

# The Dollar Auction : A Case Study

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***Index Terms***—Dollar Auction, subgame perfect equilibrium, International escalation

## I. INTRODUCTION

The Dollar Auction is a simple parlor game designed by Martin Shubik ([1]) to illustrate a paradox in non-cooperative behavior brought about by rational choice theory. In this game, players with complete information are trapped into making an ultimately irrational decision based completely on a sequence of rational choices made throughout the game.

The game goes as follow – Two players bid for an object worth a dollar on the condition that both the loser and the winner must pay their highest bids, although only winner can receive the object. The irrationality comes about in the following sense – as the games goes on and the bids are placed, the bids take very large values, even larger than the actual price of the object. Let's say that the bids can only be in the multiples of 5 cents. If player 1 bids 5 cents, player 2 will think that if he bids 10 cents, there is a chance that player 1 drops out and he can make a profit of 90 cents in that case. Thus, the game begins. Now, at some stage, say, player 1 has a bid of 25 cents and player 2 has a bid of 20 cents. Player 2 is better off placing a higher bid than the player 1 as if he drops out at this stage, he is in a loss of 20 cents, but there is a chance that player 1 dropping out, if he place a higher bid. The situation with the player 1 is identical. Both player rationally trying to avoid the loss, eventually end up placing higher and higher bids which generally go higher than the actual price of the object. This behavior is known as *escalation*.

The dollar auction resembles international escalation well in which the governments in conflicts commit resources that will never be returned. For example, the dollar auction can model perfectly the conflict between India and Pakistan over Kashmir issue. Here, Kashmir can be considered to be the dollar being auctioned. The bids in this auction are the military expenses of each nation and the role of the auctioneer can be very well ascribed to advanced industrialized countries, who are the major suppliers of arms.

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In this study, we will study subgame perfect equilibrium of Dollar Auction game. The work described here is due to Barry O'Neill ([2]) who proved that there is a rational strategy which can remove this paradox. Given the wealth of two players, there is a first player bid ( $< 1\$$ ) which will ensure that second player drops out. Thus, according to him, two rational players will never bid against each other. The paradox comes about because the game exploits some irrationality in the players' behavior.

The calculation of *subgame perfect equilibrium* here, requires the extension of our method of *backward induction* as players are faced with a large number of choices in the course of the game and it may not be possible to draw a tree for such a game always. This is done using graph method and subgame are analyzed using sectioning of graph. This is described in great detail in the text.

## II. RULES OF THE GAME

There are two players and an auctioneer in the game who wants to auction an object that has a value  $v$  units. The first bidder is chosen randomly (Call him player 1). After the first bid is placed, the move goes back and forth between the players. On a player's turn, he has to place a bid larger than the last bid made by the other player. Bids can only be in multiple of 1 units – *discrete bids*. The one who passes his move is considered to have dropped out of the game and thus, loses the game. Both winner and loser pay their last bids to the auctioneer but it is only the winner who attains the object. Also, it is assumed that each player's wealth is  $w$  units. No player can bid an amount more than his wealth.

Players are assumed to be rational and to have unlimited foresight. It is assumed that no coalition is possible between the players in this case.

## III. RATIONAL STRATEGY

### A. An Example

Before we move on to state a result due to O'Neill ([2]), we consider a simple example with precise numerical value. Let  $v = 2$ ,  $w = 3$ . We will use the idea of subgame perfect equilibrium to find an optimal solution for the problem. At the start of the game, player 1 has three choices for bidding –  $\{1, 2, 3\}$ . If

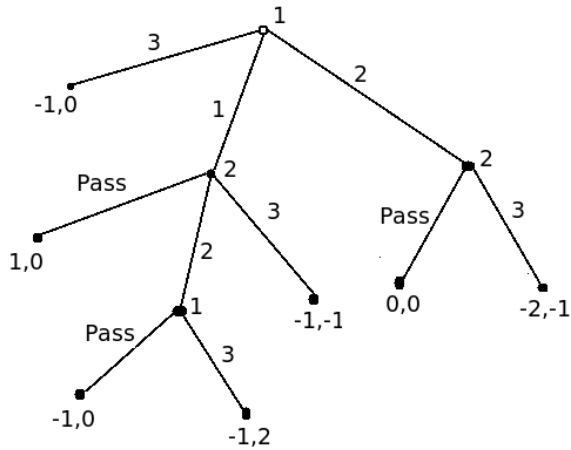


Fig. 1. Extensive form of the example of Dollar Auction considered in section III.A.

player 1 bids 3, he wins the game and the payoff profile is  $(-1, 0)$ . If player 1 chooses 2, move goes to player 2, who has two choices  $-\{3, Pass\}$ . The tree diagram for this extensive game is shown in fig. 1.

