Topological Theory of Defects (Reference: Mermin RMP)

To understand defects, we should first ask the question - Defects in what? So, to define defects, we need an ordered ~~space~~ Medium:

Ordered ~~Space~~ Medium - (X)

An ordered medium can be regarded as region of space described by a function $f(\vec{r})$ that assigns to every point of the region an order parameter.

The possible values of the order parameter constitute a space known as the order parameter space.

~~If f is unifor~~

If f is constant, we say that the medium is uniform.

We are, here, interested in a non-uniform media in which order parameter varies continuously through the space, perhaps, at isolated pts, lines, or surfaces.

The singular regions of lower dimensionality constitute defects.

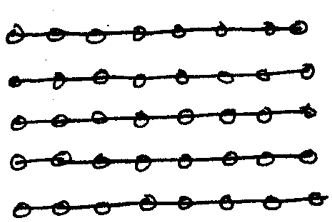
What is the most common example one can think of, for an ordered system?

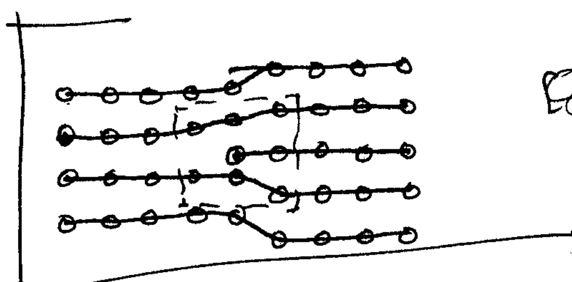
We will take it to a crystal lattice. ~~A crystal has a definite arrangement of atoms which are~~

A perfect crystal consists of a space-filling array of periodically repeated identical copies of a single structural unit.

or

In a perfect crystal, identical copies of a single structural unit are repeated periodically to fill the complete space.





~~[Point defect]~~ in 2-D

Let's define an order-parameter in the following way (for simplicity)

~~Go to a site and see~~

~~$s(\vec{r}) = \sum_i u$~~

$$s(i,j) = \frac{1}{4}\Big(u(i+1,j) + u(i,j+1) + u(i-1,j) + u(i,j-1)\Big)$$

where $u(i,j) = \begin{cases} 1 & \text{if atom is there at site } (i,j) \\ 0 & \text{otherwise.} \end{cases}$

So, what if we remove a half-row in the lattice.

Order-parameter becomes discontinuous in space and we get a defect known as Dislocations.

Following pts should be noted –

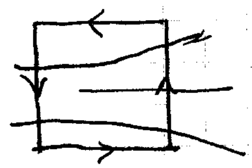
1) This defect is different from the defect in which say one atom is displaced slightly from its position.

That defect can be "cured" by doing some sort of "local" surgery. (Bring back atom to the original position)

Whereas, in the give example, if this defect has to be eliminated, it has to move through the space to great distance. ⇒ Order-parameter has to be altered to large distances, which may not be physically (energetically) favorable.
(This is a salient feature of so-called "topological" defects

2) How can we detect whether there is a defect at all?

Simple :→ Draw a loop around the "core" of the defect



Thus, when we go up, there are three rows crossing the loop & when we come down, there are 2-rows

Thus, we can figure out that there is an extra row to the right.

Any loop of whatever size, we take, there is mis-match of one-row between going-up & coming down. (⇒ This can be used to character-ize this defect)

Already, we have started to make use of loops. I guess that you might have started to have a feel of where we are heading towards, and loops characterized by same number.

Example-2 → (Planar Spins)

Consider a medium of planar spins. Spins are just arrowheads for us.

Order parameter is a vector of fixed magnitude (keep this to be unity) constrained to lie in a plane.

Thus, the OP is a circle (S^1).

Thus, the situation in hand is that we have vectors $\vec{S}(\vec{r})$ as a function of ~~space~~. plane.

→ Now, suppose, we are told that this field is continuous everywhere in plane, except possibly at a pt. P.

→ We are given the explicit form of $\vec{S}(\vec{r})$ everywhere for distances greater than d ?

Can we tell whether there is a singularity at P?

