The Structure, Stability, and Dynamics of Self-Gravitating Systems

Governing Equations for Spherically Symmetric Objects

By definition, the hydrostatic structure of a nonrotating star is a structure that must be time-independent in the inertial reference frame -- i.e., all Eulerian (partial) time-derivatives are zero -- and one whose velocity flow-field is given, very simply, by the statement $v = 0$. When $\dot{a}_t = 0$ and $v = 0$, it must also be true that the Lagrangian time derivative of all variables is zero. Hence, a mathematical model that properly describes the hydrostatic structure of a nonrotating star is obtained by setting the Lagrangian time derivatives $D$ and all velocities $v$ equal to zero in the principal governing equations.

Employing these simplifications, the time-dependent continuity equation [I.B.1] and first law of thermodynamics [I.C.1] become irrelevant, Euler's equation [I.A.1] reduces to a statement of

**Hydrostatic Equilibrium**

$$\left( 1/\rho \right) \nabla P = - \nabla \Phi,$$

[Equation III.A.1]

and the Poisson equation

$$\nabla^2 \Phi = 4\pi G\rho$$

[Equation I.D.1]

remains unchanged.

Enforcing Spherical Symmetry

In searching for the equilibrium structure of spherically symmetric stars, the task of solving this coupled pair of multi-dimensional, partial differential equations [III.A.1 & I.D.1] is greatly simplified if we express the Laplacian and gradient operators in spherical coordinates, then set all derivatives with respect to $q$ and $j$ equal to zero. (This operation is permitted because, by definition, a spherically symmetric structure will exhibit no variations in either of the angular coordinate directions $\theta$ or $\phi$.) By doing this, only the radial component of each equation survives, and the relevant multi-dimensional, partial differential equations reduce to one-dimensional, ordinary differential equations having the following form:

$$\left( 1/\rho \right) \frac{dP}{dr} = - \frac{d\Phi}{dr},$$

[Equation III.A.2]

and

$$(1/r^2) \left[ d \left( r^2 \frac{d\Phi}{dr} \right) /dr \right] = 4\pi G\rho.$$

[Equation III.A.3]

Employing a Specific Barotropic Equation of State

Up to this point, our discussion has been sufficiently general to accommodate any equation of state. As we focus in on the techniques that are generally used to solve this pair of differential equations [III.A.2 & III.A.3] in conjunction with a particular barotropic equation of state, it proves useful to tailor the discussion to the chosen $P(\rho)$ function. To proceed, select either a
- polytropic, or
- zero-temperature white dwarf equation of state.