The subgame perfect equilibrium is calculated in fig. 2. It is assumed that whenever the two outcomes gives same payoff to a player, the player chooses the one corresponding to the smaller bid. Thus, in the last subgame (subgame in fig. 2 (top)), player 1 is indifferent between *Pass* and 3. He chooses *Pass* as it represent the smaller bid between the two options. This assumption is made in calculating favorable outcomes in other subgames as well. We find that the subgame perfect equilibrium for this game is  $(1, Pass)$  i.e. player 1 bids 1 unit and player 2 drops out.

This result that there is a bid for player 1 for which player 2 drops out is not specific to this problem. There is a general result due to O'Neill ([2]) which is stated and proved below –

### B. General Result

**Theorem 1.** For a dollar auction game with stake  $v$  units and equal wealth  $w$  units for the players, the rational course of action is for player 1 to bid  $(w - 1) \bmod (v - 1) + 1$  units, and for player 2 to drop out.

**Proof**– At a given point in the course of the game, past histories are irrelevant. The game can be represented as a directed graph as shown in fig. 3. The nodes in the graph represent positions in the game and include the information of whose move it is, and what are the current level of bids. This is a shorthand form of representing an extensive form games

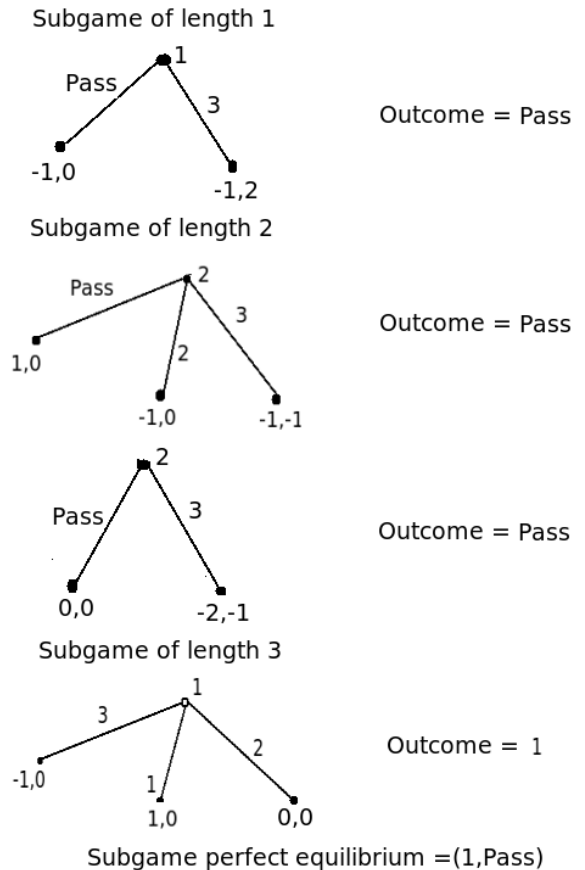


Fig. 2. Calculation of subgame perfect equilibrium of the example of Dollar Auction considered in section III.A.

where there are a large number of options available to each players at each node. The graph consists of a  $(w + 1) \times (w + 1)$  array with rows and columns labelled from 0 to  $w$ . Positions giving player 1 the move are circles, while those giving move to player 2 are squares. The moves are represented by solid and dashed lines for players 1 and 2 respectively. The drop out moves are not shown for simplicity.

To calculate subgame perfect equilibrium of the game, we need to define the concept of a win. A *win* for player  $i$  is defined as any node where player  $i$  would eventually receive the prize if the play started at that node and proceeded rationally. Let's divide the graph into sections as shown in fig. 4. The sections  $C_i$  are one node wide and  $w$  nodes long, while  $B_i$ ,  $D_i$  and  $E_i$  sections are  $v - 1$  wide. Let us now find the winners of nodes in respective sections –

- $C_i$  – All nodes are wins for player  $j$  who has the bid of  $w$ , though it is player  $i$  who has the move in this section.
- $E_i$  – Any node is a win for  $i$  as, at least, he can win  $v$  by raising to his maximum bid. It would

