Models of Spherically Symmetric Polytropes with $0 < n < \infty$

As has already been mentioned in the accompanying general introductory discussion of the equilibrium structure of spherically symmetric stars, most discussions of polytropes are couched in terms of (either analytical or numerical) solutions to

\[ \frac{1}{x^2} \left[ \frac{d}{dx} \left( x^2 \frac{dQ}{dx} \right) \right] = -Q^n. \]  

[Equation III.A.21]  
Ch67, Chapter IV, Eq. (11)

To date, an analytical solution to the Lane-Emden equation has been derived for only two values of the polytropic index in the range $0 < n < \infty$, namely, $n = 1$ and $n = 5$.

In developing a numerical algorithm that will permit us to determine the structure of polytropes with an arbitrary index $0 < n < \infty$, we have elected not to solve this second-order, nonlinear ordinary differential equation by a technique of direct numerical integration but, rather, to combine some ideas that have been presented separately in our earlier discussions entitled Solution Technique 1 and Solution Technique 3.

According to our presentation of Solution Technique 3, the equilibrium structure of a spherically symmetric polytrope is defined only by functions $\rho(r)$ and $\Phi(r)$ that simultaneously satisfy the following two equations:

\[ r = \left\{ \frac{F_{\text{surface}} - F}{(n + 1)K_n} \right\}^n, \]  

[Equation III.A.39]

and

\[ \frac{1}{r^2} \left[ \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) \right] = 4\pi G \rho. \]  

[Equation III.A.3]

Dividing the first of these equations by the corresponding expression for the density at the center of the polytrope, we find

\[ \frac{\rho}{\rho_{\text{central}}} = \left\{ \frac{(\Phi_{\text{surface}} - \Phi)}{(\Phi_{\text{surface}} - \Phi_{\text{central}})} \right\}^n. \]  

[Equation III.C.16]

So once the gravitational potential has been determined at every radial location throughout the object, the density (measured as a ratio to the central density) will be known everywhere as well.

According to our presentation of Solution Technique 1, the second of these principal equations (which is simply the Poisson equation written in a form that is relevant to spherically symmetric systems) may also be written in the following "integral" form,

\[ \frac{d\Phi}{dr} = GM \times r^2, \]  

[Equation III.A.11]
where

$$M_r = \phi^{=r} 4 \pi \rho r^2 \, dr$$

[Equation III.A.10]

is the total mass lying interior to the radius "r". Realizing that, at the surface of any spherically symmetric structure where \( r = R \) and \( M_R = M_{\text{total}} \), the gravitational potential takes the simple form,

$$\Phi_{\text{surface}} = - \frac{GM_{\text{total}}}{R},$$

[Equation III.C.17]

these three equations may be solved iteratively in reverse order ([III.A.10], [III.A.11], then [III.C.16]) to obtain self-consistent values of the gravitational potential and mass density at all radii throughout any polytope of index \( n \). The following outline describes more specifically the steps that may be taken to solve this system of equations iteratively on a discrete, spherical coordinate grid.

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**Numerical Technique**

1. **Choose a particular system of units.** Here, we will set \( G = R = \rho_{\text{central}} = 1 \).

2. **Choose a particular barotropic equation of state.** More specifically, select a desired value of the polytropic index, \( n \).

3. **Choose the number of spherical shells** \( N \) into which the equilibrium structure is to be divided, then set the radial width of each shell \( \Delta r = R/N = 1/N \) and, hence, \( r(i) = (i / N) \).

4. Throughout the volume of your computational lattice, "guess" a trial distribution of the mass density \( \rho \). (Usually an initially uniform-density distribution will suffice to start the iteration.)

5. Integrating (i.e., summing up the incremental mass contributions) from the center of the object, outward, **determine the value of the variable** \( M_r \) **at the location of each spherical shell**, \( 2 \leq i \leq N \), according to the expression,

$$M(i) = \left(\frac{4}{3}\right) \pi \rho(i) \left\{ \left[ \frac{i}{N} \right]^3 - \left[ \frac{i-1}{N} \right]^3 \right\} + M(i-1),$$

where the central-most mass element is,

$$M(1) = \left(\frac{4}{3}\right) \pi \rho(1) \left( \frac{1}{N} \right)^3.$$
Integrating (i.e., incrementing) from the surface of the object, inward, determine the value of the gravitational potential at the location of each spherical shell, \( 1 \leq i \leq N - 1 \), according to the expression,

\[
\Phi(i) = \Phi(i+1) - \Delta r \left[ \frac{G M(i)}{r(i)^2} \right] = \Phi(i+1) - \left[ \frac{M(i) N}{i^2} \right],
\]

where the value of the potential at the outermost zone is

\[
\Phi(N) = \Phi_{\text{surface}},
\]

and

\[
\Phi_{\text{surface}} = \frac{-GM_{\text{total}}}{R} = -\frac{M_{\text{total}}}{R}.
\]

From the most recently determined values of the gravitational potential (Step 6), calculate an "improved guess" of the density distribution throughout the computational grid according to the expression,

\[
\rho(i) = \left\{ \frac{\Phi_{\text{surface}} - \Phi(i)}{\Phi_{\text{surface}} - \Phi(1)} \right\}^n.
\]

Has the model converged to a satisfactory equilibrium solution?

**YES**

\( \Rightarrow \) Stop iteration.

**NO**

\( \Rightarrow \) Repeat steps 5 through 8.

The analytical solution for \( n=1 \) spherical polytropes,

\[
\Theta_{n=1} = \left( \frac{\sin \xi}{\xi} \right),
\]

[Equation III.C.7]

Ch67, Chapter IV, Eq. (45)

gives values for \( \rho_{\text{max}} / \rho_{\text{central}} \) to compare with our analytical solution described above.

\[
\rho_{\text{max}} / \rho_{\text{central}} \text{ for } n=1 \text{ polytropes} \text{ (analytical in top row, numerical in bottom row)}
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In the next table we have no analytical model to compare our values to.

\[
\rho_{\text{max}} / \rho_{\text{central}} \text{ for } n=1.5 \text{ polytropes}
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