The Structure, Stability, and Dynamics of Self-Gravitating Systems

System of Spherical Coordinates

<table>
<thead>
<tr>
<th>DEFINITIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 = r = [x^2 + y^2 + z^2]^{1/2} )</td>
</tr>
<tr>
<td>( \xi_2 = \theta = \tan^{-1}\left(\frac{(x^2 + y^2)^{1/2}}{z}\right) )</td>
</tr>
<tr>
<td>( \xi_3 = \phi = \tan^{-1}\left(\frac{y}{x}\right) )</td>
</tr>
<tr>
<td>( h_1 = h_r = 1 )</td>
</tr>
<tr>
<td>( h_2 = h_\theta = \xi_1 )</td>
</tr>
<tr>
<td>( h_3 = h_\phi = \xi_1 \sin \xi_2 )</td>
</tr>
</tbody>
</table>

Direction Cosines for Spherical Coordinates

\[
\begin{array}{ccc|ccc}
\text{i} & \text{j} & \text{k} & e_1 & e_2 & e_3 \\
\gamma_{11} & \gamma_{12} & \gamma_{13} & \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\gamma_{21} & \gamma_{22} & \gamma_{23} & \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
\gamma_{31} & \gamma_{32} & \gamma_{33} & -\sin \phi & \cos \phi & 0 \\
\end{array}
\]

Position Vector

\[
x = e_1 \xi_1 = e_r r
\]
### Evaluating: \((\partial_{\xi} e_i)\)

<table>
<thead>
<tr>
<th></th>
<th>(\partial_{\xi_1})</th>
<th>(\partial_{\xi_2})</th>
<th>(\partial_{\xi_3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1)</td>
<td>0</td>
<td>(e_2)</td>
<td>(e_3) \sin(\xi_2)</td>
</tr>
<tr>
<td>(e_2)</td>
<td>0</td>
<td>-(e_1)</td>
<td>(e_3) \cos(\xi_2)</td>
</tr>
<tr>
<td>(e_3)</td>
<td>0</td>
<td>0</td>
<td>-(e_1) \sin(\xi_2) - (e_2) \cos(\xi_2)</td>
</tr>
</tbody>
</table>

\[
A = D_{\xi_2} = \frac{v_2}{\xi_1}
\]

\[
B = \sin\xi_2 D_{\xi_3} = \frac{v_3}{\xi_1}
\]

\[
C = \cos\xi_2 D_{\xi_3} = \left[ \frac{v_3}{\xi_1} \right] \cot\xi_2
\]

### Derived Expressions

<table>
<thead>
<tr>
<th>(\frac{d}{dt} e_1)</th>
<th>(= \frac{d}{dt} e_r)</th>
<th>(= \frac{d}{dt} e_q (d\theta/dt) + e_q \sin\theta \frac{dq}{dt})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{d}{dt} e_2)</td>
<td>(= \frac{d}{dt} e_q)</td>
<td>(- e_r \frac{d\theta}{dt} + e_q \cos\theta \frac{dq}{dt})</td>
</tr>
<tr>
<td>(\frac{d}{dt} e_3)</td>
<td>(= \frac{d}{dt} e_q)</td>
<td>(- e_r \sin\theta \frac{dq}{dt} - e_q \cos\theta \frac{dq}{dt})</td>
</tr>
<tr>
<td>(v)</td>
<td>(= \frac{d}{dt} x)</td>
<td>(= e_r \frac{dr}{dt} + e_q r \frac{d\theta}{dt} + e_q r \sin\theta \frac{dq}{dt})</td>
</tr>
<tr>
<td>(a)</td>
<td>(= \frac{d}{dt} v)</td>
<td>(= \frac{d}{dt} [ e_r \left( \frac{d^2 r}{dt^2} - r (\frac{d\theta}{dt})^2 - r \sin^2\theta (\frac{dq}{dt})^2 \right) ] + e_q \left[ 2 (\frac{dr}{dt})(\frac{d\theta}{dt}) + r (\frac{d^2\theta}{dt^2}) + 2 r \sin\theta \cos\theta (\frac{dq}{dt})^2 \right] + e_q \left[ 2 \sin\theta (\frac{dr}{dt})(\frac{dq}{dt}) + r \sin\theta (\frac{d^2 q}{dt^2}) + 2 r \cos\theta (\frac{d\theta}{dt})(\frac{dq}{dt}) \right] )</td>
</tr>
</tbody>
</table>
Gradient:

\[ \nabla = e_r \frac{\partial}{\partial r} + e_q \left( \frac{1}{r} \right) \frac{\partial}{\partial q} + e_j \left[ \frac{1}{(r \sin \theta)} \right] \frac{\partial}{\partial j} \]

This application permits you to determine the gradient of virtually any analytically expressible scalar function \( G(\mathbf{x}) \) in spherical coordinates, utilizing the symbolic manipulation capabilities of Mathematica®.

