Hydrodynamic Collapse

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1. INTRODUCTION

We are now quite accustomed to the idea that the structure of main sequence stars and the initial evolution of these stars away from the main sequence can be modeled theoretically with relatively high precision. At the same time, however, we are aware that theoretical models describing the formation
of main sequence stars or models of the latest stages of stellar evolution have met with little more than qualitative success. This is primarily because the birth and death of most stars are hydrodynamic rather than hydrostatic events. Any phase of stellar evolution should be modeled as a hydrodynamic event if unbalanced accelerations within the star cause significant changes in the velocity structure of the star on a time scale that is comparable to or shorter than other time scales that govern the star’s evolution. For example, whenever gravity is unbalanced by other forces, large velocities can be generated and the structure of the star will change significantly on a dynamic time scale,

$$t_d \sim [G \rho]^{-1/2},$$

(1.1)

where $G$ is the gravitational constant and $\rho$ is the mean density of the star. This physical situation must be modeled hydrodynamically.

Hydrodynamic events erect two important barriers that impede efforts to model stellar evolution in complete generality: (1) Acceleration terms cannot be ignored in the physical equations that dictate the structure of the star, and often supersonic velocities are generated—hence, the evolution becomes computationally difficult to describe. (2) The events occur so rapidly, compared to the main sequence life of the star for example, that observations are unable to completely map out the “evolutionary tracks” along which stars of different mass evolve—hence, it is often difficult to directly compare theoretical models of hydrodynamic evolutions with observations. In spite of these barriers, significant progress has been made in recent years toward understanding some important features of the hydrodynamic events in stellar evolution. In this article an attempt is made first to review pieces of analytic work that have contributed significantly to our basic understanding of hydrodynamic collapse under the force of gravity. These works should serve as the foundation for all future models of hydrodynamic collapse. Second, a brief description of the methods employed in constructing hydrodynamic computer programs is presented so that the general reader can better appreciate the recent advances that have been made in the numerical modeling of hydrodynamic collapse. Finally, a summary is given of the new knowledge that has been gained from recent numerical models of, specifically, the hydrodynamic collapse of interstellar gas clouds to form stars. Although no direct mention is made in this article of hydrodynamic collapse in the late stages of stellar evolution, the physical processes that are discussed here are sufficiently general that those who examine these latter stages will, hopefully, find this review to be a good background and reference source.

Throughout this review, strong emphasis is placed on the role that rotation and gas pressure play during hydrodynamic collapse. The role of magnetic fields is not discussed since very little self-consistent modeling of magnetic fields during phases of hydrodynamic collapse has been done to date. The interested reader is referred to a recent review by Mouschovias (1981) of work that has been done exploring the role of magnetic fields in star formation. A brief discussion of the role that turbulent viscosity may play as an angular momentum transport agent during hydrodynamic collapse is included. A clear distinction is drawn between the hydrodynamic collapse of a gas cloud and the accretion phase that (usually) follows collapse. Although accretion is certainly an important hydrodynamic event in star formation as well as in many other stages of stellar evolution it is an extremely complex phase of evolution in itself. It is discussed only in passing here.

2. SETTING THE STAGE FOR COLLAPSE

2.1 The Jeans Criterion

Jeans (1929) was the first to point out that a gaseous medium of density $\rho$ and temperature $T$ will be unstable toward collapse under gravity if the volume occupied by the medium is sufficiently large. The Jeans criterion can be expressed simply in terms of the relative importance of thermal energy per unit mass $E_T$ and gravitational potential energy per unit mass $E_G$ of the medium:

$$|E_G| > E_T$$

(2.1)

means the medium is unstable toward collapse. For an homogeneous, perfect gas sphere of radius $R$ and mass $M = 4\pi R^3 \rho / 3$,

$$E_G = \frac{3}{5} \frac{G M}{R}$$

(2.2)

$$E_T = \frac{3}{2} \frac{\Theta}{\mu} T$$

(2.3)

where $\Theta$ is the gas constant and $\mu$ is the mean molecular weight of the gas. So at a given $\rho$ and $T$, collapse will ensue if

$$M > M_J \equiv \left[ \frac{3}{4\pi} \left( \frac{5\Theta}{2G} \right)^{3/2} (T/\mu)^{3/2} \rho^{-1/2} \right]^{1/2}$$

(2.4a)

or,

$$R > R_J \equiv \left[ \frac{15\Theta}{8\pi G} \right]^{1/2} (T/\mu)^{1/2} \rho^{-1/2}$$

(2.4b)
The Jeans mass $M_J$ and Jeans radius $R_J$ are thus defined for purposes of discussion in the remainder of this article. For a gas cloud of a given mass and temperature, condition (2.4b) can be written

$$R < R_J = \frac{2G \mu M}{3 \pi \rho T},$$

(2.4c)

but care should be taken not to confuse the direction of the inequality here with that in (2.4b). More generally, when other physical processes are important in a gaseous medium, condition (2.1) should be modified accordingly. For example, rotational kinetic energy per unit mass $E_R$ and the magnetic field energy per unit mass $E_B$ will lead to a modified criterion given approximately by

$$|E_0| > E_T + E_R + E_B$$

(2.5)

necessary for collapse (see, for example, Larson, 1972 and Mouschovias and Spitzer, 1976). Once condition (2.1) or the more general form (2.5) is satisfied, contraction under gravity will take place on the dynamic time scale given by Eq. (1.1).

The solid line in Figure 1 shows how the temperature of metal rich interstellar gas in our Galaxy is believed to vary with gas density (the bottom axis is in units of g cm$^{-3}$, the top axis is in units of number of hydrogen atoms per cm$^3$). This curve, also shown in Figure 10 and discussed further in §6, has been derived theoretically by considering detailed heating and cooling processes in a $3 M_\odot$ gas cloud during its collapse toward stellar densities. The temperatures shown at the low density portion of the figure (regions I and II) are appropriate for a wide range of cloud masses and they agree, where there is overlap, with observationally determined conditions. Using the solid $\rho$-$T$ curve of Figure 1, $M_J$ from Eq. (2.4a) has been tabulated in units of solar masses in column (3) of Table I for a range of densities. The important thing about the $\rho$-$T$ behavior of interstellar gas depicted in Figure 1 is that once a given mass cloud ($M > 0.1 M_\odot$) becomes unstable toward collapse at low densities, condition (2.1) will continue to hold until the cloud has contracted to densities $\rho > 10^{-5}$ g cm$^{-3}$ (off scale to the right in Figure 1). That is, the properties of the interstellar medium demand that Jeans unstable gas clouds of stellar mass undergo hydrodynamic collapse all the way to stellar densities in (roughly) a single dynamic time $t_d$. The appropriate values for $t_d$ are tabulated in column 4 of Table I. Gauslaid (1963) pointed out this property of the interstellar medium some time ago.

The Jeans instability, then, provides a "simple" means by which interstellar gas clouds can rapidly be transformed into stars. With this in mind, all of the hydrodynamic collapse models discussed in the remaining sections of

\begin{table}[h]
\centering
\begin{tabular}{cccc}
\hline
\hline
$\rho$ (g cm$^{-3}$) & $T(K)^*$ & $M_J/M_\odot$ & $t_d$ (yr) \\
\hline
10^{-22} & 20 & 380.0 & 10^7 \\
10^{-20} & 8 & 10.0 & 10^6 \\
10^{-18} & 5 & 0.5 & 10^4 \\
10^{-16} & 5 & 0.05 & 10^3 \\
10^{-14} & 5 & 0.005 & 10^2 \\
\hline
\end{tabular}
\caption{Table I}
\end{table}

*T is taken from Figure 1; $\mu = 2$ (pure molecular hydrogen).
this article have been evolved from initial structures that satisfy the Jeans criterion (2.4).

2.2 Choosing Specific Initial Conditions for Collapse

Unfortunately the Jeans criterion alone does not define the unique configuration from which collapse can begin. The criterion is based on globally averaged properties of a cloud, so, assuming global properties are the same, an inhomogeneous structure is just as likely to collapse as is an homogeneous structure. And the criterion sets only a limit on the conditions in a cloud that will be sufficient for collapse. For example, for a given cloud mass, Eq. (2.4c) states that any initial radius less than \( R_J \) will lead to collapse. Furthermore, the criterion is based on an initial configuration that is relatively quiescent. Hunter (1979) and Hunter and Fleck (1982) have shown that at a given density and temperature, the minimum cloud mass required to initiate collapse can be considerably lowered from the value given by criterion (2.4a) if a cloud is initially subjected to an ordered implosion velocity field of sufficient magnitude.

It would be convenient if observations of interstellar gas clouds were to reveal that clouds are relatively quiescent when they begin to collapse, that they begin collapse from a configuration near the Jeans limit \( R \approx R_J \), and that they are fairly homogeneous structures initially. Then numerical models of hydrodynamic collapse could be built from a small range of initial conditions and the endpoint of collapse would subsequently be well-defined. This is certainly not the case. Radio observations, which are primarily restricted to the lowest density region of Figure 1 (marked as region I), indicate quite clearly that massive molecular clouds \( (10^3 - 10^6 M_\odot) \) that are unstable according to criterion (2.1) are very inhomogeneous and exhibit supersonic internal motions (see the discussions of Pienzias, 1974, and Solomon and Sanders, 1979). Furthermore, many observations (see Elmegreen and Elmegreen, 1978, for a summary) and numerical simulations (see Woodward, 1978, for a review, plus Bodenheimer, Tenorio-Tagle and Yorke, 1979; Klein, Sandford and Whitaker, 1980; Tenorio-Tagle and Bedijn, 1981; and Sandford, Whitaker and Klein, 1982, for more recent work) of hydrodynamic events in the interstellar medium strongly suggest that gas clouds initially stable against collapse are, in many instances, pushed to the verge of collapse by ionization fronts or by shock waves and therefore do not approach the Jeans condition in a quiescent fashion. The study of hydrodynamic collapse, as it applies to star formation, must therefore treat a wide variety of physically realistic initial conditions.

It must be stressed at the outset of this article that models of hydrodynamic collapse have only begun to explore the wide variety of initial configurations that are demanded by observations. Generally the cloud models that have been investigated have initial structures that deviate at most only slightly from homogeneous, quiescent spheres (with and without solid body rotation). The results of these investigations are fairly well understood, though, and lay a solid footing for future models that employ more exotic (but more realistic) starting conditions. It will be quite some time before studies of hydrodynamic collapse will be complete in sufficiently general terms to allow us to understand, for example, how stellar mass functions are determined when interstellar gas clouds transform into star clusters. The study of hydrodynamic collapse is really only in its infancy. It is in this light that theoretical work on the problem of hydrodynamic collapse should be reviewed.

3. Analytic and Semi-Analytic Models of Pressure-Free Collapse

When analyzing the time evolution of any hydrodynamic event, the differential equation of overriding interest is the equation of motion

\[
\frac{dv}{dt} = - \nabla \phi - \frac{1}{\rho} \nabla P, \tag{3.1}
\]

which describes the acceleration applied to all volume elements of the gaseous, or "fluid", medium being studied. In (3.1) \( v \) is the fluid velocity, \( \phi \) is the gravitational potential, \( \rho \) is mass density and \( P \) is gas pressure (see §5.1 for other supplemental relations). When gradients of the gas pressure produce forces within a gas cloud that are negligible compared to gradients in the gravitational potential, then the acceleration of fluid elements is quite accurately described by the relatively simple relation

\[
\frac{dv}{dt} = - \nabla \phi, \tag{3.2}
\]

understandably known as the pressure-free equation of motion.

Equation (3.2) is a strict equality only if the gas temperature is exactly zero everywhere in a cloud. In an isolated gas cloud of radial dimension \( R_0, \phi \sim GM/R_0 \), so Eq. (3.2) becomes a useful approximation as long as

\[
\frac{P}{\rho R_0} \ll \frac{GM}{R_0^2} \sim GpR_0. \tag{3.3}
\]

In terms of the isothermal sound speed

\[
c_t = [P/\rho]^{1/2}, \tag{3.4}
\]
then, Eq. (3.2) is a useful approximation as long as
\[
\frac{c_s}{R_0[G\rho]^{1/2}} \ll 1.
\]
(3.5)

As will be stressed throughout the remaining sections of this article, the qualitative evolution of a gas cloud as it undergoes hydrodynamic collapse will be drastically different depending on whether or not condition (3.5) is realized in the initial cloud state. It is therefore instructive to write (3.5) in terms of the well-known Jeans criterion of Section 2. A cloud can be considered "pressure-free" in its initial state if, at a specified density and temperature,
\[
R_0 \gg R_J \quad (i.e., \ M \gg M_J).
\]
(3.6)

In the following few subsections, pressure-free conditions will be assumed initially and throughout the hydrodynamic collapse of a self-gravitating gas cloud. The simplicity of Eq. (3.2) permits an analytic determination of the time-evolving cloud structure for several instructive cases. Our understanding of the cloud collapse problem as acquired from these pressure-free models is briefly summarized in §3.3.

A considerable modification of the collapse evolution is required if \( R_0 \sim R_J \) initially. Equation (3.1) must be used instead of equation (3.2) because pressure gradients cannot be ignored during the cloud's evolution. Section 4 discusses the few analytic analyses that are presently available to describe hydrodynamic collapse when gas pressure plays a major role.

3.1 Uniform Density Sphere

Under the assumption of spherical symmetry, the variables in (3.2) are allowed to have only radial variations, having no gradients in either of the spherical coordinates \( \theta \) or \( \phi \). Hence, the Poisson Eq. (5.4) governing the gradient in the gravitational potential reduces to an ordinary differential equation of second order in \( r \):
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 4\pi G\rho.
\]
(3.7)

This equation in turn reduces to the form
\[
\frac{d\phi}{dr} = \frac{GM(r)}{r^2}
\]
(3.8)

where
\[
M(r) = 4\pi \int_0^r \rho(r) r^2 \, dr.
\]
(3.9)

Hence the Lagrangian equation of motion for a spherically symmetric, self-gravitating distribution of pressure-free gas becomes
\[
\frac{dv}{dt} = -\frac{GM(r)}{r^2}
\]
(3.10)

Hunter (1962) provides the most complete discussion of the collapse from rest of a uniform density, pressure-free sphere—a collapse evolution described entirely by Eq. (3.10). A fluid element initially at a distance \( r_i \) from the cloud center will feel an acceleration only from the matter interior to \( r_i \), \( M(r_i) \). Assuming that the total mass interior to this fluid element remains constant as the element accelerates toward the cloud center (this actually proves to be a realistic assumption in many situations, as will be demonstrated presently), Eq. (3.10) can be integrated analytically to determine the position and velocity of the fluid element as a function of time. Multiplying both sides of Eq. (3.10) by \( 2vdv \), where \( v = dr/dt \), gives
\[
dl {v^2} = -2GM(r_i) \frac{dr}{r^2},
\]
which integrates once to give
\[
v^2 = 2GM(r_i) \left( \frac{1}{r} - \frac{1}{r_i} \right) + v_i^2
\]
(3.11)

where \( v_i \) is the initial velocity of the fluid element. Therefore, the time-dependent velocity of a fluid element which collapses from rest \( (v_i=0) \) is
\[
v = -\left[ 2GM(r_i) \left( \frac{1}{r} - \frac{1}{r_i} \right) \right]^{1/2}.
\]
(3.12)

Notice that the negative sign appears in order to denote collapse and that the time dependence is still disguised in the variable \( r(t) \). Making the substitution
\[
\frac{r}{r_i} = \cos^2 \zeta
\]
(3.13)
in Eq. (3.12) gives
\[
2 \cos^2 \zeta \, d\zeta = \left[ 2GM(r_i) \right]^{1/2} \, dt,
\]
(3.14)

which integrates to
\[
\zeta + \frac{1}{2} \sin 2\zeta = \left[ \frac{2GM(r_i)}{r_i^3} \right]^{1/2} \, t,
\]
(3.15)
where the initial condition \( r=r_I (\xi=0) \) at time \( t=0 \) has been applied. Notice that generally the parameter \( \xi \) is a function of both \( r_I \) and \( t \).

Obviously, \( r \to 0 (\xi \to \pi/2) \) at a finite time

\[
l = \tau_{ff}(r_I) = \left[ \frac{3\pi}{32G \rho(r_I)} \right]^{1/2},
\]

(3.16)

where \( \rho(r_I) \equiv 3M(r_I)/(4\pi r_I^3) \) is the mean density of material interior to the fluid element at the onset of collapse. This “free fall” time \( \tau_{ff} \) is a function only of \( \rho(r_I) \). If the gas cloud is initially uniform in density, then fluid elements at all radii will have exactly the same free fall time, and the time parameter \( \xi \) will be a function only of time \( t \). This special case produces an homologous cloud collapse as the fractional radial positions of all fluid elements \( r/r_I \) are identical at each instant of time. A given fluid element will not overtake another one which was initially at a smaller radius (until all elements hit the cloud center together at exactly one free fall time), therefore guaranteeing that \( M(r_I) \) is constant as a function of time during the collapse. This was the crucial assumption that allowed the straightforward integration of Eq. (3.10).

The time-evolving density of a fluid element can also be determined analytically for this uniform density, pressure-free cloud. The Lagrangian form of the continuity Eq. (5.2) specifies that

\[
d \ln \rho = - \frac{1}{r^2} \left[ \frac{\partial}{\partial r} (r^2 \rho v) \right] dt.
\]

(3.17)

From Eqs. (3.12) and (3.14),

\[
r^2 v = - \frac{\pi}{2\tau_{ff}} r^3 \frac{\sin \xi}{\cos^3 \xi},
\]

(3.18)

and

\[
dt = \frac{4\tau_{ff}}{\pi} \cos^2 \xi \, d\xi.
\]

(3.19)

Hence, for a general initial density profile in the cloud,

\[
d \ln \rho = 6 \tan \xi \, d\xi \left[ 1 + X \right]
\]

(3.20)

where

\[
X = \frac{1}{3} \left[ r \frac{\partial}{\partial r} (3 \tan \xi + \cot \xi) - \frac{d}{dt} \frac{\tau_{ff}}{r} \right].
\]

(3.21)

But, as explained above, for an initially uniform density cloud, both \( \tau_{ff} \) and \( \xi \) are independent of radius, so \( X=0 \) in this case. Therefore Eq. (3.20) immediately integrates to give

\[
\rho/r_I = \sec^6 \xi.
\]

(3.22)

The cloud remains uniform in density, with every fluid element’s density increasing with time as prescribed by Eq. (3.22). In Figure 2, the time evolution of \( \rho \) given by (3.22) has been illustrated. Notice in particular that this cloud, which starts collapsing from rest, spends 50% of one free fall time just doubling its density, and 85% of one free fall time increasing its density by a single order of magnitude.

3.2 Centrally Condensed Sphere

If, in any region of a cloud, the gas density increases outward, then \( \rho(r_I) \) will also increase in this region and, by (3.16), \( \tau_{ff}(r_I) \) will decrease outward. This means that within this region of a cloud, fluid elements at large radii will initially collapse on a shorter time scale than their neighboring fluid elements at smaller radii and will overtake their neighbors before reaching the cloud center. As one fluid element overtakes another, one of two things will happen:

![Figure 2](image-url)

**FIGURE 2.** The gas density \( \rho \) relative to its initial value \( \rho_I \) is shown as a function of time \( t \) during the collapse of a nonrotating, homogeneous, pressure-free sphere as determined analytically by Eq. (3.22). The free fall time \( \tau_{ff} \) is defined in Eq. (3.16).
either local gas density gradients will become very large, inducing large pressure gradients as well, or, in the absence of gas pressure, the mass contained in the spherical volume that is interior to a given element’s position will change. In either case, solutions (3.12) and (3.13) will no longer accurately describe the motion of fluid elements within the cloud.

