On Beatrice M. Tinsley’s Notes from 1978

This document contains a scanned copy of one of my most cherished possessions from my time in the Astronomy Department at Yale University (mid 1978 – mid 1980): Five pages of handwritten notes from my mentor, Professor Beatrice M. Tinsley, showing that she, too, had given some thought to the implications of a 1/r force-law for gravity as early as 1978. As I recall, when I first mentioned to her that I was seriously looking into the idea that the flat rotation curves of galaxies as well as the velocity dispersion of clusters of galaxies might be simultaneously explained by invoking a 1/r force-law at large distances and was wondering whether or not it would be completely crazy to consider publishing the idea, she turned to her file cabinet, pulled out a thin folder and handed the contents to me.

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(preface dated 3/8/2015)
Grav. force \( \sim \frac{1}{r^2} \) at large \( r \)

Let the accel. be \( a_c = -\frac{GM}{r^2} \)

\( -\frac{d^2a}{G} = \left[ \rho(r) \cdot 2\pi y \cdot r \cdot d\theta \right] \cdot \frac{1}{r^2} \left( 1 + \frac{x}{r_G} \right) \cos \phi \)

where \( y = r \sin \theta \), \( \cos \phi = \frac{d - r \cos \theta}{x} \)

\( \cos \theta = \frac{r^2 + d^2 - x^2}{2rd} \), \( \sin \theta \cdot d\theta = +\frac{yd}{rd}, \ d - r \cos \theta = \frac{r(d^2 - r^2 + x^2)}{2rd} \)

\[ -\frac{d^2a}{G} = \rho(r) \cdot 2\pi r^2 \frac{x}{rd} \int \frac{x}{X^2} \frac{1}{X^2} \left( \frac{d^2 - r^2 + x^2}{2rd} \right)(1 + \frac{x}{r_G}) \]

\[ -\frac{d^2a}{G} = 2\pi \rho(r) \cdot \frac{r^2 d^2}{2rd^2} \left( \frac{d^2 - r^2 + x^2}{2rd} \right) \left( \frac{1}{X^2} + \frac{1}{r_G} \right) \int dx \]

\[ = \frac{1}{4} \frac{dM(r)}{rd^2} \left\{ \frac{(d^2 - r^2)}{(d - r)(d + r)} + 2r - \frac{d^2 - r^2}{r_G} \ln \frac{d + r}{d - r} + \frac{1}{2r_G} \left[ (d + r)^2 - (d - r)^2 \right] \right\} \]

\[ -\frac{d^2a}{G} = \frac{dM(r)}{d^2} + \frac{dM(r)}{r_G d} \left\{ \frac{d^2 - r^2}{4rd^2} \ln \frac{d + r}{d - r} + \frac{1}{2d} \right\} \]

Contrib. from shell of radius \( r \); \( dM(r) = 4\pi \rho(r) r^2 dr \).

Cf. expression if \( dM(r) \) was a point mass \( d \) away:

\[ -\frac{d^2a}{G} = \frac{dM(r)}{d^2} + \frac{dM(r)}{r_G d} \]
Radio \( R = \frac{\text{Exact result}}{\text{pt. mass value}} = \frac{1}{d^2} + \frac{1}{r_c d} \left( \frac{d^2 - r^2}{4 r d} \ln \frac{d}{d-r} + \frac{1}{2} \right) \)

Express this in terms of \( \beta = d/r_c \) and \( \gamma = \frac{r_c}{d} \) \((0 < \beta < 1)\)

\[
R = 1 + \gamma \left( \frac{1}{4 \beta} \ln \frac{1+\beta}{1-\beta} + \frac{1}{2} \right) = \frac{1 + \gamma f(\beta)}{1 + \gamma}
\]

\[f(\beta) \to 1 \text{ as } \beta \to 0 \quad (r \ll d) \quad \text{and} \quad f(\beta) \to \frac{1}{2} \text{ as } \beta \to 1 \quad (r \approx d).
\]

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<th>( \beta )</th>
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As \( \gamma \to 0 \) \((d \ll r_c)\), \( R \to 1 \) : Newtonian core.

As \( \gamma \to \infty \) \((d \gg r_c)\), \( R \to R(\beta) \).

