Converting Polytropic Units to cgs Units

1. Basics

This is intended to give the collaboration a guide for converting various physical quantities derived from a polytropic evolution into real (cgs) units. Tables 1, 2, 3, and 4 summarize results for the $q_0 = 0.4$, $q_0 = 0.5$, $q_0 = 0.7$, and $q_0 = 1.3$ models, respectively, assuming that in all cases the donor star has a mass of $0.4\ M_\odot$ and the corresponding radius of a helium WD with that mass, *i.e.*, $R = 0.0155\ R_\odot$.

Initially, the two stars were constructed using the following polytropic equation of state:

$$P = K\rho^{1+1/n}\,, \tag{1}$$

with, $n = 3/2$. According to Chandrasekhar (1967), for spherically symmetric, $n = 3/2$ polytropes,

$$K = N_n G M^{(n-1)/n} R^{(3-n)/n} = 0.42422\ G M^{1/3} R\,, \tag{2}$$

$$P_c = W_n \frac{GM^2}{R^4} = 0.77014\ \frac{GM^2}{R^4}\,, \tag{3}$$

$$\frac{\rho_c}{\bar{\rho}} = 5.99071\,, \tag{4}$$

where, M and R are the mass and radius of the star, P_c and ρ_c are the star's central pressure and central density, and the star's mean density is,

$$\bar{\rho} \equiv \frac{3M}{4\pi R^3}\,. \tag{5}$$

Note: The leading numerical coefficients in these expressions will only apply to spherically symmetric, $n = 3/2$ polytropic models. In nonspherical (e.g., binary) configurations, the same scalings with M and R will apply, but the values of these leading coefficients will change.

For a given double-white-dwarf (DWD) model simulation, we will probably want to specify the mass and radius of the donor star, M_d and R_d, then let everything else scale accordingly. More specifically, we probably want to simply specify M_d, then determine the corresponding white-dwarf radius by plugging M_d into the white-dwarf mass-radius relationship; I'll use the one provided by Nauenberg (1972), namely (see also expressions A14 and A15 from Even & Tohline 2009),

$$\frac{R_d}{R_\odot} = \frac{0.0224}{\mu_e}\left\{\frac{[1-(M_d/M_{\rm ch})^{4/3}]^{1/2}}{(M_d/M_{\rm ch})^{1/3}}\right\}, \tag{6}$$

where,

$$\frac{M_{\rm ch}}{M_\odot} = \frac{5.742}{\mu_e^2}\,. \tag{7}$$

Take, for example, the "cgs Units" model whose parameters are listed in column 3 of Table 1. In this example, I have specified $M_d = 0.4\ M\odot$ and $\mu_e = 2$. Hence, $M_{\rm ch} = 1.436\ M_\odot$ and, via the above mass-radius relationship, $R_d = 0.0155\ R_\odot = 1.086\times10^9$ cm. [NOTE: I'm using $M_\odot = 2.0\times10^{33}$ g and $R_\odot = 7.0\times10^{10}$ cm.]

Scaling various masses is straightforward because the SCF model tallies the mass of the donor in polytropic units, $M_{\rm poly}$. Hence, all other masses can be scaled using the relation,

$$\frac{M_{\rm cgs}}{M_{\rm poly}} = \frac{(M_d)_{\rm cgs}}{(M_d)_{\rm poly}}\,. \tag{8}$$

Table 1. Parameter Scalings for Model $q_0 = 0.4085$ (MFTD2007)