If, however, the gas density initially decreases monotonically with radius in a spherical, pressure-free cloud, Eqs. (3.13) and (3.12) will correctly describe the time evolution of the position and velocity of all fluid elements. The cloud’s evolution as a whole will differ drastically from that of the uniform density case just discussed, though, because fluid elements at different radii will fall toward the cloud center on different time scales. The cloud must become more and more centrally condensed as the evolution proceeds because fluid elements near the cloud center will reach \( r = 0 \) (hence the local \( \rho / \rho_i \to \infty \)) before fluid elements at large radii do.

It proves instructive to study the exact evolution of \( \rho(r) \) in an initially centrally condensed, pressure-free cloud. Working from the differential form of the continuity equation, as we did in the previous subsection, is difficult in this case because \( \zeta \) and \( \tau_{fe} \) are generally nontrivial functions of \( r \) and therefore, through the variable \( X \), the right-hand side of (3.20) is far from trivial as well. Instead, realizing that the mass \( dm \) enclosed by a given spherical Lagrangian shell remains constant with time, i.e., \( dm = dm_t \), the following useful substitute for the continuity equation can be formulated:

\[
\frac{dm}{d\zeta} = \frac{4\pi R^3 \rho dr}{4\pi R^3 \rho_t dr_t} = 1
\]  

(3.23)

which, from (3.13), becomes

\[
\rho \frac{d\rho}{dr} = \left( \frac{r}{r_t} \right)^2 \frac{dr}{dr_t} = \cos^6 \zeta \left[ 1 - 2r_t \tan \frac{d\zeta}{dr_t} \right]
\]  

(3.24)

where \( \zeta (r) \) is a unique function of time for each Lagrangian fluid element \( r_t \). In terms of the cloud’s initial central density \( \rho_c \) and its corresponding central free fall time \( \tau_{fe} = \left[ 3\pi / 32G\rho_c \right]^{1/2} \), Eq. (3.15) becomes

\[
\zeta + \frac{1}{2} \sin 2\zeta = A(r_t) \frac{t}{\tau_{fe}}
\]  

(3.25)

where

\[
A(r_t) = \left[ \frac{3M(r_t)}{4\pi R^3 \rho_c} \right]^{1/2}
\]  

(3.26)

and \( M(r_t) \) is given by (3.9). Therefore, in (3.24),

\[
\frac{d\zeta}{dr_t} = (2\cos^2 \zeta)^{-1} \frac{t}{\tau_{fe}} \frac{dA}{dr_t}
\]  

(3.27)

and the time dependent density of each fluid element is given by

\[
\rho / \rho_t = \sin^6 \zeta \left[ 1 - \frac{t}{\tau_{fe}} \frac{r_t \cos \zeta}{\cos^3 \zeta} \frac{dA}{dr_t} \right]^{-1}
\]  

(3.28)

In order to illustrate the time dependent behavior of a cloud’s density profile as given by Eq. (3.28), the following profile has been arbitrarily chosen as an initial condition for the cloud:

\[
\rho(r_t) = \rho_0 \left[ 1 - \left( r_t / R \right)^n \right]
\]  

(3.29)

where \( R \) is the cloud radius initially and \( n \) is an integer greater than zero. The density profile at four different times during the cloud’s collapse has been plotted in Figure 3 for three different values of the integer \( n \). Notice that at \( t/\tau_{fe} = 0.990 \), the central density has reached a value log_{10} (\rho / \rho_0) = 3.28 in each model, as expected from Figure 2, but the overall cloud size and the cloud’s density structure at this evolutionary stage is quite different in each case. As \( n \to \infty \), the uniform density collapse discussed in §3.1 is retrieved. Obviously, the evolution of a pressure-free cloud depends strongly on the cloud’s initial structure. Even mildly centrally condensed clouds evolve to very condensed structures.

3.3 Uniform Density Spheroid

Certainly the assumption that a gas cloud is perfectly spherical in its initial state is an oversimplification that was employed above in order to allow the analytic manipulation of the otherwise complicated vector Eq. (3.2). Here this assumption is relaxed somewhat, and a uniform density, pressure-free cloud is allowed to be deformed in an axisymmetric fashion initially, or to evolve to a spheroidal shape under the influence of centrifugal forces.

In cylindrical coordinates \( (\omega, \theta, z) \), the vector Eq. (3.2) can be split into its three velocity components:

\[
\frac{d^2\omega}{dt^2} = \omega \left( \frac{d\theta}{dt} \right)^2 = -\frac{\omega}{\omega_t} \frac{d\phi}{dt},
\]  

(3.30a)

\[
\frac{d^2\theta}{dt^2} = -\frac{\omega}{\omega_t} \frac{d\phi}{dt},
\]  

(3.30b)

\[
\frac{1}{\omega} \frac{d}{dt} \left[ \omega^2 \frac{d\theta}{dt} \right] = -\frac{1}{\omega_t} \frac{d\phi}{dt}.
\]  

(3.30c)
Assuming axial symmetry of the mass distribution, and hence also of $\phi$, derivatives with respect to $\theta$ vanish. The specific angular momentum $j=\sigma^2 (d\theta/dt)$ of a fluid element is therefore conserved and the problem becomes two-dimensional.

\[
\frac{d^2 \omega}{dt^2} - \frac{j^2}{\omega^3} = - \frac{\partial \phi}{\partial \omega} \tag{3.31a}
\]

\[
\frac{d^2 z}{dt^2} = - \frac{\partial \phi}{\partial z} \tag{3.31b}
\]

The potential inside a homogeneous spheroid having semi-major and semi-minor axes $a$ and $c$, respectively, is derived by MacMillan (1958, p. 45) and Ramsey (1949, p. 169) (see also Chandrasekhar, 1969), and its gradients
are implemented by Mestel (1965) and by Lin, Mestel and Shu (1965, here-
after LMS):\footnote{Note that a typographical error occurs in Eq. (21) of LMS. The coefficient out in front of their expression for $A(e)$ should be $2\pi(1-e^2)^{1/2}e^0$, as correctly printed in Eq. (28) of Mestel (1965) and in Eq. (3.33) here.}

\[
\begin{align*}
\frac{\partial \phi}{\partial \tilde{\omega}} &= G \rho \ A(e) \tilde{\omega} \tag{3.32a} \\
\frac{\partial \phi}{\partial z} &= G \rho \ C(e) z \tag{3.32b}
\end{align*}
\]

where $e$ is the eccentricity of the spheroid and, specifically for an oblate spheroid,

\[
A(e) = \frac{2\pi(1-e^2)^{1/2}}{e^3} \left[ \sin^{-1} e - e(1-e^2)^{1/2} \right]. \tag{3.33}
\]

\[
C(e) = \frac{4\pi}{e^5} \left[ 1 - \frac{(1-e^2)^{1/2}}{e} \sin^{-1} e \right] \tag{3.34}
\]

\[
e^2 = 1 - e^2/\rho_1^2. \tag{3.35}
\]

Notice that for a sphere, $a=c, e=0$, and $A(e)=C(e)=(4/3) \pi$, so the right-hand sides of (3.32a) and (3.32b) reduce to

\[
\frac{GM}{r^2} \left( \frac{\tilde{\omega}}{\tau} \right), \text{ and } \frac{GM}{r^2} \left( \frac{z}{\tau} \right),
\]

respectively, as expected.

Defining the following dimensionless variables,

\[
\begin{align*}
R &= \frac{\tilde{\omega}}{\omega_i} \\
Z &= \frac{z}{z_i} \\
\tau &= t/\tau_{ff}
\end{align*}
\]

where subscripts $i$ denote initial conditions and

\[
\tau_{ff} = \left[ \frac{3\pi}{32G \rho_i} \right]^{1/2}
\]

as in Eq. (3.16), Eqs. (3.31a) and (3.31b) can be rewritten as

\[
\begin{align*}
\frac{R}{R^*} &= -\frac{3\pi}{32 \rho_i} A(e) R \tag{3.36a} \\
\frac{Z}{R^*} &= -\frac{3\pi}{32} C(e) Z \tag{3.36b}
\end{align*}
\]

where $J^2=(\tau_{ff}^2 \tilde{P}^2/\omega_i^4)$ and dots mean differentiation with respect to the dimensionless time $\tau$. These equations governing the motion of an individual fluid element become manageable after one realizes, as was first pointed out by Lynden-Bell (1962), that in the case of uniform rotation the solution to the equations must produce $R(\tau)$ and $Z(\tau)$ functions that are independent of a fluid element’s initial position in the spheroid. At any time during the collapse the fractional coordinate positions $z_i/z_i$ must be the same for all elements and the fractional coordinate positions $\omega_i/\omega_i$ must be the same for all elements as well. Hence the cloud must remain spheroidal, uniform in density, and uniformly rotating as it collapses. In fact, the cloud’s density must be related to the functions $R(\tau)$ and $Z(\tau)$ as

\[
\frac{\rho}{\rho_i} = \left( \frac{a_i}{a} \right)^2 \left( \frac{c_i}{c} \right) = \frac{1}{R^2 Z} \tag{3.37}
\]

since $a$ and $c$ are not special coordinates for a fluid element. Using (3.37), Eqs. (3.36a) and (3.36b) become

\[
\begin{align*}
\frac{R}{R^*} &= -\frac{3\pi}{32 RZ} A(e) R \tag{3.38a} \\
\frac{Z}{R^*} &= -\frac{3\pi}{32} C(e) Z \tag{3.38b}
\end{align*}
\]

where, from (3.35),

\[
e^2 = 1 - (1-e^2) \frac{Z^2}{R^2}. \tag{3.39}
\]

Equations (3.38a) and (3.38b) now describe completely the time evolution of a uniformly rotating or nonrotating ($J^2=0$), pressure-free spheroid that collapses from rest. Even in the nonrotating case, though, the equations are coupled in a complicated way through the functions $A(e)$ and $C(e)$, so it is not surprising that their general solution has not yet been found through analytic techniques. The limiting spherical case ($e_i=0$) with no rotation, of course, has been solved, as presented in §3.1 above. As Mestel (1965) has
pointed out, though, in the nonrotating case one can show by expanding the right-hand sides of (3.38a) and (3.38b) in terms of powers of $e$ that even small deviations from spherical symmetry in the initial cloud structure amplify during the collapse, causing an oblate cloud to flatten into a pancake. Performing this expansion, we get

\[ \dot{R} \simeq -\frac{\pi^2}{8} \frac{1}{R^2} \left[ 1 + \frac{3}{10} e^2 + \frac{9}{56} e^4 \right] \] (3.40a)

\[ \dot{Z} \simeq -\frac{\pi^2}{8} \frac{1}{R^2} \left[ 1 + \frac{2}{5} e^2 + \frac{8}{35} e^4 \right] \] (3.40b)

so the ratio of the accelerations is,

\[ \frac{\dot{Z}}{\dot{R}} \simeq \frac{1}{10} \frac{e^2}{e^4} + \frac{53}{1400} \] (3.41)

For any nonzero eccentricity, the collapse does indeed accelerate faster in the $z$-coordinate than it does in the $\omega$-coordinate, driving the configuration through a sequence of flatter and flatter spheroids. In order to study this flattening evolution completely, the coupled second-order ordinary differential Eqs. (3.38a) and (3.38b) can be solved numerically by, for example, performing a straightforward Runge–Kutta integration. This has been done for the nonrotating case by LMS and for both the uniformly rotating and nonrotating cases by Hutchins (1976). A single table of numbers drawn from these sources serves to illustrate the behavior of the nonrotating collapse. Listed in Table II for a number of different initial eccentricities $e_0$, is the time $\tau_e$ (in units of $\tau_{ff}$) at which the spheroid reaches its maximum flattening (at $a=0$, $e=1$), the corresponding radius of the disk $R_e$ at this instant in time, and the velocities $\dot{R}_e$ and $\dot{Z}_e$ of the liquid at the instant $\tau_e$. Notice that a spheroid whose axes initially differ by only 10% in length ($e_0=0.9$) will flatten to a pancake in a sufficiently short time that its equatorial radius will have shrunk by less than a factor of 10. The asphericity amplifies quite rapidly.

An equally interesting result is obtained in the case of an initially spherical cloud that is in uniform rotation. The initial ratio $\beta_0$ of the cloud's rotational kinetic energy to the absolute value of its gravitational potential energy is (see, for example, §4.3)

\[ \beta_0 = \frac{8}{3\pi^2} J^2. \] (3.42)

Any nonzero value of $\beta_0$ causes the cloud to deform into an oblate spheroidal shape, because centrifugal forces retard the acceleration of $R$ relative to $Z$ in

\[ \dot{R} \simeq -\frac{\pi^2}{8} \frac{1}{R^2} \left[ 1 + \frac{3}{10} e^2 + \frac{9}{56} e^4 \right] \] (3.40a)

\[ \dot{Z} \simeq -\frac{\pi^2}{8} \frac{1}{R^2} \left[ 1 + \frac{2}{5} e^2 + \frac{8}{35} e^4 \right] \] (3.40b)

so the ratio of the accelerations is,

\[ \frac{\dot{Z}}{\dot{R}} \simeq \frac{1}{10} \frac{e^2}{e^4} + \frac{53}{1400} \] (3.41)

of the centrifugal force to gravity in the collapsed disk. Notice in particular that $\chi_e$ is less than unity in every case.

The discussion in this section has been restricted to clouds that receive an oblate spheroidal deformation. A similar analysis of prolate spheroidal deformations (see LMS and Hutchins, 1976) shows that this deformation also amplifies during the cloud collapse. Clearly a pressure-free gas cloud is unlikely to maintain even an approximately spherical structure if it undergoes a self-gravitating, hydrodynamic collapse.

### 3.4 Fragmentation of a Pressure-Free Spheroid

In this section it is demonstrated that density enhancements within an otherwise initially uniform density sphere or spheroid will rapidly amplify relative to the background cloud medium if the cloud is pressure-free. Amplification is sufficiently fast that the ratio of density $\rho^b$ in the perturbed region to the background density $\rho$ becomes infinite ($\rho^b/\rho \to \infty$) in less than one collapse time. Once $\rho^b/\rho \to \infty$, the perturbed region is no longer an integral part of the background, or "parent" cloud medium and its subsequent

**TABLE II**

<table>
<thead>
<tr>
<th>$c/a_0$</th>
<th>$e_0$</th>
<th>$\tau_e$</th>
<th>$R_e$</th>
<th>$\dot{R}_e$</th>
<th>$\dot{Z}_e$</th>
</tr>
</thead>
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<tr>
<td>0.99</td>
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<td>—</td>
<td>0.015</td>
<td>-8.00</td>
<td>-8.83</td>
</tr>
<tr>
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<td>0.997</td>
<td>0.068</td>
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<td>-4.18</td>
</tr>
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<td>0.991</td>
<td>0.130</td>
<td>-2.49</td>
<td>-3.06</td>
</tr>
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<td>-2.27</td>
</tr>
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<td>0.70</td>
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<td>0.954</td>
<td>0.355</td>
<td>-1.19</td>
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<tr>
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<td>0.931</td>
<td>0.459</td>
<td>-0.90</td>
<td>-1.73</td>
</tr>
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<td>0.10</td>
<td>0.995</td>
<td>0.789</td>
<td>0.917</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

* Data taken from Lin, Mesel, and Shu (1965).

$^\dagger$ When available, data taken from Hutchins (1976); otherwise taken from LMS.
interaction with that medium cannot be described by the continuum hydrodynamic equations. In this sense the perturbed region is said to “fragment” from the parent cloud once $p^*/p$ becomes large. The parent cloud structures that are perturbed in this discussion are the uniform density sphere and spheroid of §§3.1 and 3.3. First the evolution of the sphere will be considered.

By Eqs. (3.15), (3.16), and (3.22), the parent cloud’s density $\rho$ at any time

$$t = \tau_{\text{ff}}(\rho_i) G(\xi)$$

(3.43)
during the collapse will be given by

$$\rho/\rho_i = \sec^6 \xi \cdot \frac{1}{E(\xi)}. \quad (3.44)$$

During the collapse, $\xi$ increases from an initial value zero to a final value $\xi_f$ at the end of one collapse time $\tau_c$. As $\xi$ approaches $\xi_f$ it proves useful to express (3.43) and (3.44) in terms of the parameter

$$\varepsilon = \xi_f - \xi$$

(3.45)

which becomes smaller with time. By a Taylor series expansion,

$$t/\tau_{\text{ff}}(\rho_i) \simeq G(\xi_f) - \varepsilon G'(\xi_f) + \frac{1}{2} \varepsilon^2 G''(\xi_f) - \frac{1}{6} \varepsilon^3 G'''(\xi_f), \quad (3.46)$$

$$\rho/\rho_i \simeq [\cos \xi_f + \varepsilon \sin \xi_f]^{-6} \left[ E(\xi_f) - \varepsilon E'(\xi_f) \right]^{-1}. \quad (3.47)$$

For a spherical collapse in particular, $G(\xi)=(\pi/2) (\xi + \frac{1}{2} \sin 2\xi)$, $E(\xi)=1$, $\tau_c = \tau_{\text{ff}}(\rho_i)$ and $\xi_f = \pi/2$. So $G'(\xi_f)=G''(\xi_f)=0$ and for $\varepsilon \ll 1$ (i.e., $t \ll \tau_c$) (3.46) and (3.47) reduce to the simple forms

$$t/\tau_{\text{ff}}(\rho_i) \simeq 1 - \varepsilon^2 G''(\xi_f) = 1 - \frac{4}{3\pi} \varepsilon^3, \quad (3.48)$$

$$\rho/\rho_i \simeq \left( \frac{3\pi}{8} \delta_0 \right)^{-1}. \quad (3.49)$$

Near the end of its collapse, then, the cloud’s density at any time can be easily calculated through the small parameter $\varepsilon$.

If a spherically symmetric region within the parent cloud initially has a density

$$\rho^* = \rho_i (1 + \delta_0)$$

(3.50)

that is enhanced over the parent cloud’s density by a small factor $\delta_0 \ll 1$, it will collapse on a time scale

$$\tau_{\text{c}}^* = \tau_{\text{ff}}(\rho^*)$$

(3.51)

$$= \tau_{\text{ff}}(\rho_i) (1 + \delta_0)^{-1/2} \approx \tau_{\text{ff}}(\rho_i) \left( 1 - \frac{1}{2} \delta_0 \right) \quad (3.52)$$

that is slightly shorter than that of its parent cloud. Therefore, this subregion of the cloud that is enhanced in density will complete its collapse, $\rho^*/\rho^* \to \infty$, and “fragment” from the parent cloud before the parent cloud has finished its own collapse. Fragmentation of the subregion will be complete at a time $t = \tau_{\text{c}}^*$ when, from Eqs. (3.48) and (3.52),

$$\varepsilon^3 \approx \frac{3\pi}{8} \delta_0 \quad (3.53)$$

and, from (3.49), the parent cloud’s density has risen to a value

$$\rho/\rho_i \simeq \left( \frac{3\pi}{8} \delta_0 \right)^{-1}. \quad (3.54)$$

Since density enhancements most almost certainly exist at some finite amplitude in all clouds, fragmentation seems inevitable during the collapse from rest of a spherical pressure-free cloud. This result, that $\rho/\rho_i \propto \delta_0^{-2}$, was derived first, in a more general fashion, by Hunter (1962) and is discussed by Mestel (1963).

