The Structure, Stability, and Dynamics of Self-Gravitating Systems

Solution Technique 3
(Self-Consistent Field technique)

As was derived earlier, the multidimensional statement of hydrostatic equilibrium for nonrotating polytropic configurations is,

\[(n + 1)Kn \nabla (\rho^{1/n}) = -\nabla \Phi.\]

[Equation III.A.6]

By gathering all terms under a single gradient operator, this relation may also be written in the form,

\[\nabla \{ (n + 1)Kn \rho^{1/n} + \Phi \} = 0.\]

[Equation III.A.37]

Hence, it must be true that throughout a spherically symmetric, equilibrium polytropic configuration,

\[(n + 1)Kn \rho^{1/n} + \Phi = C_o,\]

[Equation III.A.38]

where \(C_o\) is a constant.

At the surface of a polytrope, \(\Phi = \Phi_{\text{surface}}\) and \(\rho = 0\). Hence, \(C_o = \Phi_{\text{surface}}\) and from the above relation [III.A.38] we deduce that,

\[\rho = \{ (\Phi_{\text{surface}} - \Phi) / [(n + 1)K_n] \}^n.\]

[Equation III.A.39]

This algebraic expression defines how the gas density may be derived from the gravitational potential throughout the structure of a nonrotating polytrope. On the other hand, the Poisson equation,

\[(1/r^2) \left[ d \left( r^2 \, d\Phi/dr \right) / dr \right] = 4\pi G\rho,\]

[Equation III.A.3]

provides an independent relationship between the gravitational potential and the gas density that must be satisfied at every point inside a nonrotating polytrope. The equilibrium structure of a spherically symmetric polytrope is defined only by functions \(\rho(r)\) and \(\Phi(r)\) that simultaneously satisfy both of these equations. In a self-consistent field technique, one iterates back and forth between solutions to these two equations until a satisfactory, "self-consistent" solution to both equations has been determined.