Parameter (1)	Poly Units[a] (2)	cgs Units (3)
μ_e	$--$	2.0
	Properties of Donor	
M_d	6.957×10^{-3}	$0.4\ M_\odot = 8.0 \times 10^{32}$
V_d	0.0618	$--$
$R_d = (3V_d/4\pi)^{1/3}$	0.245	$0.0155\ R_\odot = 1.086 \times 10^9$
κ_d	0.01904	2.74×10^{12}
$\rho_d^{\rm max}$	0.71	9.415×10^5
$(\rho^{\rm max}/\bar{\rho})_d$	6.31	$--$
$P_d^{\rm max}$	0.01076	2.47×10^{22}
$(kT/\mu m_H)_d^{\rm max}$	0.0152	$2.63 \times 10^{16} \Rightarrow T_c = 6.38 \times 10^8\ K$
	Properties of Accretor	
M_a	1.703×10^{-2}	$0.979\ M_\odot = 1.96 \times 10^{33}$
V_a	0.1041	$--$
$R_a = (3V_a/4\pi)^{1/3}$	0.292	$0.0185\ R_\odot = 1.29 \times 10^9$
κ_a	0.03119	4.48×10^{12}
$\rho_a^{\rm max}$	1.00	1.32×10^6
$(\rho^{\rm max}/\bar{\rho})_a$	6.11	$--$
$P_a^{\rm max}$	0.0312	7.17×10^{22}
$(kT/\mu m_H)_a^{\rm max}$	0.0312	$5.41 \times 10^{16} \Rightarrow T_c = 1.31 \times 10^9\ K$
	Properties of System	
a_0	0.8169	3.62×10^9
Ω_0	0.2112	0.0627
$P_{\rm orb} = 2\pi/\Omega_0$	29.75	100.0
$J_{\rm tot}$	7.794×10^{-4}	5.22×10^{50}

[a]Parameter values in polytropic units drawn from Table 1 of MFTD2007.

In particular, the mass of the accretor is,

$$(M_a)_{\rm cgs} = \left[\frac{(M_a)_{\rm poly}}{(M_d)_{\rm poly}}\right](M_d)_{\rm cgs} = 0.979\ M_\odot = 1.96\times 10^{33}\ {\rm g}\,. \tag{9}$$

Similarly, any other length in the problem, $\ell_{\rm cgs}$, should be scaled according to the relation,

$$\frac{\ell_{\rm cgs}}{\ell_{\rm poly}} = \frac{(R_d)_{\rm cgs}}{(R_d)_{\rm poly}}\,. \tag{10}$$

The subtle problem with this relation is that the SCF code does not actually tell us what the radius of the donor star is in polytropic units; instead the SCF code quotes the *volume* of the donor, V_d, in polytropic units. This actually makes sense, given that the donor nearly fills its Roche lobe and is therefore not spherical. So how do we determine $(R_d)_{\rm poly}$? Since we do not actually have to have a precise value — after all, we are only trying to get a general idea of what the temperatures and densities are in our polytropic simulations if we scale them to the characteristic size of DWDs — let's assume the donor is spherical and define, $R_d \equiv (3V_d/4\pi)^{1/3}$. As recorded in Table 1, since the SCF code reports that $(V_d)_{\rm poly} = 0.0618$, this gives $(R_d)_{\rm poly} = 0.245$. (We have recorded this number in blue to emphasize that it is not a number taken directly from the SCF model.) Similarly, we determine from the SCF value of $(V_a)_{\rm poly} = 0.1041$ that the approximate radius of the accretor is $(R_a)_{\rm poly} = 0.292$. From scaling relation (10), we therefore conclude that,

$$(R_a)_{\rm cgs} = \left[\frac{(R_a)_{\rm poly}}{(R_d)_{\rm poly}}\right](R_d)_{\rm cgs} = 0.0185\ R_\odot = 1.29\times 10^{9}\ {\rm cm}\,. \tag{11}$$

NOTE: This accretor radius is *way* off of the white-dwarf mass-radius relation. But we're stuck with it because that's the system we put together for the MFTD paper.