Incidentally, Eqs. (3.53) and (3.54) can also be used to estimate the radius that an initially centrally condensed cloud will have when its central most region has reached an infinite density. For example, the centrally condensed models whose initial density profiles are given by Eq. (3.29) and whose evolutions are shown in Figure 3 have an initial ratio of central to mean density

$$\rho_c/\rho_i = 1 + \frac{3}{n}. \quad (3.55)$$
So, by Eq. (3.50), the central region can be considered to be a region having an initial density enhancement \( \delta_0 = 3/n \). At the instant the central region finishes its collapse (it “fragments” from the rest of the cloud), the mean density in the cloud should have increased by a factor

\[
\rho / \rho_0 \approx \left( \frac{9\pi}{8n} \right)^{2/3}
\]

given by Eq. (3.54), and the outer boundary of the cloud, which shrinks on a time scale that is governed by the mean density, should be

\[
R / R_0 = (\rho / \rho_0)^{-1/3} \approx \left( \frac{9\pi}{8n} \right)^{2/3}.
\]

For the model \( n = 25 \) shown in Figure 3, the fractional radius predicted is \( R / R_0 \approx 0.27 \), which agrees with the cloud boundary shown at \( t / \tau_{ff} = 0.99 \).

Silk (1982) has recently shown that fragmentation of a freely falling spheroidal cloud occurs even more rapidly than in the perfectly spherical collapse. His result shows that a spheroidal subregion with an initial density enhancement \( \delta_0 \), as given by (3.50), will fragment from its parent spheroidal cloud by the time the parent cloud's density has risen to a value \( \rho / \rho_0 \propto \delta_0^{-1} \) instead of \( \delta_0^{-2} \) as in (3.54). He derived this result in the following way: The collapse of a nonrotating uniform density spheroid can be described with high accuracy by Eqs. (3.43) and (3.44) if, according to Lin, Mestel and Shu (1965),

\[
G(\zeta) = \left[ \frac{32}{3\pi} \right]^{1/2} \frac{1}{a_0} \left[ \zeta + \sin 2\xi \right] - \frac{a_2}{a_0} \left( \frac{1}{2} \sin 2\xi \right),
\]

(3.55)

\[
E(\zeta) = 1 + E_2 \tan^2 \zeta + E_4 \tan^4 \zeta,
\]

(3.56)

where, in terms of the spheroid's initial eccentricity \( e_t \),

\[
E_0 = 3 - \frac{4\pi}{a},
\]

\[
E_4 / E_0 = \frac{1}{3} (\psi_t - \sin \psi_t)^{-2},
\]

\[
[\psi_t^2 + \psi_t \sin \psi_t - 4(1 - \cos \psi_t)]
\]

\[
a_0 = [2 \alpha t]^{1/2}
\]

\[
\left[ \frac{8\pi}{3} \right]^{1/2} ; 0
\]

\[
\left[ \frac{4\pi}{3} ; 0 \right]
\]

\[
\left[ 0 ; \pi \right]
\]

\[
(0 ; \pi)
\]

(3.57)

where \( K = \sec^6 \zeta \cdot \left( -E(\zeta) \right)^{-1} \) depends only on the spheroid's initial eccentricity and not on the time parameter \( \epsilon \). The cloud density has a quite different dependence on \( \epsilon \) than it had in Eq. (3.49) for the spherical collapse. This is because the spheroidal collapse is essentially one-dimensional. The relationship between \( t \) and \( \epsilon \) is drastically different from its spherical counterpart (3.48) as well. In a nonspherical collapse, \( G(\zeta) \) and \( G^*(\zeta) \) are not equal to zero, so, for \( \epsilon \ll 1 \), (3.46) reduces to

\[
t / \tau_{ff}(\rho_0) \approx G(\zeta) - \epsilon G^*(\zeta).
\]

(3.58)

A region of enhanced density \( \rho = \rho(1 + \delta_0) \) within the initial cloud structure that has a spheroidal geometry which is concentric with the geometry of the parent cloud will also collapse according to (3.58) if \( \tau_{ff}(\rho_0) \) is replaced by \( \tau_{ff}(\rho) \). Both the parent cloud and the subregion will have identical \( \zeta \) values because \( \zeta \) depends only on the spheroid's initial eccentricity. Following the same train of arguments used in the spherical collapse, then, the subregion of enhanced density will complete its collapse, \( \rho / \rho_0 \to \infty \) and “fragment” from the parent cloud at a time

\[
t = \tau_{ff}(\rho_0) G(\zeta) = (1 + \delta_0)^{-1/2} \tau_{ff}(\rho_0) G(\zeta)
\]

(3.59)

that is shorter than the collapse time of the parent cloud. Equating the time \( t \) in (3.59) and (3.58) implies for the parent cloud

\[
\epsilon \approx \frac{1}{2} \frac{G}{G^*} \delta_0.
\]

(3.60)
Hence, from (3.57), the parent cloud’s density at the time of fragmentation will be

$$\rho/\rho_t \approx \frac{2K\Gamma}{G} \delta_0^{-1}.$$  \hspace{1cm} (3.61)

This dependence of $\rho$ on $\delta_0$ derived by Silk predicts that fragmentation will occur relatively more quickly in a spheroidal cloud than in the perfectly spherical collapse. Since, as we have learned, pressure-free clouds are unlikely to remain spherical, it seems that (3.61) is generally more applicable than (3.54) to studies of fragmentation. As Silk has emphasized, in a pressure-free cloud it seems difficult to avoid fragmentation during dynamical collapse.

3.5 Summary and Discussion

The models of pressure-free collapses in §§3.1–3.4 have been purposely presented relatively free of editorial comments because in this pristine format the models beautifully address single aspects of the generally quite complicated problem of hydrodynamic collapse. They lay a solid foundation for a discussion of the more general problem.

The most important concepts to be learned from these analytic/semi-analytic models are:

1) The time $t_c$ that it takes for a pressure-free cloud, starting from rest, to collapse to a configuration in which its mass density $\rho/\rho_t \to \infty$ is exactly as was suggested in the introduction to this article,

$$t_c \sim t_a \sim (G\rho_t)^{-1/2}.$$  \hspace{1cm} (3.62)

For a uniform density sphere, $t_c$ is exactly $t_a (\rho_t)$, as given by Eq. (3.16), but for nonspherical evolutions $t_{cff}(\rho_t)$ is only an approximate collapse time. For a centrally condensed sphere of initial central density $\rho_0$, the collapse time of the cloud center is $t_c = t_{cff}(\rho_0)$, but for the edge of the cloud, $t_c = t_{cfe}(\rho_0)$, where $\rho_t$ is the initial average density of gas interior to the cloud edge. These two times can differ considerably if $\rho_0 \gg \rho_t$.

2) As depicted vividly in Figure 2, a cloud which starts from rest spends most of its collapse time $t_c$ sitting near its initial state just “feeling” the gravitational acceleration. Velocities and gas densities increase very slowly at first. Not until the last 10% or so of its collapse time does the cloud’s density and overall structure (see also Figure 3) undergo substantial change. Relating this to observations, if an ensemble of clouds that are actually in the state of gravitational collapse is observed, it seems clear that a large majority of the clouds will exhibit a structure that is not far removed from their initial structure at the onset of their collapse.

3) The evolution of a pressure-free cloud is strongly dependent on initial conditions in the cloud at the onset of its collapse. Clearly, from §3.2, a cloud’s density profile at any stage of its evolution reflects its initial profile. The size of the disk to which a spheroidal cloud will evolve in one collapse time, as shown in §3.3, depends critically on the degree to which the cloud was initially flattened.

4) Perhaps most importantly of all, a cloud that is indeed “pressure-free”, as defined by Eq. (3.6), at the onset of collapse is highly unlikely to maintain even approximate sphericity or density homogeneity during its collapse. In this regard, the uniform density sphere discussed in §3.1 is clearly a singular solution of limited interest when studying the general cloud collapse and star formation problem.

One precautionary note should accompany any discussion that attempts to relate these pressure-free models to real cloud collapse problems. In each of the models, gas velocities continuously increase in magnitude as $t \to t_c$. This state is unsatisfactory if, as we are assuming here, the desired endpoint of collapse is an object (or objects) in approximate hydrostatic balance ($v=0$). Pressure gradients, serving to decelerate the gas flow, must develop near the end of the collapse in order to bring the cloud toward hydrostatic balance. Therefore, in a gaseous medium for which the hydrodynamic equations are appropriate, the solutions of §§3.1–3.4 are valid for, at most, one collapse time. (The pressure-free collapse models can, in principle, be used to describe the evolution of an “n-particle” system beyond one collapse time. See, for example, Miller and Smith’s [1979] application of the uniformly rotating spheroid solution to a stellar dynamic problem.)

It is appropriate to note here that deceleration of the gas flow will generally occur through a shock front because the velocities generated during a pressure-free collapse are generally highly supersonic with respect to the initial sound speed of the gas,

$$v \sim \frac{R_0}{t_{cff}} \sim R_0(G\rho_t)^{1/2} \gg c_s.$$  \hspace{1cm} (3.62)

4. ANALYTIC AND SEMI-ANALYTIC MODELS OF COLLAPSE WITH PRESSURE

4.1 Isothermal Spheres

Several authors (Larson, 1969; Penston, 1969; Shu, 1977; and Hunter, 1977) have shown that when isothermal pressure gradients are important during a gas cloud’s collapse, the equations governing the collapse admit a set of similarity solutions. Certain properties of these solutions can be described
analytically and are instructive models for comparison with more detailed, numerical collapse calculations. Using an isothermal equation of state

$$P = c_s^2 \rho,$$

(4.1)

the equation of motion for a spherically symmetric gas flow (3.1) becomes

$$\frac{dv}{dt} = -\frac{GM(r)}{r^2} - \frac{c_s^2}{\rho} \frac{\partial \rho}{\partial r}. \tag{4.2}$$

In Eulerian form, (4.2) is written

$$\frac{dv}{dt} + v \frac{dv}{dr} = -\frac{GM(r)}{r^2} - \frac{c_s^2}{\rho} \frac{\partial \rho}{\partial r}. \tag{4.3}$$

Here it is useful to write the usual continuity equation (5.2) in terms of the mass $M(t)$,

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + v \frac{\partial M}{\partial r} = 0 \tag{4.4}$$

where

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho. \tag{4.5}$$

Equations (4.3)–(4.5) completely specify the behavior of the three dependent variables $M$, $\rho$, and $v$ as a function of the independent variables $r$ and $t$. A similarity solution becomes possible for these equations when the single independent variable

$$\zeta = \frac{c_s t}{r} \tag{4.6}$$

is used to replace both $r$ and $t$. Then if $M$, $\rho$, and $v$ assume the following forms,

$$M(r, t) = \frac{c_s^2 t}{G} m(\zeta) \tag{4.7a}$$

$$\rho(r, t) = \frac{c_s^2}{4\pi G r^2} P(\zeta) \tag{4.7b}$$

$$v(r, t) = -c_s U(\zeta), \tag{4.7c}$$

the three coupled partial differential equations (4.3)–(4.5) reduce to two coupled ordinary differential equations for the functions $P(\zeta)$ and $U(\zeta)$,

$$\frac{dU}{d\zeta} = \frac{U(1 - P(1 + U) - 2)}{[(U + 1)^2 - \zeta^2]} \tag{4.8}$$

$$\frac{dP}{d\zeta} = \frac{\zeta P[2 - P(1 + U)]}{[(U + 1)^2 - \zeta^2]} \tag{4.9}$$

and a single equation defining $m(\zeta)$,

$$m(\zeta) = P(U + 1/\zeta). \tag{4.10}$$

The parameters $\zeta$, $m$, $P$, and $U$, and Eqs. (4.8)–(4.10) are exactly those used by Hunter (1977) in his analysis of this problem, but they differ in form from the relations used by Larson, Penston, and Shu, primarily because these authors chose to use a similarity variable $x = \pm (1/\zeta)$ instead of $\zeta$. Hunter’s analysis is the most complete and his relations will be used here, but a transformation between his presentation and those of the other authors can be easily obtained (see Hunter’s Table 1).

In Figure 4, $U(\zeta)$ is plotted for three different solutions to Eqs. (4.8) and (4.9). These solutions, taken from Hunter (1977), vary smoothly throughout the range $-\infty < \zeta < +\infty$ and obey the initial condition that the fluid velocity $v \to 0$ at $t \to -\infty$. Curve LP is a solution similar to that derived by Larson (1969) and Penston (1969); curve EW is Shu’s (1977) expansion wave solution; and curve H is Hunter’s curve b, as defined by the parameters in his Eq. (11b). Figure 5 shows the corresponding functions $P(\zeta)$, taken from Hunter.

It can be shown by analytic manipulation of Eqs. (4.8) and (4.9) that these smooth solutions have the following behaviors in various limits:

As $\zeta \to -\infty$,

$$U \approx -\frac{2}{3} \left(\frac{1}{\zeta}\right)^3 + \frac{1}{45} \left[\frac{2}{3} - \exp(Q_0)\right] \left(\frac{1}{\zeta}\right)^3, \tag{4.11a}$$

$$\ln(\zeta^2 P) \approx Q_0 + \frac{1}{6} \left[\frac{2}{3} - \exp(Q_0)\right] \left(\frac{1}{\zeta}\right)^2, \tag{4.11b}$$

where $Q_0$ is a positive constant.

For $\zeta \approx 0$,

$$U \approx U_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right) U_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right)^2 \left[1 - P(0)/6\right] U_0^2, \tag{4.12a}$$

$$P \approx P_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right) P_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right)^2 \left[1 - P(0)/6\right] P_0^2, \tag{4.12b}$$

$$m(\zeta) \approx m(0) + \frac{1}{3} \left(\frac{1}{\zeta^2}\right) m(0) + \frac{1}{3} \left(\frac{1}{\zeta^2}\right)^2 \left[1 - P(0)/6\right] m(0)^2, \tag{4.12c}$$

$$U \approx U_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right) U_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right)^2 \left[1 - P(0)/6\right] U_0^2, \tag{4.12d}$$

$$P \approx P_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right) P_0 + \frac{1}{3} \left(\frac{1}{\zeta^2}\right)^2 \left[1 - P(0)/6\right] P_0^2, \tag{4.12e}$$

$$m(\zeta) \approx m(0) + \frac{1}{3} \left(\frac{1}{\zeta^2}\right) m(0) + \frac{1}{3} \left(\frac{1}{\zeta^2}\right)^2 \left[1 - P(0)/6\right] m(0)^2, \tag{4.12f}$$

where

$$P(0) = c_s^2. \tag{4.12g}$$
\[ P \approx P_0 - \xi^2 \left[ \frac{1}{2} P_0 (P_0 - 2) \right] \]  
\[ (4.12b) \]

where \( U_0 \) and \( P_0 \) are positive constants.

As \( \xi \to +\infty \),

\[ U \approx (2 m_0 \xi)^{1/2}, \]  
\[ (4.13a) \]

\[ P \approx (m_0 / 2 \xi)^{1/3}, \]  
\[ (4.13b) \]

where \( m_0 \) is a positive constant. The values of the constants \( U_0, P_0, \) and \( m_0 \) depend on the chosen value of \( Q_0 \). For the three curves displayed in Figures 4 and 5, Hunter has numerically determined values of these constants and their values are given here in Table IV.

\[
| & Q_0 & U_0 & P_0 & m_0 \\
\hline
LP & 0.5139 & 3.278 & 8.854 & 46.915 \\
H & 11.236 & 0.295 & 2.378 & 2.577 \\
EW & +\infty & 0.0 & 2.000 & 0.975 \\
\hline
\]

In terms of the physical quantities \( v(r, t) \) and \( \rho(r, t) \), the asymptotic behaviors (4.11)-(4.13) translate into:

For \( r < 0 \) and \( r \ll c_s |t| \),

\[ v(r, t) \approx \frac{2}{3} \frac{r}{(-t)} \]  
\[ (4.14a) \]
\[ \rho(r, t) \approx \frac{\exp(Q_0)}{4\pi G} t^{-2}. \quad (4.14b) \]

For \( r \gg c_d |t| \) at any time,

\[ v(r, t) \approx -c_s U_0. \quad (4.15a) \]

\[ \rho(r, t) \approx \frac{c_s^3 P_0}{4\pi G} r^{-2}. \quad (4.15b) \]

For \( t > 0 \) and \( r \leq c_d |t| \)

\[ v(r, t) \approx -\left( \frac{2m_0}{c_s} \right)^{1/2} \sqrt{t} r^{-1/2}. \quad (4.16a) \]

\[ \rho(r, t) \approx \frac{1}{4\pi G} \left( \frac{m_0 c_s^3}{2} \right)^{1/2} t^{-1/2} r^{-3/2}. \quad (4.16b) \]

The three solutions shown in Figures 4 and 5 are purely mathematical in nature and are subject to interpretation if their behavior is to be connected to any physically realistic situation. It is in the manner of interpretation that Shu differs from Larson and Penston. Shu has discussed only the portion of the similarity solutions that cover the range \( 0 < \xi < +\infty \) claiming that, for example, the EW solution approximates quite accurately the accretion flow of an isothermal envelope on to a newly formed stellar core. Larson and Penston have discussed only the portion of the similarity solutions that cover the range \( -\infty < \xi < 0 \), claiming that the LP solution accurately describes the transformation of a uniform density sphere into a strongly centrally condensed supersonically collapsing cloud. The LP curve and the EW solution, and their respective physical interpretations, apparently mutually exclude one another because they do not join smoothly together at \( \xi = 0 \).

Hunter (1977), however, has realized that a single smooth solution, for example curve H in Figures 4 and 5 or the extension of the LP curve into the region \( \xi > 0 \), can be used over the entire range \( -\infty < \xi < +\infty \) to describe an isothermal cloud's complete evolution. He has interpreted each solution as the time evolution of an isothermal cloud where \( \xi \to -\infty \) \((t \to \infty)\) corresponds to the initial conditions (uniform density, zero velocity) in the cloud, \( \xi = 0 \) \((t = 0)\) marks the formation of a central zero velocity (zero radius) core whose mass at times \( t > 0 \) is \( M_c = c_s^3 m_0 |t| G \), and \( \xi \to +\infty \) \((t \to +\infty)\) represents very late stages of the collapse when most of the cloud mass is contained in the core. Using this scenario, the asymptotic limits of curves \( P(\xi) \) and \( U(\xi) \) shed some light on what the radial structure of the cloud should look like at any time \( t \). For any stage of the collapse prior to core formation, \( t < 0 \), there will be a region of the cloud \( r \leq c_d |t| \) that is collapsing homologously, maintaining uniform density and exhibiting a velocity profile \(-v \propto r\). There will also be a region at large radii \( r \gg c_d |t| \) exhibiting constant velocity and a density profile \( \rho \propto r^{-2} \). At exactly the same time \( t = 0 \), the entire cloud will exhibit an \( r^{-2} \) density profile. Then, after core formation, at any time \( t > 0 \), there will continue to be a region at large radii \( r \gg c_d |t| \) that exhibits \( \rho \propto r^{-2} \), \( v \propto \) constant, while near the core, \( r \leq c_d |t| \), the cloud's structure will change to \(-v \propto r^{-1/2}, \rho \propto r^{-3/2}\).