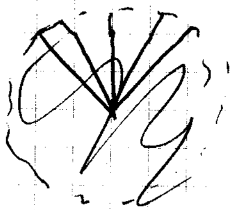
We take analogy from last ~~step~~ example -

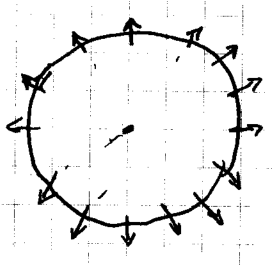
Consider any circle centred at P with radius larger than d

Field $s(\vec{r})$ is then known on the circle and we can easily measure the total angle w.r.t. to some fixed direction through which the vector $\vec{s}(\vec{r})$ turns as $\vec{r}$ traverses the complete ~~[illegible]~~ circular contour. (Sign convention).

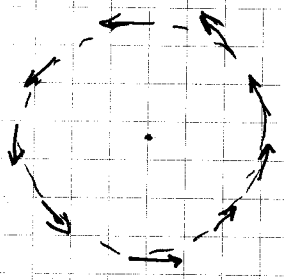
Since $s(\vec{r})$ is continuous on circle, this angle should be an integral multiple of 2π. $(\vec{s}(\vec{r}, \theta + 2K\pi) = \vec{s}(\vec{r}, \theta))$

The integer n is known as the winding number.

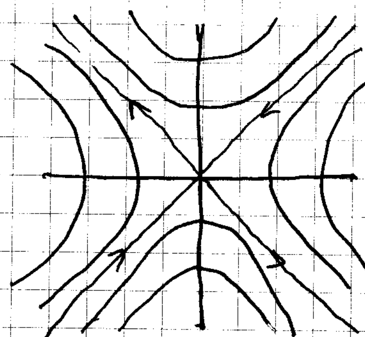




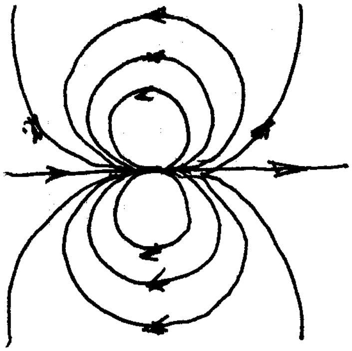
$n = 1$



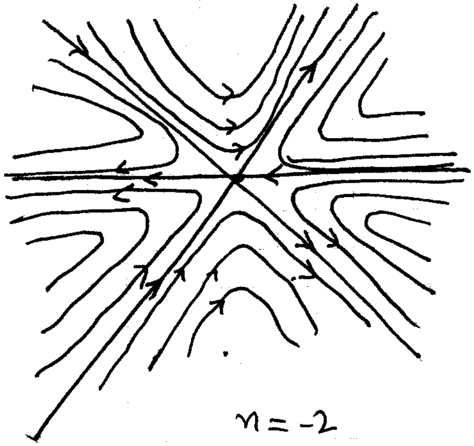
$n = 1$



$n = -1$



$n=2$.



$n=-2$

Now, Let's shrink the circle about P continuously down to an infinitesimal circle about P.

$\because$ n is discrete and $S(\vec{r})$ is continuous, n cannot change on any loop around P as if it has to change, the change will be discrete (discontinuous)

$\Rightarrow$ n is constant for any loop around P.

If $n \neq 0$, $\Rightarrow \vec{s}$ has to turn through an angle of at least 2π, no matter how small the circle becomes.

$\Rightarrow$ Derivative of $\vec{s}$ diverges at P & s is singular at P.

(It is incredible that if there is a singularity with non-zero n, it leaves its signature on the field arbitrarily away from P)

↳ Just like electric field

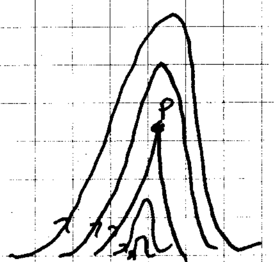
This is not a new observation. Take a point chare.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{q}{\epsilon_0}\delta^3(\vec{r})}$$

If $n = 0$, Does that mean s is non-singular at P?

