Comment on “Exact Bosonization for an Interacting Fermi Gas in Arbitrary Dimensions”

In a recent Letter [1], Efetov et al. propose an exact mapping of an interacting fermion system onto a new model that is supposed to allow sign-problem-free Monte Carlo simulations. In this Comment, we show that their formalism is equivalent to the standard approach of Blackenbecker, Scalapino, and Sugar (BSS) [2] for fermionic systems and has the same sign statistics and minus sign problem.

Our first observation is that the partition function for a given configuration of the auxiliary fields $\phi$ is the same in the standard formulation $Z_f$ [Eq. (8) in Ref. [1]] and in their new bosonized scheme $Z_b$ [Eq. (9)]:

$$Z_f[\phi] = Z_b[\phi].$$

(1)

This observation is trivial in the limit of the time step $\Delta \tau \to 0$, where both schemes reproduce the same partition function. Since $Z_b$ can be negative also in this limit [2], $Z_b$ is also not sign positive. Both $Z_f$ and $Z_b$ are positive if $\phi$ is a smooth path [2], but restricting the configuration space to smooth paths amounts to a semiclassical approximation.

We next show that $Z_f[\phi] = Z_b[\phi]$ also holds for finite $\Delta \tau$ and piecewise constant paths where the field $\phi_\tau$ is constant on the interval $[(l + 1)\Delta, l\Delta]$. Efetov et al.’s Eq. (9) is equivalent to $Z_b[\phi] = \text{Tr}[e^{-\beta H_0} U_j(\beta; 0)]$ where $\frac{\partial}{\partial \tau_1} U_j(\tau_1, \tau_0) = -H_{11}(\tau_1) U_j(\tau_1, \tau_0)$ and $H_{11}(\tau) = -\sum \phi_i(\tau) e^{r \tau H_0} m \tau e^{-r \tau H_0}$. Since $U_j(\tau_2, \tau_0) = U_j(\tau_2, \tau_1) U_j(\tau_1, \tau_0)$, $Z_b[\phi] = \text{Tr}[e^{-\beta H_0} \prod_{l=0}^{N} U_j(l\Delta, (l-1)\Delta)]$. We are left with the task of evaluating $U_j[l\Delta, (l-1)\Delta]$ on the $l$th time interval where the fields are constant and take the values $\phi_i$. For this constant in time field configuration, $U_j[l\Delta, (l-1)\Delta] = e^{l\Delta H_0} e^{-\Delta H_0} e^{-(l-1)\Delta H_0}$ with $H_l = H_0 - \sum \phi_i m \tau$. Hence,

$$Z_b[\phi] = \det\left[1 + \sum_{l=0}^{N} e^{-\Delta h_i}\right] = Z_f[\phi].$$

(2)

We also demonstrate that the proposed formalism is equivalent to that of BSS [2]. Starting from the expression for $Z$ found between Eq. (9) and Eq. (10) and using a matrix notation the partition function reads

$$\ln \frac{Z_b[\phi]}{Z_0} = \int_{0}^{1} du \sum_{\sigma} \text{Tr} \Phi G_{\sigma},$$

(3)

where $\Phi$ is a matrix with elements $\Phi(rl, r'l') = \delta_{r,r'} \delta_{l,l'}$, the trace is over the space time indices, and

$$G_{\sigma} = G^0 + u \sigma G^0 \Phi G^0,$$

(4)

where $G^0$ is the noninteracting Green function. From Eqs. (3) and (4) we get

$$\ln \frac{Z_b[\phi]}{Z_0} = \int_{0}^{1} du \sum_{\sigma} \text{Tr} \Phi G_{\sigma} = \int_{0}^{1} du \sum_{\sigma} \int_{0}^{1} du' \sum_{\sigma'} \text{Tr} \Phi G_{\sigma} G_{\sigma'}^{-1} G^0.$$