There is no reason to expect that a collapsing gas cloud will follow the time evolutionary behavior described by this simple similarity model since boundary conditions at the edge of the cloud and the detailed processes involved in the formation of a zero velocity core have been omitted in the model.

A description of the detailed collapse of protostellar clouds calculated using numerical hydrodynamic computer codes is given in §6. In summary, though, the earliest isothermal stage of collapse has been observed to transform an approximately uniform density cloud into a centrally condensed structure. The envelope of this structure, whose extent is governed by the distance over which a sound wave has had time to propagate from the edge of the cloud, has \( \rho \propto r^{-2} \) and \( v \propto \) constant; the central volume, untouched by the inward propagating sound wave, collapses homologously. When the cloud has collapsed to a sufficiently high density, its central volume departs from isothermality and a "core" in hydrostatic balance soon forms. The envelope accretes onto the core as the evolution proceeds. The outer regions of the envelope continue to possess the structure \( \rho \propto r^{-2}, v \propto \) constant, but just outside the core the structure of the flow is often observed to change into a "Bondi accretion" (Bondi, 1952) free-fall flow having \( \rho \propto r^{-3/2}, v \propto r^{-1/2} \). As Hunter (1977) has pointed out, the full similarity models depicted as LP and H in Figures 4 and 5 give an amazingly accurate qualitative description of this more detailed hydrodynamic collapse.

Quantitatively, the three solutions plotted in Figures 4 and 5 are quite different. For example, the inflow velocity at radii \( r \gg c_d |t| \) are, from (4.15a) and Table IV, \(-3.3 c_s, -0.3 c_s, \) and 0.0 for the models LP, H, and EW, respectively. Hunter discusses these quantitative differences in some detail and decides, despite comments to the contrary made by Shu (1977), that the solution of Larson and Penston is the one most appropriate to real cloud collapse evolutions.

Although Hunter has gone to some trouble to obtain solutions to Eqs. (4.8) and (4.9) that are smooth across \( \xi = 0 \), in reality the region of the solutions immediately surrounding \( \xi = 0 \) \((-e \leq \xi \leq +e, e \ll 1)\) is the region that is least likely to match the evolution and structure of a real cloud. At any time \( t \neq 0 \), this region of the solution will represent the outer fringes of the cloud that are almost certainly going to be influenced by boundary conditions.
not accounted for by the similarity equations. Likewise, at exactly the evolutionary time $t=0$ ($\zeta=0$) the similarity models LP and H predict that the cloud will instantaneously convert from a structure of infinite central density and finite inflow velocities to a structure that exhibits accretion onto a central, zero velocity core. The formation of this core in real evolutions will involve a number of complicated processes that are not accounted for in the similarity models.

Because the relatively simple similarity solutions appear to fairly describe the collapse of isothermal gas clouds, it should be instructive to analyze additional properties of the solutions. Larson and Penston have shown that the important ratio of the pressure gradient to gravity during the collapse can be expressed analytically as

$$\frac{|\text{pressure}|}{|\text{gravity}|} = \left| \frac{r^2 \zeta^2 \partial P}{GM \rho \partial r} \right| = \left| \frac{d \ln(\zeta^2 P)/d\zeta}{P(U + 1/\zeta)} \right|$$

(4.17)

This ratio is of order unity for all of the similarity solutions up to the point of core formation. Using the relations (4.11a) and (4.11b), at times $t<0$,

$$\frac{|\text{pressure}|}{|\text{gravity}|} \approx 1 - \frac{2}{3} \exp(-Q_0).$$

(4.18)

Using (4.12a) and (4.12b), at the instant of core formation,

$$\frac{|\text{pressure}|}{|\text{gravity}|} = \frac{2}{P_0}.$$

(4.19)

After core formation, relations (4.13a) and (4.13b) tell us that near the core ($\zeta \to +\infty$) the ratio

$$\frac{|\text{pressure}|}{|\text{gravity}|} \approx \frac{3}{2} \frac{1}{(m_0 \xi)}$$

(4.20)

does become quite small, in accord with the free fall nature of the accretion.

Larson (1969) has pointed out that a similarity solution to Eqs. (4.4), (4.5), and (3.1) should be possible for gas flows obeying the general adiabatic pressure relation

$$P = k_2 \rho^{\gamma}$$

(4.21)

for values of $\gamma \neq 1$. Larson has listed, without showing his derivation, the asymptotic behavior of $P$ and $U$ near $\zeta=0$ for such solutions. Cheng (1978), in an account similar to Shu's (1977), has shown that as $\zeta \to +\infty$ the similarity flow has a free fall structure $-\rho \propto r^{-1/2}$ and $\rho \propto r^{-3/2}$ independent of the adiabatic index $\gamma$. Complete similarity models covering the entire range $-\infty \leq \zeta \leq +\infty$, analogous to Hunter's (1977) isothermal models, are not currently available in the literature for arbitrary $\gamma$.

### 4.2 Fragmentation of Isothermal or Adiabatic Spheres

As was discussed in §3.4, Hunter (1962) was the first to show by an analytic perturbation analysis that a pressure-free sphere has a strong tendency to fragment during its dynamical collapse. In a perfectly pressure-free cloud, there is no preferred wavelength (size) for a perturbation; perturbations of all sizes grow at the same rate, related to their initial density enhancement $\delta_0$. In any cloud that has a nonzero gas temperature, though, the growth rate derived by Hunter will not be universally applicable. For a given gas temperature and density, there will always be perturbations of sufficiently short wavelengths $\lambda < R$ for which the pressure-free assumption, Eq. (3.6), is invalidated. These short wavelength perturbations will not grow relative to the background medium because pressure gradients will be strong enough to oppose their self-gravity. The relative stability of a given wavelength perturbation may, however, change during a collapse as pressure becomes more or less important relative to gravity.

The relative importance of pressure forces to gravity can be parameterized by the ratio $\alpha$ of thermal energy to the absolute value of gravitational potential energy. From Eq. (4.29) of §4.3, for a uniform density sphere that contracts along an adiabat given by (4.21),

$$a = \frac{5}{2} \frac{k_3}{GM} R \rho^{-2/3},$$

(4.22)

where $R$ is the cloud radius. Conserving the cloud mass during collapse, and realizing that for a spherical collapse $R \propto \rho^{-1/3}$, $\alpha$ will change according to the relation

$$\alpha / \alpha_i = (\rho / \rho_i)^{\gamma - 4/3},$$

(4.23)

where the subscript $i$ denotes initial values. Therefore, for $\gamma < 4/3$, the relative influence of pressure will decrease during a collapse, while for $\gamma > 4/3$, pressure will become relatively more important. The parameter $\alpha$ is, in fact, a direct indicator of how the size $\lambda$ (assumed to scale as $\rho^{-1/3}$, like any spherical radius) of a perturbation compares to the Jeans length at any stage during the collapse. Combining Eqs. (2.4e), (4.21) and (4.22),

$$\lambda / R_J = a^{-1/2}.$$

(4.24)
Therefore the ratio \( \lambda/R_J \) for a given size perturbation will change during the collapse according to the relation
\[
\frac{\lambda/R_J}{(\lambda/R_J)_i} = (\rho/\rho_0)^{(4/3 - \gamma)/2}.
\] (4.25)

For \( \gamma < 4/3 \), a given perturbation will become “more unstable” as the background density rises. For \( \gamma > 4/3 \), a given perturbation will become “more stable” during the collapse.

Hunter (1962) has used a first order perturbation technique to show how the fragmentation scenario for a pressure-free sphere (§3.4) is altered in the presence of adiabatic pressure gradients. His results can be summarized as follows. For \( \gamma < 4/3 \), perturbations having initial wavelengths \( \lambda_i \leq R_J \) will be able to amplify relative to the background medium during the collapse. Their growth rate asymptotically approaches that derived for the pressure-free sphere because of the lessening effects of pressure. As Hunter (1962, p. 607) has emphasized, though, even in an isothermal collapse (\( \gamma = 1 \)) pressure forces “appreciably delay the growth of disturbances with initial length scales of the order of \( R_J \) or less” because they exert a stabilizing influence early in the collapse.

For \( 4/3 < \gamma \leq 5/3 \), perturbations having initial wavelengths \( \lambda_i \leq R_J \) cannot amplify at any time during the collapse because pressure will always dominate over gravity on these scales. Analyzing the behavior of perturbations having \( \lambda_i \gg R_J \), Hunter (1962) has shown that in the adiabatic regime, their growth rate relative to the background medium is significantly altered from the rate described by the pressure-free analysis. The amplitude of these perturbations does not grow monotonically with time relative to the background medium. Instead, perturbations oscillate in amplitude. As long as \( \lambda \) remains \( \gg R_J \) during the collapse, the amplitude of the oscillations grows with time, though at a somewhat slower rate than in the pressure-free analysis. For the limiting case \( \gamma = 5/3 \), Hunter has shown that the amplitude of the oscillations does not grow at all during the collapse. As a given perturbation wavelength \( \lambda \) becomes comparable to \( R_J \), of course, growth of that perturbation will cease if \( \gamma > 4/3 \).

4.3 Endpoint of Rotating Collapse with Pressure

As was pointed out in §3.5, pressure-free analyses can at best describe only the first free fall phase of a hydrodynamic collapse. The cloud collapse cannot be stopped and a hydrostatic balance be achieved unless some forces are included in the equation of motion to oppose gravity. Once isotropic gas pressure forces are included, and a realistic temperature-density relation is combined with the appropriate equation of state for the gas, a rough prediction can be made regarding the final size to which the gaseous sphere will collapse before it can approach hydrostatic balance. The prediction can be made by way of the virial theorem (cf. Chandrasekhar, 1967, pp. 49–51) which states that for hydrostatic balance to be attainable, the cloud’s total thermal energy \( S \) must be 1/2 of the absolute value of its gravitational potential energy \( W \). That is, balance can be achieved when
\[
\alpha = S/|W| = \frac{1}{2}
\] (4.26)

For a uniform density sphere of radius \( R_0 \),
\[
S = \frac{3}{2} \frac{P}{\rho} M,
\] (4.27)

and
\[
W = -\frac{3}{5} \frac{GM^2}{R_0},
\] (4.28)

so
\[
\alpha = \frac{5}{2} \frac{R_0}{GM \rho} \frac{P}{\rho}.
\] (4.29)

Assuming that the pressure is related to the density via the adiabatic relation (4.21),
\[
\alpha = \left[ \frac{5}{6} \frac{3}{4\pi} \frac{1}{G} \frac{k_B}{M} \right] M^{-2/3} \rho^{2/3} \rho^{\gamma-4/3}.
\] (4.30)

If the cloud’s initial value \( \alpha_0 < 1/2 \), the cloud will contract under gravity. As the cloud’s density increases from an initial value \( \rho_0 \), \( \alpha \) will change according to the relation
\[
\alpha/\alpha_0 = (\rho/\rho_0)^{2/3-\gamma/3}.
\] (4.31)

Obviously \( \alpha \) can only increase toward the desired value 1/2 required for equilibrium if \( \gamma > 4/3 \). Therefore, when considering the evolution for perfectly spherical clouds, studies of isothermal collapses (\( \gamma = 1 \); see §3.6) will not bring us any closer to understanding the end point of hydrodynamic collapse than do studies of pressure-free collapses. For \( \gamma > 4/3 \), the density of the cloud \( \rho \) at the point where it can achieve virial balance is given directly by (4.31) by setting \( \alpha = 1/2 \),
\[
\rho/\rho_0 = (2\alpha_0)^{1/(4/3-\gamma)}.
\] (4.32)
Hence the cloud’s final radius $R$ must be

$$R/R_0 = (2a_0)^{1/(3\gamma - 4)}.$$  \hspace{1cm} (4.33)

Of course, virial balance is only one property that the cloud must attain at the endpoint of its collapse. It must generally develop density and pressure gradients throughout its structure in order to achieve detailed hydrostatic balance. Therefore, Eqs. (4.33) and (4.32) can only be used to give approximate values for the cloud’s radius and average density at the end of its phase of hydrodynamic collapse.

When a cloud has some rotational energy to contend with during its evolution, the predicted endpoint of collapse is generally quite different from the spherical predictions. The cloud’s structure will be spheroidal, and virial balance can be achieved for any adiabatic exponent $\gamma > 0$. The altered predictions arise because the virial equation must also include a term that accounts for the cloud’s total rotational energy $\mathcal{T}$. Using the ratio

$$\beta = \mathcal{T} / |W|,$$  \hspace{1cm} (4.34)

virial balance is achieved when

$$\frac{a}{a_0} + \frac{\beta}{\beta_0} = \frac{1}{2}.$$  \hspace{1cm} (4.35)

Tohline (1981) has generalized the above arguments, which allow the endpoint of collapse to be predicted, to the case of an homogeneous spheroid in uniform rotation. For a given oblate spheroidal eccentricity $e$ [defined by Eq. (3.35)], angular velocity $\omega$, and adiabatic exponent $\gamma$,

$$W = \frac{3}{5} \frac{GM^2}{a} \frac{\sin^{-1} e}{e},$$  \hspace{1cm} (4.36)

$$S = \frac{3}{2} k_\alpha \rho^{\gamma - 1} M,$$  \hspace{1cm} (4.37)

$$\mathcal{T} = \frac{1}{5} M a^2 \omega^2,$$  \hspace{1cm} (4.38)

$$M = \frac{4}{3} \pi a^3 [1 - e^2]^{3/2} \rho,$$  \hspace{1cm} (4.39)

where $a$ is the semimajor axis of the spheroid. As before, adiabatic collapse implies that the ratio $a$ will vary from its initial value $a_0$ as

$$\frac{a}{a_0} = \left( \frac{a}{a_0} \right)^{4-3\gamma} \left[ \frac{\sin^{-1} e_0}{e_0} \right]^{5-3\gamma} \left( \frac{1 - e_0^2}{1 - e^2} \right)^{(\gamma - 1)/2}.$$  \hspace{1cm} (4.40)

where $e_0$ is the cloud’s initial eccentricity. Assuming conservation of angular momentum [$a/\omega = (a/\omega_0)^{3-2\gamma}$] the ratio $\beta$ will change from its initial value $\beta_0$ as

$$\frac{\beta}{\beta_0} = \left( \frac{a}{a_0} \right)^{-1} \left[ \frac{e}{\sin^{-1} e} \right] \left( \frac{1 - e^2}{1 - e_0^2} \right)^{1/2}.$$  \hspace{1cm} (4.41)

These last two equations, plus relation (4.35) that defines virial equilibrium, provide three equations for the four unknown parameters $a$, $\beta$, $\omega$, and $a$ that define the endpoint structure of collapse for a rotating, adiabatic spheroid. The fourth relation that closes out this system of equations was chosen by Tohline to be the relation that specifies what the eccentricity of a Maclaurin spheroid is for a given value of $\beta$:

$$\beta = \frac{3}{2} \frac{1}{e^2} \left( \frac{1 - e^2}{3} \right) = \frac{e}{\sin^{-1} e} \left( \frac{1 - e^2}{1 - e_0^2} \right)^{1/2}.$$  \hspace{1cm} (4.42)

It is reasonable to expect that this relation must hold, at least approximately, at the endpoint of collapse for a uniform density spheroid because the relative contribution of pressure gradients to force balance in the $R$- and $Z$-directions in a spheroid is the same as that which is supplied by forces due to the incompressible nature of the fluid in a Maclaurin spheroid. Equations (4.35), (4.40), (4.41), and (4.42) can be combined to give the following algebraic formula predicting uniquely the final eccentricity to which a spheroid will collapse for a given set of initial parameters $a_0$, $\beta_0$, and $e_0$ if the collapse occurs along a specified adiabat:

$$F(e, \gamma) = 3a_0[3\beta_0]^{4-3\gamma} \left[ \frac{\sin^{-1} e_0}{e_0} \right]^{5-3\gamma} \frac{1 - e_0^2}{1 - e^2} $$  \hspace{1cm} (4.43)

where

$$F(e, \gamma) \equiv \left( 1 - e^2 \right)^{(\gamma - 1)/2} \left[ Q(e) \right]^{4-3\gamma} \left( \frac{3}{2} \frac{\sin^{-1} e}{e} - Q(e) \right)$$  \hspace{1cm} (4.44)

and

$$Q(e) = \frac{9}{2} \frac{1}{e^2} \left( \frac{\sin^{-1} e}{e} \right)^2 \left( \frac{1 - e^2}{1 - e_0^2} \right)^{1/2} - (1 - e^2)^{1/2}.$$  \hspace{1cm} (4.45)

For a cloud that is initially spherical, $e_0 = 0$ and (4.43) becomes simply

$$F(e, \gamma) = 3a_0(3\beta_0)^{4-3\gamma},$$  \hspace{1cm} (4.46)

and $\beta_0 = e_0^2/(4\pi G \rho_0)$. Once $e$ has been determined for the final configuration, the cloud’s final $\beta$ can be determined from Eq. (4.42), its final equatorial
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radius can be determined from (4.41), and its final value of $\alpha$ can be determined from (4.35). The cloud's final density will be

$$\rho/\rho_0 = \left(\frac{\alpha}{\alpha_0}\right)^{-3} \left[1 - \epsilon_0^2\right]^{1/2}$$

(4.47)

and its final pressure will be

$$P/P_0 = (\rho/\rho_0)^{\gamma},$$

(4.48)

from which its final temperature can also be determined.

Figures 6-9, taken from Tohline (1981), show for four different adiabatic indices $\gamma=5/3$, 7/5, 4/3, and 1 (Tohline used $\Gamma$ instead of $\gamma$) what the analytically determined results for the endpoint of collapse are. Equation (4.46), assuming a spherical initial cloud, was used to obtain the results shown in these figures. In each frame of these figures, equilibrium values are plotted on the horizontal axis versus $\beta_0$ on the vertical axis. Each solid line in the figures traces the locus of equilibrium values for a single value of $\alpha_0$. The symbols with error bars in frames 6a and 7a are taken from the results of numerically evolved hydrodynamic models calculated by Boss (1980d) and will not be discussed here.

Studying the behavior of the curves in Figures 6-9 in limiting cases is particularly useful in understanding the role that pressure can play in determining the endpoint of hydrodynamic collapses.

**Evolutions with $\gamma=4/3$** When $\gamma=4/3$, the cloud's temperature, and hence $S$, increases in such a way as to very nearly (and in the case of a nonrotating sphere, *exactly*) counter any increase in gravitational potential energy, thereby keeping the cloud's value of $\alpha$ approximately constant. Since $\alpha$ does not change appreciably from its initial value $\alpha_0$, then from Eq. (4.35) it is clear that the cloud's equilibrium value of $\beta$ will be simply $\beta \approx 1/2 - \alpha_0$. Figure 8a graphically illustrates the fact that $\beta$ is essentially independent of $\beta_0$ when $\gamma=4/3$.

