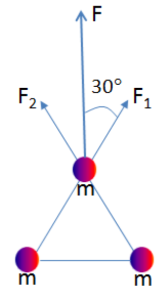
Solutions for Supplementary HW #9: Ch. 13-14

1. In deep space, three spherical objects with the same mass of m = 2.0×10^5 kg separated with equal distance of r = 2 km by forming a triangular structure.
   (a) Find the gravitational force acting on every object.
   (b) Find the total potential energy of them if we assume the potential energy is zero if they are separated with r = ∞.

   Solution:

   (a) As shown in the right figure, the net force acting on one of the mass by other two ones is given by



$$\vec{F} = \vec{F}_1 + \vec{F}_2; \quad F_1 = F_2 = G\frac{m^2}{r^2}$$

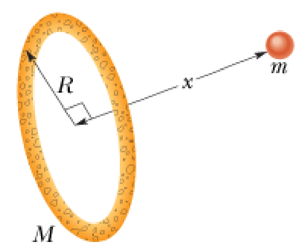
$$F = F_1\cos 30° + F_2 \cos 30° = \sqrt{3}G\frac{m^2}{r^2}$$

   (b) The total potential energy for these three masses is given by

$$U = U_{12} + U_{23} + U_{31}$$

$$= 3U_{12} = 3(-G\frac{m^2}{r}) = -3G\frac{m^2}{r}$$

2. Many galaxies contain ring-like structures. Consider a homogeneous thin ring of mass M and outer radius *R* (see the figure). In addition, there is a spherical object with mass m placed a distance *x* from the center of the ring, along the line through the center and perpendicular to the ring plane.



   (a) Calculate the gravitational potential energy *U* of the all system
   (b) Show that your answer in part (a) reduces to the expected result when $x >> R$. ($U = -GmM/x$)
   (c) Use $F_x = -dU/dx$ to find the magnitude and direction of the force on the object. ($F_x = -GmM\frac{x}{(x^2+R^2)^{3/2}}$)
   (d) Show your answer in part (c) reduces to the expected result when $x >> R$. ($F_x = -GmM/x^2$)
   (e) What are the values of *U* and F_x when $x = 0$? ($U = -GmM/R$, $F_x = 0$)

   Solution:

   (a) Consider a mass segment $dM = (\frac{M}{2\pi R})dl = (\frac{M}{2\pi R})Rd\varphi$ on the ring, where φ is the integrated angle. The potential energy between this segment and the mass *m* is given by

$$dU = -G\frac{mdM}{\sqrt{R^2+x^2}} = -G\frac{m\frac{M}{2\pi}d\varphi}{\sqrt{R^2+x^2}}$$

   so that

$$U = \int dU = -\int_0^{2\pi} G\frac{m\frac{M}{2\pi}d\varphi}{\sqrt{R^2+x^2}} = -G\frac{mM}{\sqrt{R^2+x^2}}$$

(b) As $x >> R$, $\sqrt{R^2+x^2} \approx x$ so that $U \approx -G\frac{mM}{R}$

(c) $F_x = -\frac{dU}{dx} = -\frac{d}{dx}[-G\frac{mM}{\sqrt{R^2+x^2}}] = -GmM\frac{x}{(x^2+R^2)^{3/2}}$ (for $x \neq 0$)

where the "-" represents the attractive force character.

(d) As $x >> R$, $(x^2+R^2)^{3/2} \approx x^3$ so that $F_x \approx -G\frac{mM}{x^2}$

(e) At $x = 0$, the potential energy can be obtained from (a), i.e

$U = -G\frac{mM}{R}$

But we cannot obtain the net force simply from (c) but we should realize from symmetry that the forces acting on mass m (at the center of the ring) from every segment of the mass on the ring cancel each other, thus

$F_{net}(x=0) = 0$

3. Assume the orbital of a lunched satellite around the earth is circular.
   (a) Show that there is only one speed that a satellite can have if the satellite is to remain in an orbit with a fixed radius and it does not depend on the mass of the satellite.
   (b) What is the speed and height H above the earth's surface at which all synchronous satellite (regardless of mass) must be placed in orbit, assuming the earth is spherical?
   (c) If a synchronous satellite has a mass m. How much additional kinetic energy is needed for it to escape from the earth?