Here are some other scaling relations applied to the $q_0 = 0.4085$ model, assuming $M_d = 0.4\ M_\odot$ (NOTE: $m_H/k = 1.211\times 10^{-8}$ cgs):

- $$\rho_{\rm cgs} = \rho_{\rm poly}\left[\frac{(M_d)_{\rm cgs}(R_d)^3_{\rm poly}}{(M_d)_{\rm poly}(R_d)^3_{\rm cgs}}\right] = \rho_{\rm poly}\left[1.326\times 10^{6}\right]$$
- $$\kappa_{\rm cgs} = \kappa_{\rm poly}\left(\frac{G_{\rm cgs}}{G_{\rm poly}}\right)\left[\frac{(M_d)_{\rm cgs}\cdot(R_d)^3_{\rm cgs}}{(M_d)_{\rm poly}\cdot(R_d)^3_{\rm poly}}\right]^{1/3} = \kappa_{\rm poly}\left[1.436\times 10^{14}\right]$$
- $$P_{\rm cgs} = (\kappa\rho^{5/3})_{\rm cgs} = (\kappa\rho^{5/3})_{\rm poly}\left[\left(\frac{GM_d^2}{R_d^4}\right)_{\rm cgs}\left(\frac{R_d^4}{GM_d^2}\right)_{\rm poly}\right] = P_{\rm poly}\left[2.299\times 10^{24}\right]$$
- $$\left(\frac{kT}{\mu m_H}\right)_{\rm cgs} = \left(\frac{P}{\rho}\right)_{\rm cgs} = (\kappa\rho^{2/3})_{\rm poly}\left[\left(\frac{GM_d}{R_d}\right)_{\rm cgs}\left(\frac{R_d}{GM_d}\right)_{\rm poly}\right] = \left(\frac{P}{\rho}\right)_{\rm poly}\left[1.734\times 10^{18}\right]$$
- $$\Omega_{\rm cgs} = \Omega_{\rm poly}\left[\frac{G_{\rm cgs}(M_d)_{\rm cgs}(R_d)^3_{\rm poly}}{G_{\rm poly}(M_d)_{\rm poly}(R_d)^3_{\rm cgs}}\right]^{1/2} = \Omega_{\rm poly}\left[0.2975\right]$$
- $$J_{\rm cgs} = J_{\rm poly}\left[\frac{(GM_d^3R_d)_{\rm cgs}}{(GM_d^3R_d)_{\rm poly}}\right]^{1/2} = J_{\rm poly}\left[6.702\times 10^{53}\right]$$

Table 2. Parameter Scalings for Model $q_0 = 0.500$ (DMTF 2006)

Parameter (1)	Poly Units[b] (2)	cgs Units (3)
μ_e	$--$	2.0
	Properties of Donor	
M_d	3.07×10^{-3}	$0.4\ M_\odot = 8.0 \times 10^{32}$
V_d	0.0814	$--$
$R_d = (3V_d/4\pi)^{1/3}$	0.269	$0.0155\ R_\odot = 1.086 \times 10^{9}$
κ_d	0.016	2.753×10^{12}
$\rho_d^{\max}$	0.235	9.292×10^{5}
$(\rho^{\max}/\bar{\rho})_d$	6.22	$--$
$P_d^{\max}$	0.00143	2.44×10^{22}
$(kT/\mu m_H)_d^{\max}$	0.00609	$2.62 \times 10^{16} \Rightarrow T_c = 6.35 \times 10^{8}\ K$
	Properties of Accretor	
M_a	6.14×10^{-3}	$0.7996\ M_\odot = 1.60 \times 10^{33}$
V_a	0.037	$--$
$R_a = (3V_a/4\pi)^{1/3}$	0.207	$0.01192\ R_\odot = 8.347 \times 10^{8}$
κ_a	0.016	2.75×10^{12}
$\rho_a^{\max}$	1.00	3.95×10^{6}
$(\rho^{\max}/\bar{\rho})_a$	6.02	$--$
$P_a^{\max}$	0.016	2.72×10^{23}
$(kT/\mu m_H)_a^{\max}$	0.016	$6.88 \times 10^{16} \Rightarrow T_c = 1.67 \times 10^{9}\ K$
	Properties of System	
a_0	0.8764	3.539×10^{9}
Ω_0	0.1174	0.0603
$P_{\rm orb} = 2\pi/\Omega_0$	53.52	104.2
$J_{\rm tot}$	1.97×10^{-4}	4.30×10^{50}

[b]Parameter values in polytropic units drawn from Table 5 in DMTF2006.