**FIGURE 6** Properties of the (virial) equilibrium configurations to which a rotating spheroid will evolve if it follows a $\gamma=5/3$ adiabat. Each solid line defines the locus of points for a fixed value of $\alpha_0$. (a) Equilibrium $\beta$ (bottom scale) and equilibrium $\alpha$ (top scale) are plotted as a function of $\beta_0$; vertical dashed and dotted lines locate points of dynamic instability in equilibrium models (not discussed here); symbols with error bars locate the results of individual hydrodynamic models evolved by Boss (1980d, not discussed here). (b) Ratio of the equilibrium density to the initial density plotted as a function of $\beta_0$. (c) Ratio of the equilibrium equatorial radius to the initial equatorial radius plotted as a function of $\beta_0$. Equilibrium values are calculated analytically using Eqs. (4.44)–(4.46) and the figure is taken directly from Tohline (1981).
Evolutions with $\gamma > 4/3$ Since, for $\gamma > 4/3$ collapses, $\alpha$ increases during the cloud’s evolution, pressure can play a major role in defining the cloud configuration at the endpoint of collapse. Figures 6 and 7 illustrate that the solution to (4.46) does reduce to the behavior predicted by relations (4.31)–(4.33) when rotation plays only a minor role in a $\gamma > 4/3$ collapse. For sufficiently low values of $\beta_0$, that is, the endpoint values of $\alpha$, $\rho$, and $a$ are independent of $\beta_0$.

Evolutions with $\gamma < 4/3$ As was emphasized above while discussing the endpoint of spherical, nonrotating cloud collapses, pressure forces alone cannot stop a collapse if $\gamma < 4/3$. Rotation must ultimately play a role in defining the equilibrium configuration to which all of these objects will evolve. In the limit where $\alpha$ remains small throughout an evolution, the endpoint of collapse will be a very thin disk in which $\beta \approx 1/2$ ($\alpha \approx 0$ and $e \approx 1$). In this limit, Eq. (4.41) alone predicts the final disk radius:

$$\frac{a}{a_0} = \frac{4}{\pi} \beta_0.$$  \hspace{1cm} (4.49)

Equation (4.46) reduces to a form only slightly more complicated than this when the final $e = 1$:

$$\frac{a}{a_0} \approx \frac{4}{\pi} \beta_0 \left[ 1 - \frac{8}{3} \left( \frac{a_0}{\rho_0} \right)^{1/2} \left( \frac{4\beta_0}{\pi} \right)^{(4/3-1)/2} \right].$$  \hspace{1cm} (4.50)

As is shown in Figure 9c, for $\gamma = 1$ the spheroid’s final radius is proportional to $\beta_0$ and is practically independent of $a_0$. Note, however, that the final density is sensitive to $a_0$ since, as $a_0 \to 0$, the spheroid evolves to a disk of zero thickness where conservation of mass requires that $\rho/\rho_0 \to \infty$.

4.4 Summary and Discussion

The similarity solution for the collapse of an isothermal cloud presented in §4.1 supports what has been found to be true from many numerical models of hydrodynamic collapse, namely, isothermal pressure gradients can play a major role in erasing a cloud’s “memory” of its initial structure. During its collapse an isothermal cloud will develop a $\rho \propto r^{-2}$ density profile throughout its structure, if it is given enough time. As Shu (1977) points out, the reason for this is clear. If left to itself, an isothermal sphere that is stable against collapse will try to put itself into the perfect mechanical balance that is

FIGURE 7. Properties of the equilibrium configurations to which a rotating spheroid will evolve if it follows a $\gamma = 7/5$ adiabat. Symbols and lines are described in the caption to Figure 6. This figure is taken directly from Tohline (1981).
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described by detailed hydrostatic equilibrium (cf. Chandrasekhar, 1967, p. 155). When boundary conditions are sufficiently far removed, this balanced structure will be one that is approximately \( \rho \propto r^{-2} \) throughout the cloud. The stable sphere will approach this structure in one sound crossing time \( t_s = R_0/c_s \), where \( R_0 \) is the cloud radius and \( c_s \) is its isothermal sound speed.

An isothermal sphere of a given temperature and density that is unstable toward collapse \( (R_0 > R_J) \), will also try to establish a detailed mechanical balance in one sound crossing time. The unstable cloud must, of course, contend with collapse at the same time, which occurs on a time scale \( \sim \tau_{ff} \). The cloud will approach a structure described by the similarity solution, then, only if \( t_s \leq \tau_{ff} \). Bodenheimer and Swope (1968) were the first to stress this competition between time scales. One can easily show that \( t_s \sim \tau_{ff} \) is equivalent to saying \( R_0 \sim R_J \). The similarity solutions of \( \S 4.1 \) are therefore applicable only to clouds that begin collapsing from configurations very near the Jeans limit.

Hunter's (1962) analysis indicates that fragmentation can still occur during hydrodynamic collapse in a medium that maintains isothermality, or more generally if the gas's adiabatic exponent during collapse is less than 4/3. If a perturbation wavelength is initially \( \lambda_1 \leq R_J \), its growth is delayed somewhat from what is expected in a pressure-free environment. As Hunter has stressed, however, his analysis is applicable only in regions of the cloud that maintain approximately uniform density throughout the collapse. Growth of perturbations is allowed in these regions because pressure gradients are negligible compared to gravity. As we have seen, however, an isothermal cloud that begins its collapse from a radius \( R_0 \sim R_J \) will develop a \( \rho \propto r^{-2} \) structure during its evolution and, more specifically, Eq. (4.18) shows that prior to core formation pressure gradients are comparable to gravity in the cloud envelope. Therefore, Hunter's perturbation analysis is not applicable to clouds that collapse from configurations near the Jeans limit.

To complement the summary of pressure-free collapse models that was presented in \( \S 3.5 \), the most important concepts to be learned from the hydrodynamic collapse models of this section are:

1) If a cloud collapses from a configuration near the Jeans limit, its evolution is not strongly dependent on initial conditions. This has been demonstrated explicitly for an isothermal collapse, where independent of initial conditions, the cloud assumes a similarity profile \( \rho \propto r^{-2} \) during its collapse. Physically, it is reasonable to expect that other adiabatic indices \( \gamma > 1 \) allow the flow to wipe out its memory of initial conditions, although the similarity

FIGURE 8 Properties of the equilibrium configurations to which a rotating spheroid will evolve if it follows a \( \gamma = 4/3 \) adiabat. Lines are described in the caption to Figure 6. This figure is taken directly from Tohline (1984).
profile will have properties different from that of an isothermal flow (Larson, 1969 and Cheng, 1978).

2) If a cloud collapses from a configuration near the Jeans limit, it will not be susceptible to the Lin, Mestel, and Shu (1965) flattening instability. In the absence of rotation it can therefore maintain rough sphericity during collapse, but will almost certainly become centrally condensed. Its structural properties during collapse will almost certainly inhibit fragmentation during its hydrodynamic collapse.

Tohline's (1981) work, summarized in §4.3, provides a very useful way of predicting, from initial conditions alone, at least qualitatively what the endpoint of a hydrodynamic collapse will be. This work is completely divorced from the other analytic analyses that have been presented in this section as it does not address the characteristics of the gas flow during hydrodynamic collapse. The ultimate goal of studies of hydrodynamic collapse, of course, is to understand how the collapse flow eventually leads to equilibrium structures such as those described in §4.3. As was pointed out in §3.5, the supersonic velocities generated during collapse must generally be arrested by passage of the flow through a shock front. This event alone severely complicates models of hydrodynamic collapse and adequate treatment of the event generally requires numerical, rather than analytic, techniques. With the analytic models of this section as background, it is certainly easier to understand and evaluate the physical phenomena that are encountered in numerical models of hydrodynamic collapse.

5. COMPUTER CODES TO STUDY HYDRODYNAMIC COLLAPSE

In order to make significant progress toward understanding the general problem of hydrodynamic collapse, one must inevitably use numerical schemes to integrate a strongly coupled set of physical equations. Numerical codes vary in complexity from those which are designed to integrate a set of two, coupled ordinary differential equations that describe relatively simple physical collapse [e.g., Eqs. (3.38a) and (3.38b) describing the collapse of pressure-free spheroids or Eqs. (4.8) and (4.9) describing an isothermal similarity flow] to those which are designed to integrate a set of four or more partial differential equations that describe the nonspherical, inhomogeneous collapse of more realistic physical systems. A complete discussion of the

FIGURE 9 Properties of the equilibrium configurations to which a rotating spheroid will evolve if it collapses isothermally ($\gamma=1$). Lines are described in the caption to Figure 6. This figure is taken directly from Tohline (1981).
variety of numerical techniques that have been used to study hydrodynamic collapse is beyond the scope of this paper. There are a few fundamental principles that are generally adhered to in designing these codes, however. An understanding of these principles along with some knowledge of the basic terminology that accompanies discussions of numerical hydrodynamic codes is needed before anyone can critically discuss the results of hydrodynamic collapse calculations. An attempt will be made here to briefly outline these fundamental principles and to explain some of the most commonly used terminology before a review of recent numerical calculations is made in §6.

5.1 The Basic Equations

There are four classic equations of fluid flow that form the foundation for all studies of self-gravitating, hydrodynamic collapse. They are: the vector equation of motion

\[ \rho \frac{dv}{dt} = -\rho \nabla \phi - \nabla P, \]  
(5.1)

the equation of continuity (conservation of mass)

\[ \frac{dp}{dt} + \rho \nabla \cdot v = 0, \]  
(5.2)

the energy equation

\[ \frac{dU}{dt} + P \frac{d}{dt} (\frac{1}{\rho}) = \frac{dQ}{dt}, \]  
(5.3)

and the Poisson equation (self-gravity)

\[ \nabla^2 \phi = 4\pi G \rho. \]  
(5.4)

In these equations, \( v \) is fluid velocity, \( \rho \) is mass density, \( P \) is gas pressure, \( \phi \) is gravitational potential, \( G \) is the gravitational constant, \( U \) is specific internal energy for the gas, \( dQ/dt \) is a term governing heat losses or gains by the system, and the independent variables are time \( t \) and the spatial coordinate vector \( \mathbf{r} \). In the absence of a gravitational field \( (\phi=0) \), Eq. (5.1) is known as Euler’s equation. It plus Eq. (5.2) are invariably the first equations to be introduced in any classical hydrodynamics text (e.g., Lamb, 1945 or Landau and Lifshitz, 1959). Equation (5.3), used for gaseous media, is simply the first law of thermodynamics.

To complete the set of equations, an expression for \( U \) and \( dQ/dt \) in terms of the other variables \( v \), \( \rho \), \( P \), and \( \phi \) must be supplied. For example, in a perfect gas,

\[ U = c_v (\mathcal{R} \rho)^{-1} (P/\rho), \]  
(5.5)

where \( c_v \) is the specific heat at constant volume for the gas, \( \mu \) is its mean molecular weight, and \( \mathcal{R} \) is the gas constant. And, for example, for an adiabatic evolution, \( dQ/dt = 0 \). Generally speaking, \( U \) and \( dQ/dt \) will be much more complicated than this as molecular species in the gas introduce a temperature-dependent \( c_v \) and a temperature-density-dependent \( \mu \) (cf. DeCampli et al., 1978; Gerola and Glassgold, 1978) or as positional variations in the source and sink functions for the radiation field require a detailed radiation transport treatment (cf. Tscharniuch and Winkler, 1979; Winkler and Newman, 1980b; Gerola and Glassgold, 1978).

The right-hand side of the equation of motion (5.1) includes only the two terms that are of greatest concern to this limited review. It can be considerably more complicated in hydrodynamic collapse problems as it should include all forces which may contribute to the acceleration of fluid elements. If radiation pressure, magnetic fields, and fluid viscosity are expected to play a significant role during a collapse calculation then (5.1) must assume the form

\[ \frac{dv}{dt} = -\rho \nabla \phi - \nabla P + \frac{\kappa \rho}{c^2} F + \frac{1}{4\pi} (\nabla \times B) \times B + \eta \nabla^2 v, \]  
(5.6)

where \( F \) is the flux from the radiation field, \( \kappa \) is the gas opacity, \( c \) is the speed of light, \( B \) is the magnetic field vector, and \( \eta \) is the dynamic viscosity of the fluid. [If the divergence of the velocity field makes an important contribution to the flow, then an additional viscosity term should appear in (5.6); cf. Landau and Lifshitz, 1959, p. 49.] When these forces are included, supplemental relations which control the behavior of \( F \), \( \kappa \), \( B \), and \( \eta \) must also be supplied before the set of working physical equations can be complete. The reader is referred to the works of Winkler and Newman (1980b), Scott and Black (1980), Regev and Shaviv (1980), and Tscharniuch (1978, 1981) for specific applications of these additional forces in investigations of the hydrodynamic collapse problem.

The terms in Eqs. (5.1)–(5.3) can be manipulated in such a way as to allow each equation to be written in the form

\[ \frac{\partial}{\partial t} D + \nabla \cdot (Dv) = S_D \]  
(5.7)

where \( D \) is a volume density of either mass \( (D=\rho) \), momentum \( (D=\rho v) \), or
energy \((D = \rho U + \frac{1}{2} \rho v^2)\), and \(S_D\) is a source term. This form is known as the conservation form for the equations governing fluid motion because in the absence of any source terms \((S_D = 0)\), Eq. (5.7) simply states that the quantities mass, momentum, and energy must be conserved in the system. When written in this form, the respective source terms for Eqs. (5.1)–(5.3) are

\[
S_D = -\rho \nabla \phi - \nabla P, \tag{5.8}
\]

\[
S_D = 0, \tag{5.9}
\]

\[
S_D = -\rho \mathbf{v} \cdot \nabla \phi - \nabla \cdot (P \mathbf{v}) + \rho \frac{dQ}{dt}. \tag{5.10}
\]

The conservation Eq. (5.7) proves to be a useful form from which accurate “finite difference” representations of the fluid differential equations can be constructed for use in numerical simulations of hydrodynamic flows.

### 5.2 Eulerian versus Lagrangian Formalism

A relation that plays a crucial role in the transformation of Eqs. (5.1)–(5.3) to the conservation form (5.7) is the relation between partial and total time derivatives of any function \(f(r, t)\):

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f. \tag{5.11}
\]

There is a fundamental difference between total and partial time derivatives in the hydrodynamic equations. This difference must be understood before the equations can be applied effectively to describe physical fluid flows. In words, a total time derivative represents the rate of change of a Lagrangian variable; a partial time derivative is an Eulerian representation of the time rate of change of a variable. The physical concepts associated with the differences are:

- **Eulerian time derivatives** denote how a given quantity, say momentum, density, or energy, is changing in time at a fixed point in space.

- **Lagrangian time derivatives** denote how the momentum, density, or energy, say, of a given fluid element is changing in time. Generally, a Lagrangian fluid element will not remain at a fixed point in space during an evolution.

When an Eulerian representation of the hydrodynamic equations is used, the independent variables are time \(t\) (partial time derivatives must be used) and the fixed coordinate positions \(r\). When a Lagrangian formalism is used, the independent variables are time \(t\) (total time derivatives must be used) and the coordinate positions \(r_0\) which identify each Lagrangian fluid element's position at an initial time \(t_0\). (Actually, since associated with each fluid element \(r_0\) is also the mass \(m_0\) that uniquely identifies the fluid element and is unchanging in time, the quantity \(m_0\) can sometimes be used effectively as the Lagrangian coordinate in place of \(r_0\).) The spatial operators \(\nabla\) and \(\nabla^2\) in Eqs. (5.1)–(5.4) and in (5.6)–(5.11) are understood to be with respect to the Eulerian reference frame \(r\), so strictly speaking Eqs. (5.1)–(5.3) are written in mixed Eulerian–Lagrangian form. Direct use of relation (5.11) can quickly transform them to a pure Eulerian formalism, as is the form of Eq. (5.7). In multidimensions, it is usually not so easy to transform the hydrodynamic equations to a pure Lagrangian form. A given Eulerian coordinate position \(r\) will see many different Lagrangian fluid elements \(r_0\) as the fluid flows by, so \(r\) is in general a nontrivial function \(r(r_0, t)\). A transformation of the spatial operators \(\nabla\) and \(\nabla^2\) to the independent variable \(r_0\) must take this functional dependence into account. Lamb (1945, pp. 12–14) shows how in Cartesian coordinates the Euler equation and the continuity equation can be written in a pure Lagrangian formalism. Suffice it to say here that the form of the multidimensional hydrodynamic equations is generally considerably messier in Lagrangian form than in Eulerian form.

There are certain advantages to choosing one formalism over the other. Analytic solutions to hydrodynamic flows are nearly impossible to derive if the equations are written in Eulerian form because they are then partial differential equations. In a Lagrangian formalism, the equations are analytically integrable in time, at least in principle, because time derivatives are total rather than partial.

In numerical calculations, Lagrangian formalism is advantageous because certain quantities which should be strictly conserved in each fluid element, for example mass and specific angular momentum, can be strictly conserved because the dependent variables identify with unique fluid elements for all time. This "local" conservation of quantities is not guaranteed in an Eulerian formalism. As mentioned earlier, the Eulerian conservation Eq. (5.7) can be computationally programmed in such a way as to ensure conservation of important quantities over the entire fluid system, but this "global" conservation is not as desirable as strict local conservation.

Also, if material stacks up to a high density in a small volume during the collapse evolution, the Lagrangian coordinates \(r_0\) automatically stack up in that volume as well. In this way, spatial resolution of the fluid flow is optimized. In a strict Eulerian formalism, since the coordinates \(r\) remain fixed in space, the material can sometimes completely vacate certain coordinate positions. The numerical program may then waste time integrating the fluid flow at spatial locations where there is no fluid. In order to overcome
this disadvantage, some computer codes have incorporated a “moving
Eulerian” coordinate system attempting to mimic one of the favorable
aspects of the Lagrangian formalism.

In multidimensional hydrodynamic calculations, Eulerian formalism has
one major advantage over Lagrangian formalism: the geometric shape of
individual fluid elements remains fixed in time, set by one’s initial speci-
fication of the Eulerian coordinates r. “Difference” approximations to
gradients which relate the character of the flow in one fluid element to that of
its neighbors can therefore assume a form that need not change during an
evolution. Programming of the “difference” equations (see §5.3) that
approximate the differential Eqs. (5.1)–(5.4) is straightforward. Since the
geometric shape of individual Lagrangian fluid elements can become quite
distorted from their original chosen shapes during an evolution, deciding
how to represent spatial gradients in a “difference” form that will be accurate
throughout an evolution is not easy. Also, a dependent variable that is
supposed to describe average conditions within the volume of a single fluid
element may not accurately represent such conditions in a Lagrangian fluid
element that becomes severely distorted. In order to overcome the dis-
advantages that arise due to the distortion of fluid element shapes, many
Lagrangian computer codes permit the operator to rezone the Lagrangian
mesh—reassign the fluid to new Lagrangian elements whose shapes are not
distorted—intermittently during an evolution. Rezoning, of course, destroys
the identity of the original fluid elements and, along with it, the advantageous
property of maintaining strict local conservation. A spectacular example of
distorted Lagrangian grid cells is presented in Figure 6 of Woodward (1976).
The problem of distorted fluid elements does not arise in one-dimensional
hydrodynamic calculations.

