \mathbf{v} \cdot \mathbf{v} ) - \mathbf{v} \times ( \nabla \times \mathbf{v} )
\]

\[
= (1/2) \nabla ( \mathbf{v}^2 ) + \zeta \times \mathbf{v},
\]

[Equation I.A.7]

where the fluid vorticity

\[
\zeta = \nabla \times \mathbf{v} .
\]

[Equation I.A.8]

This relationship [I.A.7] may be combined with the standard Eulerian representation of Euler's equation [I.A.2] to derive

\[
\text{Euler's Equation in terms of the Fluid Vorticity}
\]

\[
\partial_t \mathbf{v} + \zeta \times \mathbf{v} = -(1/\rho) \nabla P - \nabla \Phi - (1/2) \nabla ( \mathbf{v}^2 ) .
\]

[Equation I.A.9]

**Rotating Reference Frame**

At times, it can be useful to view the motion of a fluid from a frame of reference that is rotating with a uniform (i.e., constant in time) angular velocity \( \Omega_f \). In order to transform Euler's equation from the inertial reference frame to such a rotating reference frame, the operator \( d/dt \) which denotes Lagrangian time-differentiation in the inertial frame must everywhere be replaced as follows:

\[
d/dt \rightarrow d/dt + \Omega_f \times .
\]

[Equation I.A.10]

Notice that here we have used the color orange to distinguish time-differentiation in the rotating frame of reference from time-differentiation in the inertial frame. Throughout this H_Book we frequently will adopt this "marking" technique to clearly identify variables or operators that are associated with the rotating frame of reference in situations where confusion with inertial-frame variables or operators might arise.

Performing this transformation implies, for example, that

\[
\mathbf{v}_{\text{inertial}} = \mathbf{v} + \Omega_f \times \mathbf{x}.
\]

[Equation I.A.11]

\[\text{BT87, Appendix 1.D, Eq. (1D-38)}\]

\[\text{Tassoul '78, §3.3, Eq. (50)}\]
and

\[ \mathbf{Dv}_{\text{inertial}} = \mathbf{Dv} + 2 \Omega_\ell \times \mathbf{v} + \Omega_\ell \times (\Omega_\ell \times \mathbf{x}) \]

\[ = \mathbf{Dv} + 2 \Omega_\ell \times \mathbf{v} - \nabla \left[ (1/2) \| \Omega_\ell \times \mathbf{x} \|^2 \right]. \]

[Equation I.A.12]
Tassoul '78, §3.3, Eq. (51)

[If we were to allow \( \Omega_\ell \) to be a function of time, one additional term involving the time-derivative of \( \Omega_\ell \) also would appear on the right-hand-side of this last expression [I.A.12] (cf., BT87, expression 1D-42). Throughout this H_Book, we will restrict our flow analyses to either the inertial reference frame or to rotating frames in which \( \Omega_\ell \) is time-invariant, so this additional term will not appear.]

Using this last expression [I.A.12] in conjunction with the standard representations of Euler's Equation [I.A.1 & I.A.2] given above, we derive

**Euler's Equation**

*in a Rotating Reference Frame*

(Lagrangian Representation)

\[ \mathbf{Dv} = - \frac{1}{\rho} \nabla P - \nabla \Phi - 2 \Omega_\ell \times \mathbf{v} - \Omega_\ell \times (\Omega_\ell \times \mathbf{x}), \]

[Equation I.A.13a]

or,

(Eulerian Representation)

\[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{1}{\rho} \nabla P - \nabla \Phi - 2 \Omega_\ell \times \mathbf{v} - \Omega_\ell \times (\Omega_\ell \times \mathbf{x}), \]

[Equation I.A.13b]

or,

(in terms of the Fluid Vorticity)

\[ \partial_t \mathbf{v} + (\zeta + 2 \Omega_\ell) \times \mathbf{v} = - \frac{1}{\rho} \nabla P - \nabla \Phi - (1/2) \nabla (\mathbf{v}^2) - \Omega_\ell \times (\Omega_\ell \times \mathbf{x}). \]

[Equation I.A.13c]

Show that the relationship between the fluid vorticity \( \zeta_{\text{inertial}} \) as measured in the inertial reference frame and the fluid vorticity \( \zeta \) as measured in a reference frame that is rotating with constant angular velocity \( \Omega_\ell \) is,

\[ \zeta_{\text{inertial}} = \zeta + 2 \Omega_\ell. \]

[Equation I.A.14]
Following along the lines of the discussion presented in Appendix 1.D, §3 of Binney and Tremaine (1987), in a rotating coordinate system the Lagrangian representation of Euler's equation [I.A.13a] may be written in the form,

$$Dv = -(1/\rho)\nabla p - \nabla \Phi + a_{\text{fict}},$$

[Equation I.A.15]

where,

$$a_{\text{fict}} = -2 \Omega_f \times v - \Omega_f \times (\Omega_f \times x).$$

[Equation I.A.16]

When comparing this form of Euler's equation [I.A.15] to the standard Lagrangian representation that is valid in the inertial reference frame [I.A.1], we see that in a rotating coordinate system material moves as if it were subject to two *fictional accelerations* which traditionally are referred to as the

**Coriolis Acceleration**

$$a_{\text{Coriolis}} = -2 \Omega_f \times v$$

[Equation I.A.17]

and the

**Centrifugal Acceleration**

$$a_{\text{Centrifugal}} = -\Omega_f \times (\Omega_f \times x) = \nabla\left[\frac{1}{2} \Omega_f^2 x^2\right].$$

[Equation I.A.18]
Prove that, for any orientation of the vector $\Omega_f$, this last relation holds true. That is, show that

$$\Omega_f \times (\Omega_f \times \mathbf{x}) = -\nabla \left[ \frac{1}{2} | \Omega_f \times \mathbf{x} |^2 \right].$$

[Equation I.A.19]

Then show that if the angular velocity of the rotating reference frame is aligned with the $z$-axis of the rotating coordinate system, that is, if $\Omega_f = k \Omega_f$, then the centrifugal acceleration may be written in the form,

$$a_{\text{Centrifugal}} = \nabla \left[ \frac{1}{2} \Omega_f^2 \omega^2 \right],$$

[Equation I.A.20]

and, in terms of the fluid vorticity in the rotating reference frame, Euler’s equation may be written in the form,

$$\partial_t \mathbf{v} + (\zeta + 2 \Omega_f) \times \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \left[ \Phi + \frac{1}{2} \rho \mathbf{v}^2 - \frac{1}{2} \Omega_f^2 \omega^2 \right].$$

[Equation I.A.21]