Table 3. Parameter Scalings for Model $q_0 = 0.7000$ (Wes Even, unpublished)

Parameter (1)	Poly Units[b] (2)	cgs Units (3)
μ_e	$--$	2.0
	Properties of Donor	
M_d	9.761×10^{-3}	$0.4\ M_\odot = 8.0 \times 10^{32}$
V_d	0.10084	$--$
$R_d = (3V_d/4\pi)^{1/3}$	0.289	$0.0155\ R_\odot = 1.086 \times 10^9$
κ_d	0.02512	2.738×10^{12}
$\rho_d^{\max}$	0.6077	9.371×10^5
$(\rho^{\max}/\bar{\rho})_d$	6.28	$--$
$P_d^{\max}$	0.01095	2.46×10^{22}
$(kT/\mu m_H)_d^{\max}$	0.01802	$2.62 \times 10^{16} \Rightarrow T_c = 6.35 \times 10^8\ K$
	Properties of Accretor	
M_a	1.394×10^{-2}	$0.5714\ M_\odot = 1.14 \times 10^{33}$
V_a	0.08501	$--$
$R_a = (3V_a/4\pi)^{1/3}$	0.273	$0.01465\ R_\odot = 1.029 \times 10^9$
κ_a	0.02732	2.98×10^{12}
$\rho_a^{\max}$	1.00	1.54×10^6
$(\rho^{\max}/\bar{\rho})_a$	6.10	$--$
$P_a^{\max}$	0.02732	6.13×10^{22}
$(kT/\mu m_H)_a^{\max}$	0.02732	$3.97 \times 10^{16} \Rightarrow T_c = 9.63 \times 10^8\ K$
	Properties of System	
a_0	0.8394	3.156×10^9
Ω_0	0.20144	0.0646
$P_{\rm orb} = 2\pi/\Omega_0$	31.19	97.2
$J_{\rm tot}$	8.938×10^{-4}	3.32×10^{50}

[b]Parameter values in polytropic units drawn from Wes Even's SCF model in April 2009; see http://gawaine.phys.lsu.edu/sph-grid/wes/q7.pdf.

Table 4. Parameter Scalings for Model $q_0 = 1.323$ (DMTF2006)

Parameter (1)	Poly Units[b] (2)	cgs Units (3)
μ_e	$--$	2.0
	Properties of Donor	
M_d	1.76×10^{-2}	$0.4\ M_\odot = 8.0 \times 10^{32}$
V_d	0.1810	$--$
$R_d = (3V_d/4\pi)^{1/3}$	0.351	$0.0155\ R_\odot = 1.086 \times 10^9$
κ_d	0.0372	2.741×10^{12}
$\rho_d^{\max}$	0.600	9.211×10^5
$(\rho^{\max}/\bar{\rho})_d$	6.17	$--$
$P_d^{\max}$	0.01588	2.39×10^{22}
$(kT/\mu m_H)_d^{\max}$	0.02646	$2.59 \times 10^{16} \Rightarrow T_c = 6.29 \times 10^8\ K$
	Properties of Accretor	
M_a	1.33×10^{-2}	$0.3023\ M_\odot = 6.05 \times 10^{32}$
V_a	0.0799	$--$
$R_a = (3V_a/4\pi)^{1/3}$	0.267	$0.01181\ R_\odot = 8.266 \times 10^8$
κ_a	0.0264	1.95×10^{12}
$\rho_a^{\max}$	1.00	1.54×10^6
$(\rho^{\max}/\bar{\rho})_a$	6.01	$--$
$P_a^{\max}$	0.0264	3.97×10^{22}
$(kT/\mu m_H)_a^{\max}$	0.0264	$2.59 \times 10^{16} \Rightarrow T_c = 6.27 \times 10^8\ K$
	Properties of System	
a_0	0.8882	2.748×10^9
Ω_0	0.2113	0.0676
$P_{\rm orb} = 2\pi/\Omega_0$	29.74	92.9
$J_{\rm tot}$	1.40×10^{-3}	1.95×10^{50}

[b]Parameter values in polytropic units drawn from Table 4 in DMTF2006.