Table V summarizes the advantages that one formalism has over the other
when the hydrodynamic equations are integrated numerically.

<table>
<thead>
<tr>
<th>TABLE V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advantages of one formalism over the other</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>Eulerian</td>
</tr>
<tr>
<td>Fluid element shapes do not distort during an evolution</td>
</tr>
<tr>
<td>Differential equations are simpler in form.</td>
</tr>
</tbody>
</table>

The two different time derivatives \( \partial f / \partial t \) and \( d f / d t \) each carry one additional important physical connotation. If the partial time derivative of a variable \( f \) equals zero, this implies that the value of \( f \) at a fixed point in space is constant with time, although \( f \) for each Lagrangian fluid element may still change with time. When the partial time derivative is zero, the solution is said to be “steady-state”. If, however, the total time derivative of \( f \) is zero, this implies that the value of \( f \) for each Lagrangian fluid element is constant in time.
The conditions \( \partial f / \partial t = 0 \) and \( df / dt = 0 \) clearly carry quite different meanings.

5.3 Implicit Versus Explicit Finite Difference Techniques

There exist a variety of methods for numerically solving a set of coupled,
partial differential equations. The most widely used and most completely
tested methods are those which employ finite difference approximations to
the differential equations. All aspects of these methods cannot be discussed
here; the interested reader should consult the book by Roache (1976) which
is an outstanding instruction and reference text on the subject. Finite
difference methods can be subdivided into two techniques—explicit and
implicit—which, upon practical application, exhibit fundamentally different
properties. In this section the basic mathematical difference between the
two techniques is briefly outlined, then a summary of the practical advantages
and disadvantages of each technique is presented. An understanding of the
basic attributes of implicit and explicit techniques allows one to appreciate
the practical limitations that are currently imposed on numerical models of
hydrodynamic collapse.

The principle of finite difference techniques can be effectively illustrated by
referring to the general conservation equation Eq. (5.7). In Cartesian coor-
dinates, and restricting it to two dimensions, this Eulerian equation becomes

\[
\frac{\partial}{\partial t} D(x, y, t) + \frac{\partial}{\partial x} [D(x, y, t) v_x(x, y, t)] + \frac{\partial}{\partial y} [D(x, y, t) v_y(x, y, t)] = S_D(x, y, t),
\]

(5.12)

where \( S_D \) may possess spatial derivatives but no time derivatives. Numeri-
cally, a continuous medium can be represented only by labeling values of the
quantities \( D, v_x, v_y, \) and \( S_D \) at discrete points in space and therefore spatial
derivatives of these quantities can only be determined approximately.
Through a Taylor series expansion the discrete spatial gradient \( \delta f / \delta x \) of any
variable \( f \) can be, for example,

\[
\frac{\delta f}{\delta x} \equiv \frac{f(x + \Delta x, y, t) - f(x, y, t)}{\Delta x} \approx \frac{\partial f}{\partial x},
\]

(5.13)
where $\Delta x$ is the spacing between adjacent Eulerian fluid elements. An analogous expression can be written to approximate $d\phi/dy$. Likewise, the behavior of each quantity as a function of time cannot be numerically determined in a continuous fashion, so time must be sliced in a discrete fashion as well. Letting the superscript "$n$" denote values of quantities at time $t^n$ and superscript "$n+1$" denote values of quantities at a later time $t^{n+1}=t^n+\Delta t$, the time derivative in (5.12) can be approximated through a Taylor series expansion as

$$\frac{\delta D}{\delta t} \approx \frac{D^{n+1}(x, y) - D^n(x, y)}{\Delta t}.$$ \hspace{1cm} (5.14)

Therefore, given that you know the value of a physical quantity $D$ at an initial time $t^n$, the value of that quantity at a later time $t^{n+1}$ can be numerically determined by substituting (5.13) and (5.14) into (5.12):

$$D^{n+1} \approx D^n + \Delta t \left[ S_D - \frac{\delta Dv_x}{\delta x} - \frac{\delta Dv_y}{\delta y} \right].$$ \hspace{1cm} (5.15)

This is the basic principle employed by finite difference techniques to "integrate" the hydrodynamic flow forward in time. The approximations for $d\phi/dx$ and $dD/dt$ need not be exactly those illustrated in (5.13) and (5.14), for example higher order terms in the Taylor series expansion can be retained to improve the accuracy of the approximations, but the principle employed remains the same.

When attempting to code up (5.15) in a computer program, a decision must be made as to how the bracketed quantity in the relation should be evaluated. There are no-time derivatives in this bracketed quantity, so all terms inside the bracket must be expressed at the same time slice. But which time slice should you use? In practice, the choice makes a great deal of difference and differentiates between the two techniques termed explicit and implicit.

**Explicit technique** If the bracketed term in (5.15) is evaluated at time $t^n$, then all quantities on the right-hand side of (5.15) are known initially (time $t^n$) and the determination of $D^{n+1}$ is straightforward.

**Fully implicit technique** If the bracketed term in (5.15) is evaluated entirely in terms of the advanced time $t^{n+1}$, then the determination of $D^{n+1}$ is quite complicated. Its value will not only depend on the known value $D^n$ but also on the unknown values of $v_x^{n+1}$, $v_y^{n+1}$, and $S_D^{n+1}$. (Evaluating the bracketed term at some time intermediate between $t^n$ and $t^{n+1}$, known as a partially implicit technique, will suffer from this same complication of dependence on unknown values of variables at the advanced time.)

The main advantage of the explicit technique over the implicit one is obvious. Even in a multidimensional calculation the physical equations Eqs. (5.1)–(5.4) can be solved in series: For example, from the known values of $u^n$, $\phi^n$, $\rho^n$ and $U^n$ [and a relation similar to (5.3)], the new mass density $\rho^{n+1}$ can be determined from (5.15) with $S_D=0$, the new momentum density can be determined from (5.15) using $S_D$ as given in (5.9), and the new energy density can be gotten from (5.15) using $S_D$ from (5.10). From the new mass density, the Poisson equation (5.4) gives $\phi^{n+1}$. This series method of solution does not require much computer storage or computational finesse, and a single integration time step can be computed comparatively quickly. In a multidimensional calculation, determining the solution to the Poisson equation itself causes perhaps more difficulty that the solution of all the other equations combined, because of its $\nabla^2$ operator. Various techniques that have been employed to solve the multidimensional Poisson equation in hydrodynamic collapse calculations are discussed in the papers that have presented numerical results from collapse calculations (cf. Black and Bodenheimer 1975, Tohline 1980a, Boss 1980a).

By comparison, the implicit technique demands that the four Eqs. (5.1)–(5.4) be solved simultaneously. It requires considerably more computational finesse than does the explicit method. In a one-dimensional calculation involving $N$ discrete spatial points, a matrix composed of $4N$ number of coupled, linear difference equations must be inverted in some manner to determine the $4N$ unknown variables at the new time. The matrix inversion is not a formidable operation in a one-dimensional calculation because its nonzero elements are confined to a central band that is [for a first order accurate difference equation like (5.15)] at most $12$ numbers wide. In multidimensions, however, the implicit technique becomes computationally cumbersome. If, for example, a two-dimensional problem involves $N$ discrete spatial points in both dimensions, the number of coupled linear difference equations grows to $4N^2$, and at the same time nonzero matrix elements sparsely populate a band that must be about $2N$ numbers wide. The situation is proportionally worse in a three-dimensional problem. This technique, then, can demand an enormous amount of computer storage for the matrix alone, and a single time step can be computationally quite expensive. An implicit "spectral" technique used, for example, by Tscharnuter and Winkler (1979) can significantly reduce the band width of the matrix in multidimensional calculations, but certain approximations regarding the flow structure must be made to be successful with this technique. An "alternating-direction implicit" (ADI) technique used, for example, by Black and Bodenheimer (1975) to solve the Poisson equation (see Douglas and Gunn, 1964, for a general description of the technique) avoids matrix inversion.
The main disadvantage of the explicit technique compared to the fully implicit one is not readily apparent, but must be pointed out because it can impose prohibitive restrictions on the technique's practical applicability. In principle there is no restriction on the size of the integration time step that can be used with a fully implicit finite difference scheme. In practice, the time step is usually chosen to adequately resolve physically interesting changes in the fluid flow. In contrast to this, there is a strict upper limit to the size of the integration time step $\Delta t$ that can be used with an explicit finite difference scheme. This limit is, in one dimension, set by the requirement

$$\Delta t \leq \frac{\Delta x}{c_s + |v_x|}, \quad (5.16)$$

where $c_s$ is the sound speed in the gas. If $\Delta t$ is taken larger than this value, then the explicit method is numerically unstable, "meaning that chaotic solutions with no relation to the continuum solution will be generated" (Roache 1976, p. 41). If $\Delta t$ by (5.16) has a different value at the different discrete spatial locations in the fluid, then the time step to be used to advance the entire system must be no larger than the smallest of all the limiting $\Delta t$'s. In a multidimensional calculation, a similar restriction on $\Delta t$ must apply in the other dimensions as well. Physically, relation (5.16) demands that the time step be small enough to prevent a supersonic flow from moving completely across any grid cell of size $\Delta x$ in a single time step, or to prevent a subsonic flow from being able to "communicate" via sound waves beyond a neighboring fluid element in a single time step. The subsonic form of restriction (5.16) was first realized by Courant, Friedrichs and Lewy (1928) (see Courant, et al., 1967, for an English translation) and is often referred to as the Courant condition.

From the free fall solutions discussed in §3, it is clear that for a system that is undergoing a self-gravitating, hydrodynamic collapse, significant changes in the structure of the entire system will occur on a time scale $\Delta t \ll \tau_{ff}$ which, from (3.62) is $\Delta t \ll R_0/\rho$. So during an implicit calculation of the collapse, the chosen time step should in practice not be too different from that imposed on an explicit calculation by the Courant condition. Therefore, as a tool for studying hydrodynamic collapse, the advantages of an explicit integration technique over an implicit one generally far outweigh the disadvantages imposed by a time step restriction on the technique. This statement does not apply to hydrodynamic studies of the endpoint of collapse. Since the endpoint of collapse is usually obtained by following the accretion of material onto a hydrostatic or quasistatic, centrally condensed core, significant changes in the structure of the physical system as induced by the accretion process will occur on a time scale that can be much longer than the soundcrossing time across the core. The Courant condition severely limits the usefulness of the explicit integration technique in this situation, as many explicit time steps, requiring a large amount of computing time, have to be made to get to the same point in the accretion evolution that could be reached by the implicit technique in only a few time steps.

In summary, Table VI lists the major disadvantages of one method over the other. Keep in mind that each disadvantage can be considerably amplified in multidimensional models.

### 5.4 Finite-Size Particle Techniques

A new, and therefore relatively untested numerical technique for studying multidimensional, self-gravitating hydrodynamic phenomena has been developed by Lucy (1977). The technique, referred to by Lucy as a finite-size particle technique (FSP) and by Gingold and Monaghan (1977) as a smoothed-particle hydrodynamics technique, incorporates certain features of $n$-body dynamic calculations (as used, for example, in simulations of stellar dynamic systems) into the fluid dynamic equations. At each integration time step, $N$ number of particles of equal mass are distributed over the volume occupied by the hydrodynamic fluid in a (Monte Carlo) fashion such that their relative packing represents as accurately as possible the relative density of the fluid. The acceleration of each particle is then determined from the Lagrangian equation of motion, and each particle's new velocity and position is appropriately updated. The technique differs from $n$-body dynamic calculations in that each particle has a finite size and itself possesses properties of a fluid. The fluid properties are essential in that they allow determination of, for example, forces due to pressure gradients between neighboring particles. The size $h$ of each particle, compared to the size $R_0$ of the entire fluid system, is appreciable:

$$h \sim R_0/N^{1/3}.$$

### Table VI

Disadvantages of one integration technique compared to the other

<table>
<thead>
<tr>
<th>Implicit</th>
<th>Explicit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computationally more difficult to program.</td>
<td>Integration time step can be severely limited by the Courant condition.</td>
</tr>
<tr>
<td>Matrix can require a large amount of computer storage.</td>
<td></td>
</tr>
<tr>
<td>Execution of each integration time step can require a considerable amount of computer time.</td>
<td></td>
</tr>
</tbody>
</table>
The length scale $h$ actually characterizes the width of each particle's own Gaussian-like density distribution. Because the particles are each quite large in extent, there is usually considerable overlap among their structures. This overlap is essential for mimicking the attributes of a continuum fluid.

The FSP technique benefits from the advantages of a Lagrangian formalism, listed in Table V, but does not suffer from the main problem encountered in pure fluid Lagrangian schemes because the particle shapes are assumed to remain the same (usually spherical) at all times. The time integration is explicit, so the technique is restricted by a condition similar to the Courant condition.

By far the greatest asset of the FSP technique is that by packing the particles specifically in regions where most of the fluid mass resides, considerably fewer fluid elements are required to resolve the fluid flow in a three-dimensional calculation than are required in finite-difference schemes, which usually employ Eulerian prescriptions of the fluid. With a small number of particles, however, the scheme by nature cannot accurately represent the continuum structure of relatively diffuse regions of the fluid. This means that the envelope, which often plays a particularly significant role in the late stages of hydrodynamic collapse evolutions, cannot be correctly represented with the FSP technique unless many particles are used.

The FSP technique carries with it two primary disadvantages. It cannot accurately resolve discontinuities in the flow as an example, when accretion shocks develop at the end of a hydrodynamic collapse. And it, like most $n$-body numerical simulations, suffers from the accumulative effect of 2-body particle encounters. These encounters collisionally relax the system (cf. Lucy's 1977 discussion) and will, over time, drive the solution away from a true gas dynamic behavior.

In direct comparisons with the results of three-dimensional finite difference calculations, the FSP technique has received somewhat mixed reviews. Durisen, et al. (1982) have shown in a study of nonaxisymmetric instabilities in equilibrium polytropes that the two techniques compare very favorably over an extended evolution. However, in a study of hydrodynamic collapse, Gingold and Monaghan (1981, 1982a), using their FSP technique, have found the endpoint of collapse to differ markedly from the results of Boss and Bodenheimer (1979) who have used a finite difference technique. Wood (1981) has developed perhaps the most sophisticated version of a FSP technique that has been applied to the multidimensional hydrodynamic collapse problem. His results (see also Wood, 1982) agree in qualitative fashion more closely with the endpoint of collapse found by Boss and Bodenheimer (1979) and by other investigations that have employed finite difference techniques than with the FSP results of Gingold and Monaghan (1981, 1982a). As suggested by Bodenheimer and Boss (1981), the discrepancy in the collapse results can probably be blamed on the size of the finite particles that Gingold and Monaghan used. They used a single length scale $h$ throughout the cloud whereas Wood allowed this scale to vary with density, permitting him to obtain superior resolution of density clumps within the collapsing gas cloud. A variable $h$ was evidently not crucial in the comparison calculations described by Durisen et al., perhaps because the physical system they examined did not undergo the enormous density variations that are encountered in collapse calculations.

The FSP technique shows great promise as a useful tool in studies of the early phases of multidimensional hydrodynamic collapse. Certainly additional comparisons between the results obtained from the FSP technique and finite-difference techniques would be beneficial.

5.5 Viscosity: Fluid, Turbulent, Numerical, and Artificial

One particular term that is used frequently by workers in the field of numerical hydrodynamics, but is often confusing to the layman is "viscosity". In a simulation of hydrodynamic flows one may encounter any or all of the four following types of viscosity: fluid, turbulent, numerical, and artificial. The difference between these is briefly explained here so that the terms can be used freely in §6.

Fluid viscosity, most commonly realized in the form of molecular viscosity, is an inherent physical property of any fluid (or gas) that allows momentum transport to occur in a direction perpendicular to the direction of the fluid flow. It arises because along with the ordered, macroscopic motion of a fluid (described by velocity $v$ in Eq. (5.1)) there always exists some random, microscopic motion in the fluid as well. Fluid viscosity acts in such a way as to minimize differential, transverse flow. It can, for example, transform differential rotation of any medium into solid body rotation. Its contribution to the equation of motion is illustrated by the $\eta \nabla^2 v$ term in Eq. (5.6). The time scale $t_\eta$ on which its effects are felt is determined by the size of the coefficient $\eta$ and the size of the physical system $R_0$:

$$t_\eta \sim R_0^2/(\eta/\rho).$$

(5.18)

Since typical values of $\eta$ that have been measured in the laboratory for real fluids are $\eta/\rho \sim 10^{-1} - 10^{-2}$ cm$^2$ s$^{-1}$, fluid viscosity's importance as a transport mechanism in astrophysical phenomena is usually negligible, by orders of magnitude.

Turbulent viscosity, at least as it is most often used in the astrophysical literature, is simply a "scaled up" version of fluid viscosity. When angular momentum transport, for example, is believed to be occurring on a rapid
time scale in an astrophysical system (as, for example, in accretion disks), the value of $\eta$ in the viscous term in the equation of motion is artificially increased to a value that will allow viscous effects to become important on the appropriate time scale. When turbulent viscosity is employed to explain an astrophysical phenomenon, the belief is usually voiced that the astrophysical system is unstable toward the development of macroscopic random motions in what is otherwise a smooth flow and that these macroscopic "turbulent eddies" contribute to momentum transport in much the same way that microscopic random motions contribute in laboratory fluids. The instability that gives rise to the turbulent eddies is not usually demonstrated, and without that, the chosen astronomical sized values of $\eta$ lack convincing physical justification. It will probably be some time before turbulent viscosity on astrophysical scales is fully understood.

**Numerical viscosity** is a completely artificial phenomenon that arises in essentially all numerical simulations of hydrodynamic flows. It generally has the same transport properties as real fluid viscosity. The phenomenon arises because the finite differences that are used in numerical schemes to approximate derivatives in the equations governing fluid flow are only approximate. An algebraic analysis of the difference equations (cf. Roache, 1972, or Hirt, 1968) clearly demonstrates that their approximate behavior introduces an artificial diffusive term into the fluid flow that mimics the $\eta \nabla^2 \nu$ term in Eq. (5.6). When only low order Taylor series expansions are used to approximate derivatives, the equivalent numerical viscosity $\eta$ can have a significant size. To avoid undesired effects that can arise from numerical viscosity, fine spatial resolution must be used in a calculation and/or high order difference approximations to the differential operators must be employed.

**Artificial viscosity**, not to be confused with numerical viscosity, has properties that are quite different from the other viscosities just discussed. It arises from a term that is purposely introduced into the computational equation of motion in finite difference schemes to stabilize the numerical scheme when shocks arise in supersonic flows. The sharp discontinuity in flow variables that must occur across a shock is difficult to treat correctly with numerical difference schemes. The physical scale over which the discontinuity should occur (set by microscopic properties of the fluid) is usually orders of magnitude smaller than the spatial resolution of the numerical scheme can allow. Artificial viscosity smooths the shock discontinuity over a few spatial zones in the computational grid, but is designed to correctly preserve the properties of the macroscopic flow through the shock. Artificial viscosity is designed not to influence the fluid flow in any way except in regions of strong (supersonic) compression.