If \( M(r) \) is centrally conc. so that say \( \beta < \frac{1}{2} \), \((r < \frac{d}{2})\), then

\[ f(\beta) = .912 \text{ and } R \geq \frac{1.912}{2} = .96.
\]

:: A first approach (with systematic errors) is to treat a centrally conc. sphere \((r < r_c)\) or one with radius \( R \ll d \) as a point mass.

\[ \mathbf{U}^2 \text{ circ.}(r) = \left| a_c \right| r = \frac{G M(r) + G M(r)}{r} \]

So if \( M(r) = \text{const.} \), inner total mass \( M \) at \( r > R \), and if \( r \gg r_c \), \( \mathbf{U}^2 \text{ circ.} \to \frac{G M}{r_c} = \text{const.} \)
Collapse time for a sphere

Considering force \( p + \) mass still, mass \( m \) fixed:

\[
\frac{d^2 r}{dt^2} = -\frac{Gm}{r^2} - \frac{Gm}{r_0 r}
\]

Usual:

\[
\frac{dt}{dr} = \left(\frac{dr}{dt}\right)^{-1}
\]

\[
\frac{dr}{dt} = \left(\frac{dt}{dr}\right)^{-1}, \quad \frac{d^2 r}{dt^2} = \frac{dt}{dr} \left(\frac{dt}{dr}\right)^{-1} = \frac{d}{dr} \left(\frac{dt}{dr}\right)^{-1} \times \frac{dr}{dt}
\]

Put \( y = \left(\frac{dt}{dr}\right)^{-1} \Rightarrow \frac{dy}{dr} = \frac{dr}{dt} \Rightarrow \frac{d^2 r}{dt^2} = \frac{dy}{dr} \times \frac{dy}{dr} \cdot \frac{dr}{dy} \cdot y \Rightarrow y = \frac{1}{r}, \; y(0) = 0.

\[
\frac{y}{\frac{dy}{dr}} = -\left(\frac{Gm}{r^2} + \frac{Gm}{r_0 r}\right) dr
\]

\[
\frac{1}{2} y^2 = \frac{Gm}{2} \left(\frac{1}{r} - \frac{1}{r_0}\right) - \frac{Gm}{r_0} \ln \frac{r}{r_0}
\]

Let \( r \to \infty \): Hence

\[
\frac{dr}{dt} = y = -\left(\frac{Gm}{r} - \frac{2Gm}{r_0} \ln \frac{r}{r_0}\right)
\]

At \( r = r_0 \):

\[
\frac{dr}{dt} = -\left(\frac{2Gm}{r_0} \ln \frac{r_0}{r_0}\right)
\]

Very roughly, \( t \approx (r_0 + r_0) \sim \frac{r_0}{\sqrt{\frac{2Gm}{r_0} \ln \frac{r_0}{r_0}}}
\]

b) \( r < r_0, \; r_0 \):

\[
\frac{dr}{dt} = \left(\frac{2Gm}{r}\right)^{1/2} \Rightarrow r^{1/2} dr = -\frac{2Gm}{r} t
\]

\[
\frac{2}{3} r_0^{3/2} - \frac{2}{3} r^{3/2} \approx \sqrt{2Gm} (t - t_0)
\]

\[
\text{total } t \sim \frac{r_0}{\left(\frac{2Gm}{r_0} \ln \frac{r_0}{r_0}\right)^{1/2}} + \frac{2}{3} \frac{r_0^{3/2}}{(2Gm)^{1/2}} \quad (r_0 > r_0)
\]

\[
r \approx \frac{2}{3} \frac{r_0^{3/2}}{(2Gm)^{1/2}} \quad (r_0 \ll r_0)
\]

\[
t \approx \sqrt{\frac{1}{6\pi G\rho_0}}
\]
Call the Newton mass collapse time \( t_N \) & the exact time \( t \), then

\[
\frac{t}{t_N} = \left( \frac{R_g}{R_0} \right)^{3/2} + \frac{3}{2} \frac{1}{R_0^{1/2}} \left( \frac{\ln R_0}{\ln R_0} \right)^{1/2} \left( \frac{1}{R_0} \right)
\]

\[
= R_0^{-3/2} + \frac{3}{2} R_0^{-1/2} \frac{1}{\ln R_0} \quad \text{where} \quad R_0 = \frac{R_0}{R_0}.
\]

This \( \downarrow \) as \( R_0 \uparrow \).