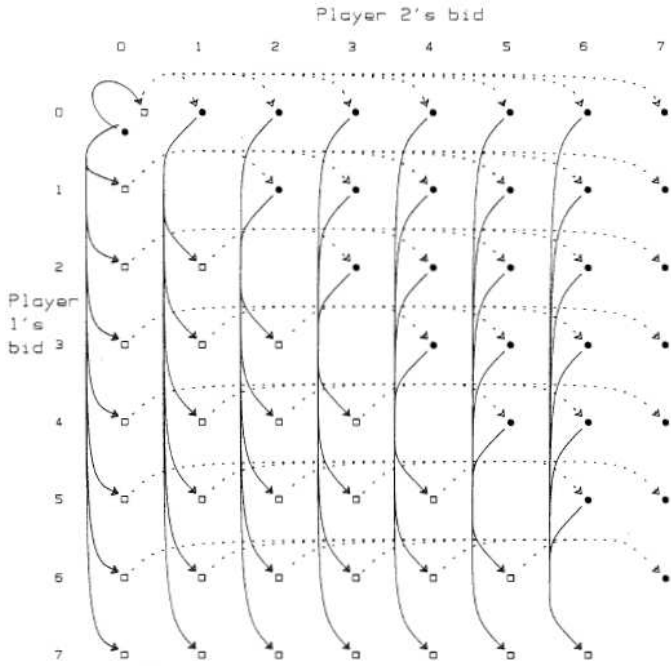


Fig. 3. The graph of a dollar auction game with  $w = 7$ . Positions giving player 1 the move are circles, while those giving move to player 2 are squares. The moves are represented by solid and dashed lines for players 1 and 2 respectively. Drop-out moves are not shown.

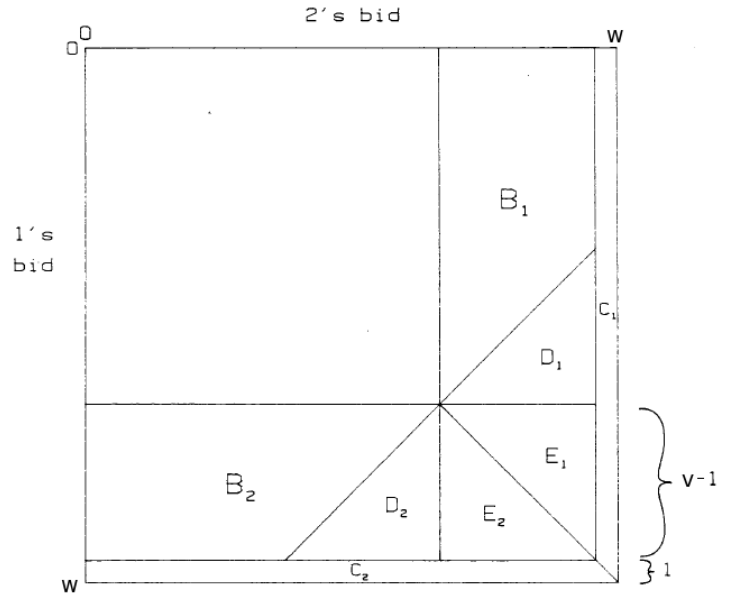


Fig. 4. Sectioning of the graph for Dollar Auction game. It is generalization of the process of analyzing last game in calculating a subgame perfect equilibrium.

be better for  $i$  to make this raise, since the raise is less than  $v$ .

- $D_i$  – At any node, bid of  $i$  cannot exceed the bid of  $j$ , thus, it is better for  $i$  to drop out. Thus, in region  $D_i$ ,  $j$  has the win.
- $B_i$  – Similarly, here also  $j$  is the winner.

Now is the time to apply *backward induction*. Let's construct further sections of the graph as shown in fig. 5. We get areas whose analysis is identical to that of previous ones. This sectioning is repeated until we are left with an unlabelled square that is  $v - 1$  or less wide. This cannot be analyzed in the same manner as we analyzed previous cases as it will either be less than  $v - 1$  wide or if it is exactly  $v - 1$  wide, it will have  $(0, 0)$  (the diagonal element) which was not present in earlier cases.

This square would have dimension  $(w + 1) - (v - 1) - (v - 1) - \dots - (v - 1) = (w + 1) \bmod (v - 1)$  if this quantity is positive, otherwise  $v - 1$ , if it is zero. The choice with player 1 at  $(0, 0)$  is shown in fig. 6. By the earlier arguments,  $C_1^*$  and  $C_2^*$  are wins for 2 and 1 respectively, therefore,  $D_1^*$  and  $D_2^*$  are wins for 1 and 2 respectively. The best move, thus, for player 1 is to bid  $(w + 1) \bmod (v - 1) + 1$  since this is the minimum bid that still represents a win. Player 2 will then drop