Enter Desired Function Expression†:

In the space provided, type in the analytical expression for the scalar function of interest, \( G \), in terms of the spherical coordinates \((r, \theta, \phi) \rightarrow (Rr, \text{Theta}, \text{Pphi})\). (For example, when you first load this HTML page, you will find in the space provided the Mathematica String Equivalent of the function \( G = \sin \theta \cos \phi / r^2 \), written in spherical coordinates.)

\[ G(Rr, \text{Theta}, \text{Pphi}) = \text{Sin[Theta]*Cos[Pphi]/(Rr^2)} \]

Evaluate \( \nabla G \):

\[ \text{<= Press this button.} \]

†If you are uncertain how to deal with or interpret the idiosyncrasies of Mathematica's input/output formats, read the accompanying page of "Mathematica hints." In the present implementation of this application, you must limit your input string to 50 characters, including embedded blanks.
Divergence:
\[ \nabla \cdot F = \left[1/r^2\right]\partial_r (r^2 F_r) + \left[1/(r \sin \theta)\right]\partial_\theta (\sin \theta F_\theta) + \left[1/(r \sin \theta)\right]\partial_\varphi F_\varphi \]

### Mathematica® Application

Developed by:
David Sherfesee
July, 1997

This application permits you to determine the divergence of virtually any analytically expressible vector function \( F(\mathbf{x}) \) in spherical coordinates, utilizing the symbolic manipulation capabilities of Mathematica®.

#### Enter Desired Function Expression:

In the spaces provided, type in the analytical expression for the three components of the vector function of interest, \( F \), in terms of the spherical coordinates \((r, \theta, \varphi) \rightarrow (Rr, Ttheta, Pphi)\). (For example, when you first load this HTML page, you will find in the space provided the Mathematica String Equivalent of the vector, \( F = e_r r + e_\theta r \sin \theta + e_\varphi r \sin \theta \cos \varphi \), written in spherical coordinates.)

\[
F(Rr, Ttheta, Pphi) = e_r Rr + e_\theta Rr \sin [Ttheta] + e_\varphi Rr \sin [Ttheta] \cos [Pphi]
\]

#### Evaluate \( \nabla \cdot F \):

\[ \text{<=} \quad \text{Press this button.} \]
Laplacian:
\[ \nabla^2 G = \frac{1}{r^2} \partial_r (r^2 \partial_r G) + \frac{1}{(r^2 \sin \theta)} \partial_\theta (\sin \theta \partial_\theta G) + \frac{1}{(r^2 \sin^2 \theta)} \partial_q (\partial_q G) \]

Mathematica® Application
Developed by:
David Sherfesee
July, 1997

This application permits you to determine the Laplacian of virtually any analytically expressible scalar function \( G(\mathbf{x}) \) in spherical coordinates, utilizing the symbolic manipulation capabilities of Mathematica®.

Enter Desired Function Expression:
In the space provided, type in the analytical expression for the scalar function of interest, \( G \), in terms of the spherical coordinates \((r, \theta, \phi)\) \(\rightarrow\) \((Rr, \Theta, \Phi)\). (For example, when you first load this HTML page, you will find in the space provided the Mathematica String Equivalent of the function \( G = \sin \theta \cos \phi / r^2 \), written in spherical coordinates.)

\[ G(Rr, \Theta, \Phi) = \frac{\sin \Theta \cos \Phi}{(Rr)^2} \]

Evaluate \( \nabla^2 G \):

\[ \langle=\rangle \text{ Press this button.} \]

\[ (\mathbf{v} \cdot \nabla) \mathbf{F} = \mathbf{e}_r \left\{ v_r \partial_r F_r + \left( \frac{v_\theta}{r} \right) \partial_\theta F_r + \left[ \frac{v_q}{r \sin \theta} \right] \partial_q F_r - \left( \frac{v_\theta F_\theta + v_q F_q}{r} \right) \right\} + \]
\[ \mathbf{e}_\theta \left\{ v_r \partial_r F_\theta + \left( \frac{v_\theta}{r} \right) \partial_\theta F_\theta + \left[ \frac{v_q}{r \sin \theta} \right] \partial_q F_\theta - \left( \frac{v_\theta F_r + v_q F_q \cot \theta}{r} \right) \right\} + \]
\[ \mathbf{e}_q \left\{ v_r \partial_r F_q + \left( \frac{v_\theta}{r} \right) \partial_\theta F_q + \left[ \frac{v_q}{r \sin \theta} \right] \partial_q F_q + \left( \frac{v_\theta F_q + v_q F_\theta}{r} \right) + \right\} \]
\[ \mathbf{l} = \mathbf{e}_1 \hat{\xi}_1 \times \mathbf{v}. \]

Hence, the magnitude of the angular momentum,

\[ I_2 = \{ \mathbf{l} \mathbf{l} \}^{1/2} = \{ [\hat{\xi}_1 \cdot \mathbf{D}\hat{\xi}_2]^2 + [\hat{\xi}_1 \cdot \mathbf{D}\hat{\xi}_3]^2 \}^{1/2}, \]

provides a second scalar integral of the motion, and the z- and x-components of the angular momentum,

\[ I_3 = \mathbf{k} \cdot \mathbf{l} = [\hat{\xi}_1 \sin \hat{\xi}_2] \mathbf{D}\hat{\xi}_3, \]
\[ I_4 = \mathbf{i} \cdot \mathbf{l} = -\hat{\xi}_1^2 \sin \hat{\xi}_3 \mathbf{D}\hat{\xi}_2 - \hat{\xi}_1^2 \sin \hat{\xi}_2 \cos \hat{\xi}_2 \cos \hat{\xi}_3 \mathbf{D}\hat{\xi}_3, \]

respectively, provide two additional independent scalar integrals of the motion. The y-component of the angular momentum, of course, must also be conserved but it does not provide an integral of the motion that is independent of the integrals \( I_2, I_3, \) and \( I_4. \)