The Structure, Stability, and Dynamics of Self-Gravitating Systems

Physical Properties of Polytropes

Once the function $\Theta(\xi)$ has been determined for a specified polytropic index $n$, other physical properties of the model can be determined straightforwardly.

From the definition of $\Theta$ [III.A.17] and the polytropic equation of state [II.C.1], show that the radial distribution of the gas density and the gas pressure inside a spherically symmetric polytrope is given, respectively, by the expressions,

\[ \rho = \rho_{\text{central}} \Theta^n, \]
\[ P = P_{\text{central}} \Theta^{n+1}. \]

[Equation III.A.28]
[Equation III.A.29]

Utilizing Form A of the ideal gas equation of state [II.A.3], show furthermore that the radial distribution of the gas temperature is given simply by the expression,

\[ T = T_{\text{central}} \Theta. \]

[Equation III.A.30]

Generally speaking, the function $\Theta(\xi)$ starts from the value $\Theta = 1$ at $\xi = 0$ then, as $\xi$ increases (i.e., as one moves radially away from the center of the model), $\Theta$ decreases monotonically toward zero. Following Chandrasekhar (1967), we identify $\xi_1$ as the radius at which the polytropic function $\Theta$ first reaches zero. Then, from the above relations [III.A.28, III.A.29, & III.A.30], it is clear that $\xi_1$ also marks the radius at which $\rho$, $P$, and $T$ fall to zero. Hence, $\xi_1$ may naturally be identified as a dimensionless measure of the total radius of the polytrope.

Mathematically, the function $\Theta(\xi)$ may be defined and evaluated at values of $\xi > \xi_1$. Generally speaking, however, $\Theta(\xi)$ becomes negative at radii immediately outside $\xi_1$ implying, for example, that the gas temperature [III.A.30] must become negative. Clearly, then, the polytropic function $\Theta$ loses its physical relevance at radii $\xi > \xi_1$. For this reason, it is customary to analyze the behavior of $\Theta(\xi)$ only over the range $0 \leq \xi \leq \xi_1$.

Radius:
Once $\xi_1$ has been determined for a given polytropic index, it is clear from the definition of $\xi$ [III.A.19 & III.A.20], that the total radius of a spherical polytrope is,

$$R = a_n \xi_1 = \{(n + 1)K_n / (4\pi G)\} (\rho_{\text{central}})^{(1/n - 1)/2} \xi_1,$$

[Equation III.A.31]

Ch67, Chapter IV, Eq. (62)

or, expressing $K_n$ in terms of the model's central (maximum) density and pressure [II.C.2],

$$R = \{ P_{\text{central}} (4\pi G p^2_{\text{central}})^{-1} \}^{1/2} (n + 1)^{1/2} \xi_1.$$

[Equation III.A.32]

When the central values of $T$ and $\rho$ do not differ significantly from the mean values of the model's temperature and density, one finds$^2$ that $\xi_1 \sim 1$. Under this condition, show that the derived expression [III.A.32] for the radius of a polytrope is simply a statement that, in equilibrium, the sound travel time from the surface to the center of the model must approximately equal the model's gravitationally defined free-fall time.

As defined earlier, the total mass lying interior to the radius "r" of a spherically symmetric object is,

$$M_r = \int_0^r 4 \pi \rho r^2 \, dr.$$

[Equation III.A.10]

Combining this expression with the definition of $\Theta$ and the definition of $\xi$, show that for a spherical polytrope, the mass interior to $\xi$ is,

$$M_\xi = 4\pi a_n^3 \rho_{\text{central}} [- \xi^2 d\Theta / d\xi].$$

[Equation III.A.33]

Ch67, Chapter IV, Eq. (67)
Evaluating this last expression at the surface of the polytrope, we see that the total mass of a spherical polytrope is,

\[ M = 4\pi a_n^3 \rho_{\text{central}} \left[ - \xi^2 \frac{d\Theta}{d\xi} \right]_{\xi = \xi_1}. \]

[Equation III.A.34]

**Ratio of central to mean density:**

Because the mean density of any sphere is,

\[ \rho_{\text{mean}} = \frac{3M}{4\pi R^3}, \]

[Equation III.A.35]

it is also clear that the ratio of the central density to the mean density in a spherical polytrope is,

\[ \frac{\rho_{\text{central}}}{\rho_{\text{mean}}} = \frac{\xi_1}{\left[ - 3 \frac{d\Theta}{d\xi} \right]_{\xi = \xi_1}}. \]

[Equation III.A.36]

The following Table, drawn directly from Table 4 of Ch67, lists several "constants of the Lane-Emden function" that may be used in conjunction with the preceding expressions to determine the total radius, total mass, and ratio of central to mean density in polytropes having a variety of different polytropic indices.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \xi_1 )</th>
<th>( \xi_1^2 )</th>
<th>( \frac{\rho_{\text{central}}}{\rho_{\text{mean}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sqrt{6} )</td>
<td>2( \sqrt{6} )</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>2.7528</td>
<td>3.7871</td>
<td>1.8361</td>
</tr>
<tr>
<td>1</td>
<td>( \pi )</td>
<td>( \pi )</td>
<td>( \pi^2/3 )</td>
</tr>
<tr>
<td>1.5</td>
<td>3.65375</td>
<td>2.71406</td>
<td>5.99071</td>
</tr>
<tr>
<td>3</td>
<td>6.89685</td>
<td>2.01824</td>
<td>54.1825</td>
</tr>
<tr>
<td>5</td>
<td>( \infty )</td>
<td>( \sqrt{3} )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

**Footnotes**

1, 2 As has been derived elsewhere, for example, for an \( n = 0 \) polytrope, \( \Theta_{n=0} = 1 - \xi^2/6 \) and \( \xi_1 = \sqrt{6} \). Hence, \( \Theta_{n=0} \) becomes negative at all radii \( \xi > \sqrt{6} \).