Point inside a spherical shell:

\[
y = r \cos \vartheta
\]

\[
\cos \vartheta = \frac{r^2 - d^2 - x^2}{2rd}
\]

\[
m \vartheta \ d\vartheta = \frac{2 \, dx}{rd}
\]

\[
\cos \varphi = \frac{r \cos \vartheta - d}{x} = \frac{r^2 - x^2 - d^2}{2 \, dx}
\]

Central accel: (go away from center)

\[
\frac{d^2 a}{G} = [g(r) \, dr \, 2 \pi y \, r \, d\vartheta] \frac{1}{x^2} \cos \varphi \left( 1 + \frac{x}{R_0} \right)
\]

\[
= \frac{\pi pr^2 \, dr \, \frac{x}{rd} \, \frac{r^2 - x^2 - d^2}{2 \, dx} \left( 1 + \frac{x}{R_0} \right)}{x^2}
\]

\[
= \frac{\pi pr^2 \, \frac{x}{rd} \, (\frac{r^2 - d^2}{x^2} - 1) \left( 1 + \frac{x}{R_0} \right) \, dx}{d^2}
\]

\[
= \frac{dM(r)}{4 \, rd^2} \int_{r-d}^{r+d} \left( \frac{r^2 - d^2}{x^2} - 1 + \frac{r^2 - d^2}{x^2} - \frac{x}{R_0} \right) \, dx
\]

\[
= \frac{dM(r)}{4 \, rd^2 \, R_0} \left[ \left( r^2 - d^2 \right) \ln \frac{r+d}{r-d} - \frac{1}{2} \left( (r+d)^2 - (r-d)^2 \right) \right]
\]

\[
= \frac{dM(r)}{4 \, rd^2 \, R_0} \left\{ \frac{r^2 - d^2}{4 \, rd} \ln \frac{r+d}{r-d} - \frac{1}{2} \right\} = \frac{dM(r)}{R_0^2} \left\{ \frac{1 - d^2}{4 \, \alpha} \ln \frac{1 + \alpha}{1 - \alpha} - \frac{1}{2} \right\}
\]

where \( \alpha = \frac{d}{r} \quad \Rightarrow \quad 0 \leq \alpha < 1 \).
\[ \frac{\text{d}a}{\text{d}t} = \frac{M(r)}{\text{rad}} \left\{ f(x) - 1 \right\} \quad \text{where } f(x) \text{ is the function defined on p.2: } f(x) \to 1 \text{ as } x \to 0 \]
\[ \frac{\text{d}a}{\text{d}t} \to 0 \text{ as } \frac{d}{r} \to 0, \quad r \gg d \]
and \[ \frac{\text{d}a}{\text{d}t} \to \frac{M(r)}{\text{rad}} \left( -\frac{1}{2} \right) \quad \text{as } d \to r \]

ie a force toward from the center, because the parks on the opposite side are relatively more effective than normal.

Not force in a medium of uniform density:

\[ M(d) = \frac{4\pi}{3} pd^3 \]
\[ dM(r) = 4\pi r^2 dr \]

Result (below): \[ a = -\frac{G M(d)}{d^2} - \frac{G M(r)}{\text{rad}} \]

ie less central attraction than expected.

Particle at \( d \) experiences net accel:

\[ \frac{\text{d}a}{\text{d}t} = \frac{M(d)}{d^2} - \frac{4\pi}{\text{rad}} \left\{ \int_0^d \text{dr} r^2 \left[ \frac{d^2 - r^2}{4rd} \ln \frac{d}{r} + \frac{1}{2} \right] - \frac{d}{r} \int_0^d \text{dr} r^2 \left[ \frac{r^2 - d^2}{4rd} \ln \frac{d}{r} - \frac{1}{2} \right] \right\} \]

\[ = -\frac{M(d)}{d^2} - \frac{4\pi}{\text{rad}} \int_0^d \text{dr} r^2 (f' - 1) \]

\[ \frac{\text{d}a}{\text{d}t} \to \frac{M(d)}{d^2} + \frac{3 M(d)}{\text{rad}} \left\{ \int_0^1 \text{d}x (f' - 1) \sum \text{approx} \right\} \]

\[ \frac{\text{d}a}{\text{d}t} \to \frac{M(d)}{d^2} + \frac{M(d)}{\text{rad}} - \frac{6 M(d)}{\text{rad}} \left( \int_0^1 \text{d}x (f' - 1) \sum \text{approx} \right) \]

\[ \frac{\text{d}a}{\text{d}t} \to \frac{M(d)}{d^2} + \frac{M(d)}{\text{rad}} - \frac{6 M(d)}{\text{rad}} \left( \int_0^1 \text{d}x (f' - 1) \sum \text{approx} \right) \text{since } (f' - 1) \text{ is negligible.} \]