### 5.6 Summary

The most desirable numerical technique for studying all phases of hydrodynamic collapse, including defining the endpoint of collapse, is an *implicit Lagrangian* technique. The advantages of a Lagrangian formalism, as outlined in Table V, are very desirable features for any numerical technique to have. The endpoint of a collapse, which will generally be found only after following a long accretion phase, cannot be discerned by explicit integration because of the formidable time barrier set by the Courant condition. Not surprisingly, the definitive study by Winkler and Newman (1980a, 1980b) of the one-dimensional collapse of a $1 M_\odot$ protostar was performed using a highly sophisticated implicit Lagrangian numerical code.

In multidimensions, the task of developing a useable implicit Lagrangian code is a formidable one. The hydrodynamic equations take on a relatively complicated form, dealing in a general fashion with a distorted computational mesh, even in only two dimensions, is difficult; and even moderate spatial resolution of the fluid leads to extremely large matrices whose inversion requires an enormous amount of computational time, even on the fastest computers (CRAY I's) that are available today. Understandably, the majority of multidimensional hydrodynamic collapse calculations to date have been performed using explicit Eulerian techniques. They can be programmed in a relatively straightforward manner and, for a given spatial resolution, demand considerably less computer storage and less execution time per time step than implicit techniques. The information gained to date from multidimensional collapse calculations has been limited accordingly. With the exception of the two-dimensional models studied by Tscharnuter (1981) using a spectral implicit technique, essentially nothing has been learned about multidimensional accretion onto collapsed cores (see also Boss and Black, 1982). A great deal has been learned about the hydrodynamic collapse phase itself, however, and this new knowledge will be highlighted in the discussion of the following section.

### 6. NUMERICAL CALCULATIONS OF HYDRODYNAMIC COLLAPSE

Throughout this section considerable use will be made of the two dimensionless ratios $\alpha$ and $\beta$ that have been introduced in §4.3. From Eq. (4.29) and the definition of a Jeans mass (2.4a), it is clear that for a uniform density
sphere of mass $M$,

$$a = (M/M_\odot)^{-2/3}. \tag{6.1}$$

So, for $a<1$, gravity will dominate over thermal pressure forces and by the Jeans criterion collapse must ensue.

This criterion for collapse is actually slightly inconsistent with the definition of possible equilibrium configurations discussed in §4.3. Virial equilibrium in a self-gravitating gas is possible only if

$$a + \beta = 1/2. \tag{6.2}$$

In the absence of rotation, this relation states that $a=1/2$ denotes an equilibrium state; $a<1/2$ indicates that the configuration is unstable toward collapse. The discrepancy between this criterion for collapse and the Jeans criterion ($a<1$) simply reflects the approximate nature of the Jeans criterion. As far as the discussion in this section is concerned, the condition $a=1/2$ is as good as $a<1$ in defining a configuration that is barely unstable toward collapse.

As was mentioned in §§3 and 4, the condition $a<1$ means that hydrodynamic collapse will occur in a pressure-free fashion. In contrast to this, $a<1$ means that initially the sound crossing time in the cloud is comparable to the free fall collapse time. Pressure gradients therefore exert an enormous influence on the structure and evolution of a gas cloud that is barely Jeans unstable. With the support of the analytic models of §§3 and 4, the discussion of this section will focus on the differences that arise between numerical models that collapse from configurations near the Jeans limit and models that collapse from configurations that are very Jeans unstable.

6.1 Spherical Collapse

The analytic models of §§3 and 4 all employed either the assumption of pressure-free conditions or the assumption of a fixed adiabatic exponent during collapse. When proper account is taken of the heating and cooling mechanisms that prevail in a collapsing interstellar cloud (see Larson, 1969, 1973a, b, and Gerola and Glassgold, 1978, for a complete discussion), one finds that gas clouds having masses in the range $0.1 M_\odot \leq M \leq 50 M_\odot$ follow a variable adiabat similar to the solid line shown in Figure 10. The data for the curve in Figure 10 were kindly given to the author by P. Bodenheimer (1980, private communication) and represent specifically the $\rho-T$ behavior of gas in the center of a collapsing $3 M_\odot$ protostar. A similar curve for a $1 M_\odot$ protostar is shown in Winkler and Newman (1980b). For reference, the dashed lines in Figure 10 have slope $d \ln T/d \ln \rho=1/3$, corresponding to an adiabatic exponent $\gamma=4.3$. This $\rho-T$ behavior for the gas is well established and is not a point of contention among investigators who numerically study the details of hydrodynamic collapse. The curve can be described quite accurately in a simple, piecewise manner:

1) At low densities, $10^{-19} \text{ g cm}^{-3} \lesssim \rho \lesssim 10^{-18} \text{ g cm}^{-3}$, the molecular cloud collapses isothermally ($\gamma=1$) because the gas is optically thin to its primary cooling radiation (infrared radiation from dust grains) and the compressional heat generated by collapse can be readily radiated away.

2) At densities above $\sim 10^{-18} \text{ g cm}^{-3}$ (point A in the figure), the cloud is opaque to this radiation, so the gas heats up, following an adiabat $\gamma \approx 7/5$ that is appropriate for its primary constituent, molecular hydrogen (see...
DeCampli et al., 1978). The exact density at which adiabatic collapse sets in is governed by how much material surrounds and obscures the center of the cloud. As shown, for a 3 \(M_\odot\) cloud it occurs at \(\rho_s \approx 10^{-11} \text{ g cm}^{-3}\); for a 1 \(M_\odot\) cloud it is closer to \(10^{-12} \text{ g cm}^{-3}\); for a cloud more massive than 3 \(M_\odot\), complete obscuration sets in at \(\rho_s < 10^{-12} \text{ g cm}^{-3}\).

3) At a temperature \(\sim 2000\) K (point B in the figure), molecular hydrogen dissociates by an endothermic reaction, so over the range of densities \(10^{-8} \text{ g cm}^{-3} \lesssim \rho \lesssim 10^{-4.5} \text{ g cm}^{-3}\), much of the compressional heat of collapse is used up dissociating \(H_2\). The resulting adiabatic exponent in this range of densities is \(\gamma \approx 1.1\).

4) Once \(H_2\) has been completely dissociated (point C in the figure), further compression of the essentially monatomic gas causes the central temperature to rise along a \(\gamma \approx 5/3\) adiabat.

Beginning with Larson (1969), many detailed numerical models have followed the spherical hydrodynamic collapse of a 1 \(M_\odot\) gas cloud along a variable adiabat similar to the one shown in Figure 10. These models will not be discussed individually here; they have been adequately summarized by Winkler and Newman (1980a, b) and in the introduction to a series of papers on this collapse problem by Stahler, Shu and Taam (1980a).

The spherical collapse of a molecular cloud having a mass \(M \sim 1\) \(M_\odot\) can be divided into two parts: the hydrodynamic collapse phase itself, which begins from a Jeans unstable, low density gas cloud and ends with the formation of a small, hydrostatic core at densities \(\sim 10^{-4} - 10^{-2} \text{ g cm}^{-3}\); and the accretion phase during which a free-falling envelope, perhaps containing a substantial fraction of the cloud mass, accretes through a shock front onto the hydrostatic core. As alluded to in §5, the accretion phase is difficult to model accurately. An accurate treatment demands an implicit hydrodynamic scheme with spatial resolution sufficient to resolve the strong accretion shock that separates the hydrostatic core from the free-falling envelope. Winkler and Newman (1980a, b) have provided a definitive analysis of the spherical collapse problem from a configuration initially near the Jeans limit through the entire accretion phase. The accretion phase by itself has been studied in detail by Stahler, Shu, and Taam (1980a, b, 1981) using a unique mathematical approach to the problem. Basically, these groups have shown that although Larson's (1969) original models of spherical collapse encompassed some major simplifying assumptions, the models predict fairly accurately the observable properties of a protostar during its final accretion phase. The work of these two groups should be consulted for a complete discussion of this important phase of protostellar collapse.

The variable adiabat of Figure 10 introduces some interesting features into the hydrodynamic collapse phase of a protostellar cloud's evolution. In order to aid in the physical interpretation of these features, lines of constant Jeans mass have been plotted in the figure for \(M_J = 100\) \(M_\odot\), 1 \(M_\odot\), and 0.01 \(M_\odot\), assuming a mean molecular weight \(\mu = 2\) in each case. If, for example, a 1 \(M_\odot\) molecular cloud lies somewhere on the solid \(\rho-T\) curve to the left of the \(M_J = 1\) \(M_\odot\) dashed line, that cloud will be prevented from collapsing because \(M < M_J\), or equivalently \(\alpha < 1\) for the cloud. If the cloud's properties put it somewhere to the right of the \(M_J = 1\) \(M_\odot\) line, \(\alpha < 1\) for the cloud and it will collapse along its appropriate \(\rho-T\) curve.

A 1 \(M_\odot\) cloud that is barely Jeans unstable begins its evolution at a density \(\rho_1 \sim 10^{-18} - 10^{-19} \text{ g cm}^{-3}\) near the intersection of the solid \(\rho-T\) curve and the dashed \(M_J = 1\) \(M_\odot\) line in Figure 10. As its density increases, it remains isothermal at about 10 K until it becomes opaque to its cooling radiation near point A on the \(\rho-T\) curve. Many investigators, beginning with Penston (1966) and Bodenheimer and Sweigart (1968), have found that during this isothermal phase of collapse the cloud develops a strongly centrally condensed structure (\(r \propto r^{-2}\)) independent of its initial density structure. At point A the centralmost, highest density region of the cloud is the first volume of material to become opaque. A hydrostatic core, possessing only a small fraction of the total cloud mass, forms. This core then accretes the infalling envelope material through a shock front. As the core increases in mass, its temperature rises following the \(\gamma \approx 7/5\) adiabatic part of the solid \(\rho-T\) curve in Figure 10. At the point (B) of \(H_2\) dissociation, the central part of this core becomes unstable toward a second collapse. This second collapse stops when dissociation is complete (point C in the figure) and a second (inner) hydrostatic core forms. Accretion of envelope material through a second shock onto this inner core allows the core to grow in mass and causes the central cloud temperature to increase along the \(\gamma \approx 5/3\) adiabatic part of the \(\rho-T\) curve in Figure 10. The formation of this second core begins what has been termed above the accretion phase of the collapse.

The reason a double core structure develops during this collapse can be understood in relatively simple physical terms. At point A in Figure 10 the Jeans mass is \(M_J \sim 0.004\) \(M_\odot\) (assuming \(\mu = 2\)). Therefore, from Eq. (6.1), if the entire one solar mass of gas reached this point in the \(\rho-T\) diagram at exactly the same time, the value of \(\alpha\) for the cloud would be \(\alpha \sim 0.025\), and the cloud would be far from virial equilibrium. However, in the collapse just described, the cloud is very centrally condensed by the time it evolves to point A so much less than 1 \(M_\odot\) of the gas hits point A initially. The centralmost volume enclosing ~0.004 \(M_\odot\) is actually able to approach virial balance at point A because (1) \(\alpha \sim 1\) for that small volume of gas, (2) the adiabatic exponent for further compression of that volume is greater than 4/3, and (3) its own dynamic readjustment time scale is short compared to the collapse time of the rest of the lower density cloud. Accretion of material onto this
small mass core increases its mass and pushes it up the $\gamma \sim 7/5$ adiabat to point B in Figure 10. At point B, $M_B \sim 0.017 \, M_\odot$ (assuming $\mu = 2$), so when more than $\sim 0.017 \, M_\odot$ of gas has entered the core, a drops below unity and $\gamma$ becomes less than $4/3$, so the core itself becomes Jeans unstable toward further dynamic collapse. Because the central core is centrally condensed, its centralmost volume leads the collapse to point C where again virial balance is attainable by a fraction of the mass. $M_C \sim 0.002 \, M_\odot$ (assuming $\mu = 1$) at point C, so for the same reasons that the first core formed, a second (inner) core can form when $\sim 0.002 \, M_\odot$ hits the $\rho-T$ conditions of point C. Accretion then pushes this inner core up the $\gamma \sim 5/3$ adiabat to $\rho-T$ conditions that allow a larger and larger mass core to maintain virial balance. This simple “virial balance” explanation for why the double core structure forms gives a surprisingly accurate prediction of what the value of the mass in the first and second cores should be when they first form. Larson (1969) found in his calculations that the first core initially had a mass $\sim 0.005 \, M_\odot$ and the second core had a mass $\sim 0.0015 \, M_\odot$.

The formation of the double core structure clearly hinges on the fact that the cloud collapses in an extremely nonhomologous fashion. Its centrally condensed, nonhomologous structure developed, in turn, because the isothermal collapse began from a configuration near the Jeans limit. The double core structure can probably be bypassed by any cloud that (1) collapses from an initial configuration that is very Jeans unstable ($a_i < 1$) and (2) is not extremely centrally condensed initially. The negligible effects of pressure in such a model will prevent the centrally condensed, isothermal similarity flow from developing and the comparatively homologous nature of the pressure-free collapse will allow a substantial part of the cloud’s central volume to reach point A at essentially a single instant in time. If this central volume encloses a mass $\geq 0.017 \, M_\odot$, it will be unable to attain virial balance at any temperature below that required to dissociate $\text{H}_2$ and force the cloud to bypass the formation of a low density, first core. Narita, Nakano and Hayashi (1970) have presented results from the collapse of an initially very Jeans unstable $1 \, M_\odot$ gas cloud where the formation of a low density hydrostatic core did not occur. This result is not surprising in light of the physical arguments just given.

The formation of the double core structure in itself is not particularly significant in a spherical collapse calculation. Bodenheimer (1972, p. 34) has pointed out that the central conditions at the time of the formation of the final “stellar” core are practically independent of initial conditions and specifically do not depend on the presence of a first hydrostatic core. The double core structure has been emphasized here, however, because as we shall see its role may be significant in multidimensional evolutions.

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The accretion phase of an evolution can have substantially different observable properties depending on whether $a_i \sim 1$ or $a_i \ll 1$ in a cloud. A very Jeans unstable model, such as the one evolved by Nakano, Narita and Hayashi (1970), will possess a much more luminous and much shorter accretion phase than will a model that is only barely Jeans unstable initially, such as the one evolved by Larson (1969) and by Winkler and Newman (1980a, b). The accretion luminosity $L_{\text{acc}}$ is different basically because it is proportional to the rate $dM/dt$ at which the cloud’s envelope mass accretes onto the hydrostatic core (Winkler and Newman, 1980a). This rate in turn is (Stahler, Shu and Taam, 1980a)

$$\frac{dM}{dt} \sim M \tau_{ff}^{-1}$$

(6.3)

where $M$ is the cloud mass. Since $\tau_{ff}^{-1} \propto a_i^{1/2}$, it is clear that an initially high density cloud ($a_i \ll 1$) will collapse faster and have a larger accretion luminosity than an initially low density cloud ($a_i \sim 1$).

Since an important observable property of the protostellar collapse does depend on the conditions in the cloud at the onset of collapse, it would be nice to have some idea of which model, $a_i \ll 1$ or $a_i \sim 1$, is more likely to arise in the interstellar medium. An answer to this is not yet available. In light of the conclusions of §§3 and 4, it should be emphasized that only the $a_i \sim 1$ model stands a chance of undergoing a reasonably spherical collapse. Being essentially pressure-free initially, any $a_i \ll 1$ model would be susceptible to flattening and would have a tendency toward fragmentation. The results of spherically symmetric collapse calculations seem physically appropriate only for gas clouds that begin their collapse from configurations near the Jeans limit.

6.2 Multidimensional Collapse

In an analogous fashion to the analytic models of multidimensional collapse (§3.3), the first step in trying to generalize numerical collapse calculations to include nonspherical effects has been to assume axial symmetry of the fluid variables and conservation of angular momentum, and to produce a two-dimensional model of hydrodynamic collapse. Larson (1972) was the first to examine two-dimensional models of collapse with and without rotation. Bodenheimer (1981) has given a detailed review of individual models from numerous subsequent investigations (see Boss and Haber, 1982 for more recent references). As in the spherical collapse, the evolution of these models can be divided into two phases: the hydrodynamic collapse phase and the subsequent accretion phase. Qualitatively, the hydrodynamic collapse phase of rotating models behaves as predicted by the pressure-free, semi-analytic
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models of Hutchinson (1976), discussed in §3.3. The cloud collapses preferentially in one direction—parallel to its rotation axis. Unfor- generalists have incorporated many simplifying physical assump- mitter densities at the larger and faster centers. The simplifying on the self-gravity and dust adiabatic equations exposed have been only consid- ered as a first approximation. By combining the results of these three-dimensional models with the knowledge that has been gained from the detailed spherical collapse calculations, it is clear picture of the large-scale properties of the collapse can be established. As the collapse proceeds, the gas cloud begins to flatten, and the collapse becomes more spherical. The collapse is halted by the formation of a protostar.

A second result of substantial impact is that gas clouds that form in high $\beta$ environments break up on a time scale that is shorter than the cooling time of the gas. This finding has important implications for the formation of multiple protostellar systems. Unlike the behavior of gas in a hydrostatic equilibrium, the collapse in a gas cloud with $\beta > 0.1$ is not self-similar, and the collapse time scale is significantly shorter than the cooling time. This suggests that the formation of multiple protostellar systems is more likely in high $\beta$ environments.

The broad generalization of the work of Toth (1978) is that in environments where $\beta > 0.2$, the collapse of gas clouds is not self-similar and the collapse time scale is significantly shorter than the cooling time. This suggests that the formation of multiple protostellar systems is more likely in high $\beta$ environments. The value of $\beta$ at the endpoint of an axi-symmetric configuration will be estimated and hence the likelihood that a given initial configuration will be estimated. From this, it is possible to understand the effects of hydrodynamic collapse and fragmentation. To be sure,
Figure 9 it is clear that most isothermal collapses should evolve to very high $\beta$ configurations, so it is not surprising that essentially all of the models evolved by Bodenheimer, Tohline and Black (1980) ended in fragmentation. From Figure 6 it is understandable why Boss (1981a) found that adiabatic collapses ($\gamma = 5/3$) end in fragmentation only if $\alpha_i$ is small ($\beta_i$ must also be relatively large to permit fragmentation at the end of an adiabatic collapse).

Since rotation is believed to be important in interstellar clouds, the implications of this second major empirical result are: (1) Gas clouds having masses $M \geq 1 M_\odot$ and $\alpha_i \sim 1$ will find it difficult to avoid fragmentation before reaching stellar densities if they conserve angular momentum because their initial collapse is primarily isothermal. (2) As Bodenheimer (1978, 1981) has often emphasized, the fragmentation of high $\beta$ configurations can relieve gas clouds of their "angular momentum problem". This can happen because during fragmentation a fraction of the cloud's angular momentum goes into orbital motion of the fragments.