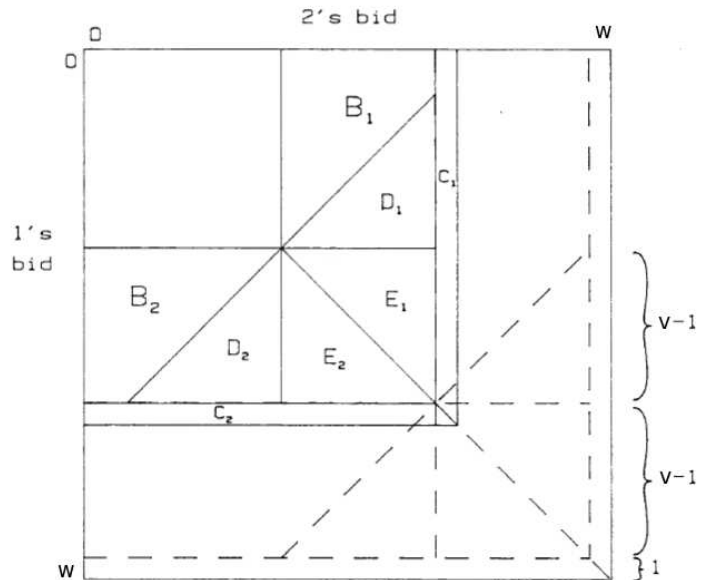


Fig. 5. Further sectioning of the graph for Dollar Auction game. This is a generalization of backward induction process used to calculate subgame perfect equilibrium of the game.

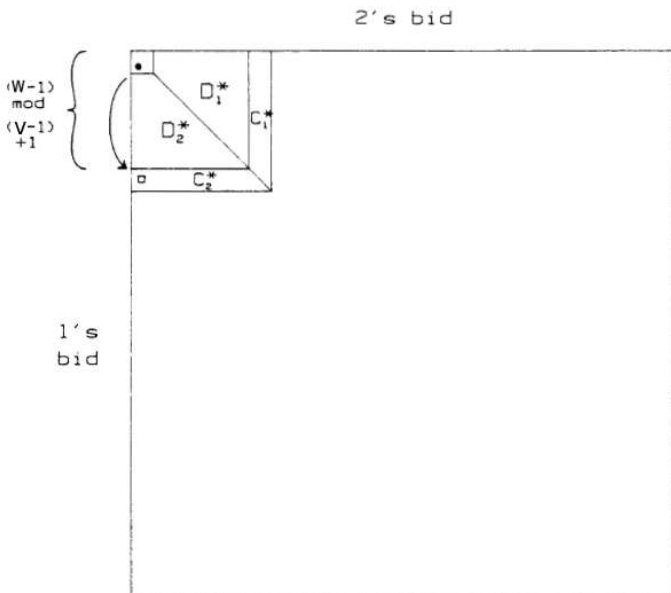


Fig. 6. Analysis of the choices with player 1 at  $(0,0)$ . The analysis is analogous to the analysis of the top game in evaluating subgame perfect equilibrium using induction process.

out since the game has moved to a node that is a win for player 1.

**Corollary.** Let  $v$  and  $w$  approaches infinity with  $w/v = a$ ,  $a$  a constant. Then the best bid for player 1 expressed as a fraction of  $v$  approaches the fractional part of  $a$ . If  $a$  is an integer, the best bid for player 1 in absolute terms equals exactly  $a \forall v > a$ .

Suppose the game sets in, with players bidding differently than prescribed by theorem 1. What would be the subgame perfect equilibrium strategies for players to follow in any subgame? This is given by theorem 2 –

**Theorem 2.** Suppose the current bids by players  $i$  and  $j$  are  $x_i$  and  $x_j$  respectively. Then, if player  $i$  has the move,  $i$  should bid  $(w - x_j - 1) \bmod (v - 1) + x_j + 1$ , if this quantity is less than  $x_i - v$  and drop out otherwise.

This model assumes that bids can be increased in discrete steps only. Leininger ([3]) has considered the case with continuously increasing bids. In that case, they found that there is equilibrium in which rational players can compete against each other and outcome of such a competition is a draw.

#### IV. SUMMARY

We considered a simple game known as Dollar Auction game. It is simple enough for our intuition to visualize what is happening. But at any point in the

course of the game, each player has a large number of choices to choose from. This makes the calculation of subgame perfect equilibrium a cumbersome using the tradition method and even prohibiting for large value of  $w$ . It is the great geometrical ingenuity of O'Neill ([2]) which made our life easy in finding the subgame perfect equilibrium of the game.

We found that the optimal course of action for rational players in the case of discrete bids is not to compete. Competition is possible as an equilibrium outcome in the continuous case where the the final result of the game is a “draw”.

#### REFERENCES

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