Solution:

(a) Since $G\frac{M_E m}{r^2} = m\frac{v^2}{r}$ is the centripetal force for a satellite with mass m,

$v = \sqrt{\frac{GM_E}{r}}$, which is dependent only on radius r but independent on mass m

(b) For a synchronous satellite,

$$24\text{h} = T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM_E}} = \frac{2\pi r^{3/2}}{\sqrt{GM_E}}$$

$$r = R_E + H = (\frac{\sqrt{GM_E}T}{2\pi})^{2/3}$$

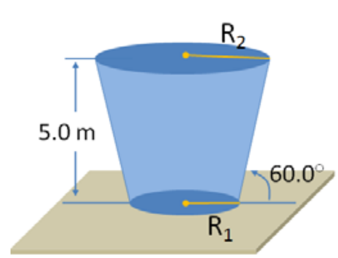
$$H = (\frac{\sqrt{GM_E}T}{2\pi})^{2/3} - R_E$$

(c) Based upon the condition that $\left(\frac{1}{2}mv_{escape}^2 - \frac{GmM_E}{r}\right) = 0$

$v_{escape} = \sqrt{\frac{2GM_E}{r}}$ with $r = (\frac{\sqrt{GM_E}T}{2\pi})^{2/3}$ for asynchronous satellite, so that

$$v_{escape} = \sqrt{2}(\frac{2\pi GM_E}{T})^{1/3}$$

4. As shown in the figure, a pond has the shape of an inverted cone with the tip sliced off and has a depth of 5.0 m. The atmospheric pressure above the pond is 1.01 × 10[5] Pa. The circular top surface (radius = R_2) and circular bottom surface (radius = R_1) of the pond are both parallel to ground. The magnitude of the force acting on the top surface is the



same as the magnitude of the force acting on the bottom surface. Find (a) R_2 and (b) R_1.

Solution:

Since $F^{top} = P_{atm}A^{top} = \pi R_2^2 P_{atm}$; $F^{bottom} = P_{bottom}A^{bottom} = \pi R_1^2 P_1$

$P_1 = P_{atm} + \rho gh$

$R_1 = R_2 - h\cot 60°$

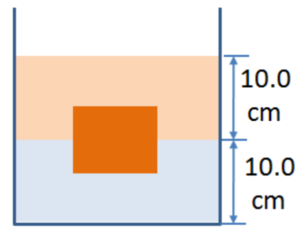
Because $F^{top} = F^{bottom}$

$P_{atm}R_2^2 = P_1 R_1^2 = P_1(R_2 - h\cot 60°)^2$

$(P_1 - P_{atm})R_2^2 - 2hP_1\cot 60° R_2 + P_1 h^2 \cot^2 60° = 0$

Find R_2 from above quadratic equation and then R_1

5. A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (see the cross section view in the figure). The density of the oil is 790 kg/m³. (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure of the lower face of the block? (c) What is the mass and density of the block? (m = 0.821 kg, ρ = 821 kg/m³)



Solution:

(a) The gauge pressure at the top is

$P_{top} - P_0 = \rho_{oil} gh = 790kg/m^3 \cdot 9.8m/s^2 \cdot 0.015m = 116\text{Pa}$

(b) The gauge pressure at the top is

$P_{bottom} - P_0 = \rho_{oil} gh_{oil} + \rho_{water} gh_{water}$

$= 790kg/m^3 \cdot 9.8m/s^2 \cdot 0.1m + 1000kg/m^3 \cdot 9.8m/s^2 \cdot 0.015m$

$= 921\text{Pa}$

(c) Since $\sum F_y = 0$

$P_{bottom}A - P_{top}A - mg = 0$

$$m = \frac{(p_{bottom} - P_{top})A}{g} = \frac{(911\text{Pa} - 116\text{Pa}) \cdot 0.1m^2}{9.8m/s^2} = 0.821kg$$

$\rho = m/V = 821kg/m^3$