Figures 11 and 12 show graphically an isothermal collapse evolution that is typical of those evolved to date by researchers using three-dimensional hydrodynamic computer codes. Specifically, the model is identical to the one evolved by Boss and Bodenheimer (1979) in a comparison calculation (a 50% $m = 2$ density perturbation is initially imposed on an $a = 0.25$, $\beta = 0.20$ uniformly rotating, homogeneous $1 M_\odot$ sphere), and the results are from a recent run made by the author on a grid quadrupling the angular resolution over that used by Boss and Bodenheimer. Figure 11 shows how an isodensity surface drawn a factor of 16 down from the cloud's maximum density changes during the initial collapse to an approximately virialized disk. The initial binary-shaped density perturbation does amplify during the collapse, but the flattened disk enclosed by the surface in part c of the figure is fairly homogeneous and does not yet show signs of fragmentation. In Figure 12 the scale of the plots has been changed to illustrate clearly the evolution of the rotationally flattened disk following the initial collapse shown in Figure 11. In each frame the isodensity contour plotted is a factor of 300 down from the maximum density. The disk rapidly breaks up into two distorted chunks of gas that are connected by a common bar. The maximum density in the final frame shown is located inside the two very centrally condensed, spherically shaped knobs at the end of the bar and is a factor of $3 \times 10^4$ higher than the cloud's density at the onset of its collapse. The binary fragments are orbiting about the cloud center at a distance that is roughly 28% of the initial cloud radius.

In many instances, the individual cloudlets that have fragmented from clouds at the end of the hydrodynamic collapse phase are themselves found to be out of virial balance and therefore susceptible to further collapse—perhaps also further fragmentation. Coarse spatial resolution in three-
dimensional hydrodynamic computer codes has prevented the further collapse of individual fragments to be followed. Multidimensional models have therefore not actually been able to address in a comprehensive fashion the question of the ultimate endpoint of hydrodynamic collapse and have been able to say essentially nothing about the accretion phase of a cloud's evolution (see, however, Boss and Black, 1982).

Ultimately, we should expect multidimensional calculations to give considerably more detailed answers to questions about fragmentation and the endpoint of hydrodynamic collapse than are given by the two general results just discussed. However, given the current limitations of the computer codes, particularly their inability to provide desirable spatial resolution at a reasonable financial cost, we cannot yet put much weight in quoted quantitative attributes of "(numerically) observed" cloud fragmentation. Essentially all collapse calculations that have been performed using finite-difference techniques have been aimed at answering the question, "Does fragmentation occur at all?", given a certain limited set of initial cloud conditions. In order to answer this question with a minimum amount of computing, the initial cloud structure has generally received a nonaxisymmetric "kick" or finite "perturbation" in either the cloud's velocity or density structure and growth or damping of the perturbation has subsequently been recorded. When an initially "binary-shaped" density perturbation results in fragmentation of the cloud into a binary system, as presented for example by Bodenheimer, Tohline and Black (1980), Boss (1981a) and here in Figures 11 and 12, the claim can be made that the cloud is susceptible to breakup, but nothing can be said about whether, in the absence of such a well-defined perturbation, the cloud would have chosen to break up into only two pieces.

Using a finite-sized particle technique, Wood (1982) has begun to address the question of what "mode" the cloud prefers to fragment in. In his isothermal collapses, which start from configurations having low-amplitude, random fluctuations in density, he found that as a$_I$ was chosen to have lower values, fragmentation produced a larger number of cloud fragments. It is tempting to suggest that a Jeans mass argument can explain this trend in his results. Figure 9 shows that lower values of a$_I$ cause an isothermal cloud to collapse toward an end point axisymmetric disk of lower final a (higher final $\beta$) which, by Eq. (6.1), contains more Jeans masses of material and hence is susceptible to fragmentation into more pieces. This interpretation is not satisfying, however, because (1) a calculation of the Jeans mass that is based only on the contribution of thermal pressure in opposing gravity is not appropriate in a configuration that obtains a substantial amount of its support from rotational energy, and (2) the dynamical instability that leads to the growth of nonaxisymmetric structure in equilibrium, high $\beta$ configurations (see above) is not directly tied to a Jeans-type instability.
Goldreich and Lynden-Bell (1965) have shown that the wave number of the most unstable mode increases for flatter (higher $\beta$) disks. Wood has suggested that this work may explain the fragmentation trend seen in his results.

Larson (1978) found a fragmentation trend qualitatively similar to that discussed by Wood (1982), but Wood does not confirm the quantitative predictability of collapse fragmentation that has been claimed by Larson. Claims that extend beyond much more than the statement of simple trends cannot be believed with currently available computational techniques and with the limited budgets that are currently at our disposal. Coarse spatial resolution currently prohibits all researchers from making reliable statements about configurations that want to break up into more than, say, four or five fragments.

One final aspect of multidimensional calculations deserves some comment, not because it contributes fundamentally to our understanding of cloud fragmentation but because it has become a much-discussed phenomenon in the recent astrophysical literature. It is ring formation. Beginning with Larson’s (1972) work, many two-dimensional axisymmetric calculations (see above) have shown that a cloud that collapses to a very high $\beta$ configuration evolves dynamically to a self-gravitating, toroidal structure that possesses a density minimum at its center. In isothermal calculations, this torus, or ring, evolves through a stage that resembles in many respects the equilibrium isothermal ring structure modeled by Ostriker (1964; see Black and Bodenheimer, 1976). Tohline (1980b) has used a semi-analytic model of rotating collapse to show that such a ring can be produced as a result of an axisymmetric density wave that naturally propagates outward from the cloud’s rotation axis during the dynamic readjustment phase of the “first disk”. Boss’s (1980b) analytic work supports this model of ring formation. The ring is strongly susceptible to dynamical nonaxisymmetric fragmentation (Norman and Wilson 1978), so in fully three-dimensional calculations it is never a long-lived structure. The appearance of the ring is noticeable as an intermediate stage preceding nonaxisymmetric deformation in many three-dimensional collapse evolutions (even in $n$-body simulations of galaxies; see Miller and Smith, 1979) and its presence has actually been used to categorize the path by which fragmentation of clouds occurs (Bodenheimer et al., 1980; Boss 1981a). Its actual physical importance in any fragmentation scenario is minimal, however, and the astronomical community would certainly not suffer if henceforth reference to the phenomenon of ring formation was minimized accordingly.

Although the physical role of ring formation in the process of star formation is probably minimal, its role in helping numerical analysts to develop accurate finite difference schemes has been enormous. The formation of a ring can be inhibited or enhanced if a hydrodynamic computer code does not accurately conserve angular momentum locally. Bodenheimer and Tschaknuter (1979), for example, discovered that the difference scheme originally used by Tschaknuter (1975) inhibited ring formation, evidently because it artificially transported (via numerical viscosity, see §5.5) angular momentum away from the symmetry axis of the computational grid. In turn, Norman et al. (1981) have shown that the first order accurate “donor cell” difference scheme, used by Black and Bodenheimer (1976) and by some of the currently active multidimensional finite difference programs, artificially transports angular momentum toward the symmetry axis of the computational grid and it may, in situations where the gas cloud is not adequately resolved radially, enhance ring formation. Norman et al. (1981) should be consulted for a description of an Eulerian finite difference technique that improves local conservation of angular momentum. Despite his criticism of the donor cell scheme, Norman (1980) has acknowledged that ring formation does occur in certain realistic physical situations, so the formation of a ring during a collapse evolution in itself does not indicate that poor numerical techniques have been employed. The physical reality of the ring is not in doubt. As mentioned above, however, its physical importance in collapse calculations has certainly been diminished by the general three-dimensional studies which permit fragmentation to occur with or without the formation of a ring.

6.3 Forming Single Stars, Like the Sun

Multidimensional hydrodynamic collapse calculations have served to magnify the importance of angular momentum as a basic hindrance to the formation of single stars, like the sun. From a wide range of reasonable initial conditions it has been found that single gas clouds collapse to rotationally flattened disks that in turn spontaneously break up into binary or multiple stellar systems. Forming a single, slowly rotating star is not easy. In order to avoid fragmenting into a multiple stellar system, a gas cloud must arrive at stellar densities ($\rho \sim 1 \text{ g cm}^{-2}$) with rotation making only a small contribution to its total virialized structure ($\beta < 0.3$). If the cloud strictly conserves angular momentum during collapse, this final structure can be realized only if rotation plays an extremely negligible role initially. Specifically, using Eq. (4.41) and assuming the cloud radius scales as $\rho^{-1/2}$, a one solar mass cloud initially near the Jeans limit ($\rho \sim 10^{-19} \text{ g cm}^{-2}$) must have

$$\beta_0 \lesssim 10^{-7}. \quad (6.4)$$

From Eqs. (4.34), (4.36) and (4.38), this value of $\beta_0$ implies $\omega_0 \sim 10^{-16} \text{ s}^{-1}$, which is unreasonably small. It is two to three orders of magnitude smaller than the angular velocities observed in molecular clouds (see Bodenheimer,
1978, for a summary of these observations) and is even smaller than the angular velocity of our Galaxy’s disk.

Rather than collapsing from configurations that intrinsically have extremely low $\beta$’s, single stars probably form from higher $\beta$ configurations that do not preserve their initial distribution of specific angular momentum during their hydrodynamic collapse and/or subsequent accretion phase. The cloud can either get rid of some of its angular momentum by transporting it outward to the more diffuse interstellar gas that surrounds the cloud as, for example, should occur if magnetic field lines of sufficient strength thread between the cloud and the interstellar medium, or the cloud can redistribute angular momentum within its own volume. Successful redistribution can be achieved by, for example, the action of turbulent viscosity (see §5.5) which transports angular momentum from the cloud center outward, and creates proportionately more low angular momentum material that can fall to the cloud center.

Unfortunately, neither magnetic fields nor turbulent viscosity are expected to be important transport mechanisms during most of the hydrodynamic collapse phase of molecular gas clouds. Once ambipolar diffusion of the magnetic field begins to occur (evidence suggests it starts at densities $10^{-20} \text{g cm}^{-3} \lesssim \rho \lesssim 10^{-18} \text{g cm}^{-3}$; see Mouschovias, 1977), the magnetic field is not tied to the neutral gas sufficiently well to be an effective transport agent (see Mouschovias, 1981, for a recent detailed discussion). Scott and Black (1981) have begun to develop numerical models of collapse including a magnetic field during the phase when ambipolar diffusion is expected to occur, so perhaps more quantitative statements regarding the role of magnetic fields during hydrodynamic collapse will be available in the near future. Also, as has been pointed out by Black and Bodenheimer (1976) and Tschauer (1981), it is unlikely that the viscosity coefficient appropriate for any turbulence will be large enough to provide a significant degree of angular momentum transport during the hydrodynamic collapse phase, as has been envisioned by, for example, Regev and Shaviv (1980). Reasonable estimates imply that the time scale for turbulent transport to be effective $t_0$ [see Eq. (5.17)] is longer than the dynamical time $t_d \sim \frac{[G \rho_0]}{1/2}$.

A single star is therefore likely to form only if significant redistribution of angular momentum occurs during the accretion phase of a cloud’s evolution, and then only if the accretion phase has a substantial portion of the cloud’s evolution. If, for example, an initially low $\alpha$ (very Jeans unstable) cloud evolves from a relatively homogeneous initial configuration, the entire cloud will collapse on a single dynamic timescale and the accretion phase will not last very long. In this case, no angular momentum redistribution can occur and the virialized structure to which the cloud collapses will probably fragment. If, however, a low $\alpha$ cloud is initially quite centrally condensed or if a cloud is initially only barely Jeans unstable ($\alpha \sim 1$) so that it rapidly evolves to a centrally condensed structure (as in the spherical collapses discussed in §6.1), then the first virialized structure—“first core”—to form will possess only a small fraction of the total cloud mass and the accretion phase will be relatively long. The material in this first core, coming from near the cloud’s rotational axis, should have a relatively small specific angular momentum (hence a relatively small $\beta$) and therefore will not fragment immediately. The first core will be able to reach a virialized structure of low $\beta$ ($\alpha \sim 1/2$) only because it has a small mass, as in the spherical collapses discussed in §6.1. Transport mechanisms like turbulent viscosity will have time to become effective in the core because the core is no longer evolving on a dynamic time scale. As the core accretes higher angular momentum material, it can avoid fragmentation into a multiple system, but only if outward transport of angular momentum occurs at a fast enough rate to keep the core at a value of $\beta < 0.3$.

This scenario is the most attractive one developed to date to explain how single stars form from rotating interstellar gas clouds. It is difficult to model numerically in a completely self-consistent fashion, however, because (1) a multidimensional implicit computer program must be used to satisfactorily follow the core accretion phase of a cloud’s evolution and (2) mechanisms that will efficiently transport angular momentum at the rates required by the model are only poorly understood. Tschauer (1981) has begun to study the accretion problem in two dimensions using his spectral implicit technique and a simple representation of turbulent viscosity. Cameron (1978) has also tried to deal with the problem of the accretion of high angular momentum material onto a relatively high $\beta$ solar nebula. He has suggested that non-axisymmetric instabilities in a high $\beta$ disk may exert gravitational torques on the material in such a way as to transport angular momentum outward. This suggestion has been supported by the detailed work of Lin and Papaloizou (1979, 1980) and by the numerical results reported by Durisen and Tohline (1980). Finally, Lin and Bodenheimer (1982) have presented a detailed model of how the late stages of accretion in a solar nebula can be controlled by the action of turbulent viscosity.

It is interesting to note that the angular momentum transport mechanisms that allow a single, low $\beta$ star to form from a rotating cloud will also produce a high angular momentum disk outside the central star that might in turn allow the formation of a planetary system. This scenario is only qualitative in nature at this point. A great deal more work on angular momentum transport processes during the hydrodynamic collapse and accretion phases of a cloud’s evolution must be done before the endpoint of hydrodynamic collapse can be fully understood.
7. COMMENTS ON FRAGMENTATION

There are currently two opposing schools of thought regarding how susceptible gas clouds are to fragmentation during a phase of hydrodynamic collapse. The analytic analyses presented in §3.4 indicate that when pressure gradients are initially negligible, fragmentation should run rampant during a collapse. Hoyle (1953) and many others since him (see Tohline, 1980c, for a review) have, in the absence of a rigorous analysis, tried to generalize this result to gas clouds that begin their evolution near the Jeans limit but evolve along adiabats having $\gamma < 4/3$ so that the pressure-free assumption becomes valid late in the collapse. The opinion is expressed that in the late stages of collapse fragmentation should readily occur in these clouds as well. On the other hand, multidimensional numerical analyses have strongly indicated that clouds which collapse from moderately Jeans unstable configurations ($M_0/M_J \leq 100$) are not susceptible to fragmentation during the initial dynamic collapse phase even if they collapse along $\gamma = 1$ (isothermal) adiabats.

It is vitally important that we understand which one of these two pictures of fragmentation is correct. The population and dynamical characteristics of any resulting star cluster will depend crucially on whether fragmentation of a gas cloud occurs during its phase of hydrodynamic collapse or after the collapse motion has been dissipated and a virialized gas structure attained.

The answer is that both conclusions are probably correct over a restricted range of conditions. As stated by Hunter (1962) and emphasized here in §4.4, the analytic analyses do not apply if $M_0 \sim M_J$ in the initial cloud because strong density gradients develop during the cloud's collapse and pressure gradients are never negligible—even if $\gamma = 1$. The lack of fragmentation in numerical models that collapse from barely Jeans unstable configurations is therefore quite understandable. However, in an initially moderately Jeans unstable cloud—$M_0 \sim 100 M_J$, say—there will be a substantial fraction of the cloud mass that will never feel the pressure wave and its accompanying density gradient impinging from the cloud surface. This central region of the cloud will indeed become more and more closely described by pressure-free conditions during an isothermal collapse and some fragmentation may occur. But it should be remembered that since perturbations require some finite time to grow, the complete isolation of individual fragments from the "parent" cloud as envisioned by Hoyle will succeed only if the (isothermal) evolution proceeds to high enough densities. The multidimensional numerical models have not observed fragmentation during isothermal collapse from moderately Jeans unstable initial configurations, probably for the simple reason that with reasonable amounts of rotation the models have flattened and stopped collapsing at relatively low densities. Along these same lines, it should be emphasized that while Silk (1982) has shown that a collapsing spheroid should fragment sooner (at a lower density) than a corresponding spherical collapse (see §3.4), this effect can be offset by the fact that a flattened spheroid's collapse will be halted by pressure gradients at a lower density than will the sphere's collapse.

The work of Buff, Gerola and Stellingwerf (1979) stands as a unique and useful bridge between the analytic pressure-free analyses and the multidimensional numerical studies of collapse with pressure. In this work a detailed numerical stability analysis was performed during the collapse of a nonrotating isothermal sphere. In a model (their model I) that was initially uniform in density and had $M_0/M_J \sim 35 (\alpha_0 \sim 0.09)$, they found that the central part of the cloud was unstable toward the growth of an off-center density enhancement during the collapse. Being restricted to spherical symmetry, the calculation could not follow the isolation of individual fragments of gas, but the spontaneous growth of a density enhancement in an off-center radial shell strongly suggests that fragmentation of the isothermal model is possible. This work also clearly indicates, however, that complete isolation of a density enhancement will occur only if a collapse remains isothermal through many orders of magnitude increase in the cloud's density. In the specific, moderately Jeans unstable model investigated, the most unstable off-center mode obtained less than a factor of two enhancement over the background cloud density during a period when the background density increased by three orders of magnitude. The collapse would presumably have to proceed to much higher densities before the enhancement became a distinct entity that subsequently was not influenced by the flow of the background medium. At high enough densities, though, a gas cloud will invariably depart from isothermal conditions and enter a period during which pressure gradients become progressively more important upon further compression of the gas. During this period additional amplification of off-center density enhancements may be stifled and/or the hydrodynamic collapse itself may be stopped before fragmentation of the enhancements can be realized.

The multidimensional models discussed in §6.2 actually agree qualitatively with the results of Buff, Gerola and Stellingwerf. Growth of off-center density enhancements has been observed during the hydrodynamic phase of collapse from moderately Jeans unstable configurations (see, for example, Figure 11 of §6.2). But complete isolation of fragments has never been seen before the overall collapse has been stopped and an approximate virial-equilibrium disk established. To date, rotation has caused the multidimensional models to stop collapsing after the background cloud density has increased by at most five or six orders of magnitude over its initial value. Therefore, these numerical models have not really tested fragmentation in regimes where the analytic analyses of Hunter (1962) and Silk (1982) and the work of Buff, Gerola and Stellingwerf have predicted that it should readily
occur. It would be useful, and indeed would be a crucial test of a multidimensional hydrodynamic code, to numerically study fragmentation in a totally pressure-free environment during a collapse that proceeds to high densities. Growth rates of perturbations could be directly compared with analytic predictions and then in subsequent investigations quantitative statements could be made about the retarding effects of pressure under more realistic conditions.

From Figure 10, it appears that the isothermal phase of low mass ($M \leq 3 M_\odot$) protostellar cloud evolutions spans a sufficiently small range of densities that fragmentation will never occur during hydrodynamic collapse from moderately Jeans unstable configurations. It is of interest to note, however, that the $\rho-T$ path for zero metal gas in the early universe is substantially different from the path depicted in Figure 10 (see, for example, Silk, 1977). The cooling, or isothermal phase probably spanned a range of ten to fifteen orders of magnitude in density for protogalactic clouds, so marginally Jeans unstable configurations may very well have experienced some degree of fragmentation during their collapse. It is now clear, though, that the important influence of finite pressure gradients at both the onset and termination of the isothermal phase of collapse would certainly have prevented marginally Jeans unstable protogalactic clouds from fragmenting with the efficiency originally envisioned by Hoyle (1953).

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