① No, $\vec{s}$ itself can be discontinuous at P. For ex-



But this can be cured by a technique called local surgery which we will examine soon. » without affecting continuity in the far region

② » If we have remove $n \neq 0$ singularity, winding number must be reduced to zero on all contours surrounding P.

Thus, to remove such singularity, singularity has to be moved across every contour encircling P, no matter how remote.

Let's look at winding number in a slightly different way

specifying the order parameter along a contour in real space is mapping of that contour into OP space.

The winding number is just the number of times that the mapping wraps the closed curve around the circle.

It is quite intuitive - ~~A~~ let us take a rubber band and wrap it around a cylinder. Any mapping with winding number n can be deformed to any other mapping with winding number n but that two mappings with distint winding nos. cannot be deformed into one another.

$\Rightarrow$ <u>Homotopy</u>.

Thus, if two mapping can be continuously deformable from one to another, we say that these are homotopic to each other.

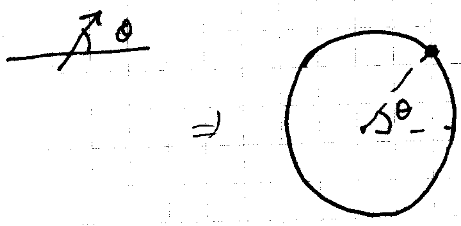
More formally,

$f \simeq g$ if

$\exists\ H : X \times I \to Y$

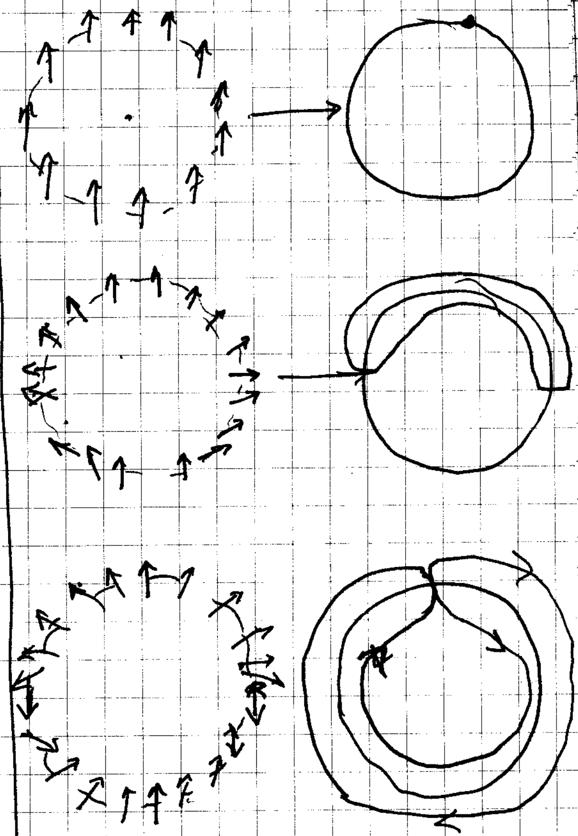
s.t. $H(-, 0) = f$

$H(-, 1) = g$.



Local Surgery:

~~The~~
Result: Let $\vec{S}(\vec{r})$ and $\vec{S}'(\vec{r})$ be two configurations of the order parameter both singular only at P and with same winding number. Then the structure of $\vec{S}$ in the nbd of P can be replaced by that of structure of $\vec{S}'$.



We have thus, succeeded in grouping all singularities into classes (indexed by winding number n) with the property that two singularities in the same class can be deformed into one another by localized alterations in the order parameter, while the singularities in the distinct classes cannot.

→ Topologically equivalent.

Talk about combination of defects

anihilation of defects

Since these classes ~~corresponds to~~ consists of homotopic maps of closed loops, they form a group called homotopy group.

∵ these are maps of closed loops, we get what is known as Fundamental group or First homotopy group.
→ Examples of ① 3-d spins in 2-d ② Nematic Liquid Crystal.
If we consider a pt. defect in 3-d, ~~and consider~~

Then, we have to consider spheres as our contours in place of loops. Then, we would require second homotopy group instead of fundamental group to classify these defects. For example.

① Spins in 3-D → Point Defects.

$\pi_2(S^2) = \mathbb{Z}$ → Hedgehog Defect.