Sonic band structure in fluids with periodic density variations

Jonathan P. Dowling
Weapons Sciences Directorate, AMSMI-RD-WS-ST, Research, Development, and Engineering Center, U.S. Army Missile Command, Redstone Arsenal, Alabama 35898-5248

(Received 19 December 1991; accepted for publication 4 February 1992)

In direct analogy to the electronic band structure found in semiconductors and the photonic bands for light in a medium with a periodic dielectric constant, a periodic density variation in a fluid can give rise to sonic frequency passbands and band gaps. Hence, a fluid medium can be constructed that prohibits sound propagation at certain frequencies while allowing practically free propagation at others. The effect of a sonic band medium on a monopole acoustic source is discussed in a simple one-dimensional model. In particular, the complete quenching of radiated power is seen for a harmonic radiator at a frequency that corresponds to a band gap—in analogy with a similar effect that is predicted for the atomic emission of electromagnetic waves in a photonic band structure. The ability to construct a medium that selectively prohibits sound propagation and emission in a certain range of frequencies, while allowing transmission and enhanced radiation rates at others, could have interesting practical applications.

PACS numbers: 43.20.Rz, 43.20.Fn, 43.20.Ks, 43.30.Jx

INTRODUCTION

One of the most fruitful concepts in solid-state physics is that of the electronic band structure of semiconductors.\(^1\) The basic idea is that the wave function of an electron moving through a periodic three-dimensional potential cannot have an arbitrary wave number \(\mathbf{k}\), but rather is confined to move in electronic energy bands where the dispersion relation connecting \(\mathbf{k}\) to the electron total energy \(E\) has in fact a solution. Although many important features of electronic band theory depend explicitly on a quantum treatment, the existence of the band structure itself depends on quantum mechanics only indirectly in that an electron is treated as a scalar wave with a wavelength given by the De Broglie relation \(\lambda = h/p\), where \(h\) is Planck's constant and \(p\) the electron momentum. Since the existence of passbands and band gaps depends only on the solution of a scalar wave equation in a periodically varying potential, one would expect to find band structure generally in the study of wave propagation in a periodic medium.

Very recently there has been a great deal of excitement in the optical community concerning the extension of these ideas to electromagnetic waves, a process that has led to the theory and experimental realization of photonic band materials and photon localization in these materials.\(^2\) Even though the word "photon" appears, the effect is primarily classical and depends upon the solution of the electromagnetic wave equation in a material with a periodically varying dielectric constant.\(^3\)\(^4\) So saying, it would seem that one should be able to realize "sonic" bands and band gaps in, say, a fluid with a periodically varying density. (Alternatively, one could consider pressure variations so long as the net effect is that the speed of sound varies periodically.) We shall show here that, in principle, this is indeed possible. In the case of photonic materials, one of the more interesting and perhaps practical properties of the photonic band structure is its ability to alter the radiative properties of excited atoms that are undergoing decay. In fact, atoms that would normally radiate at a frequency corresponding to a photon band gap will have their radiation rate quenched—they cannot decay. Atoms in the photon band on the other hand can enhance emission rates.

In this work, we shall derive the dispersion relation showing sonic band structure in a simple one-dimensional structure similar to the Kronig–Penney model of solid state theory.\(^1\) We then analyze the power radiated by a monopole harmonic source in the sonic band material and show that emission quenching occurs in the band gaps while both suppression and enhancement takes place in the bands. This is in very close analogy with what occurs in the electromagnetic situation.

The ability of a sonic band material to conduct sound waves only in certain frequency ranges and to inhibit and enhance the radiation rate of acoustic sources could have many interesting applications. It seems likely also that the results presented here are easily extendable to sound waves in solids with a latticelike bulk modulus variation. In the next section, we derive the existence of sonic bands and band gaps for the simple one-dimensional fluid model. Then in Sec. II, we consider the effect of this sonic band structure on a monopole radiator as a function of the monopole's frequency and location in the periodic medium.

I. SONIC BANDS: A SIMPEL MODEL

In analogy to the Kronig–Penney model for electronic bands in semiconductors, we present a simple one-dimensional model for a sonic band material. We imagine an infinitely long and thin tube of fluid that has periodic (of period \(d\)) \(\delta\) functionlike increases in density along its length (see Fig. 1). (We could just as well consider decreases in density of the same form.) The normal-mode solutions \(\varphi_n(x)\) along the tube obey the Helmholtz equation,

\[
\varphi''_n(x) + F(x)\varphi_n(x) = 0,
\]

where
FIG. 1. A one-dimensional model of a sonic band structure is considered in an infinitely long thin tube of fluid in which sharp increases of fluid density occur periodically with period $d$. The effect of the periodic density increases on the Helmholtz equation is modeled by a periodic array of $\delta$ functions, $\delta(x - nd)$, $n = 0, \pm 1, \pm 2, \ldots$. In Sec. I, we consider the power radiated by a harmonically oscillating monopole sound source located between $x = 0$ and $x = d$. We represent the monopole radiator here by a longitudinally oscillating cylinder aligned with the tube and with position $x_0$. For certain frequencies $\omega$ of the oscillator the emitted radiation will lie in a forbidden frequency band gap and no net power will be radiated. Emissions at passband frequencies can be either enhanced or inhibited—depending on the location of the radiator in the tube.

$$F(x + nd) = F(x) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

is a periodic function of $x$ with period $d$, as indicated in Fig. 1. This function expresses the periodic density changes in terms of periodic fluctuations of the eigenvalues of the Helmholtz equation. The explicit form of $F(x)$ we shall take as

$$F(x) = K^2 \left( 1 + gd \sum_{n = -\infty}^{\infty} \delta(x - nd) \right),$$

where $g$ is a measure of the strength of $\delta$ functions located at $nd$, $n = 0, \pm 1, \pm 2, \ldots$, and $K = \omega/c$ corresponds to the wave number in the bulk medium between the $\delta$ functions. From the Bloch–Floquet theorem concerning the solutions of a periodic potential problem, the two independent solutions of Eq. (1) have the form,

$$\Psi_{\pm}(x) = e^{\pm ikx} u_{\pm}(x),$$

where $u_{\pm}(x)$ are periodic with period $d$. Concentrating on the two intervals $I = (0, d)$ and $II = (-d, 0)$, we now specify the boundary conditions at $x = 0$. We have

$$\Psi_I(0^+) = \Psi_{II}(0^-),\quad (5a)$$

$$\Psi_I(0^+) - \Psi_{II}(0^-) + gdK^2 \Psi(0) = 0,\quad (5b)$$

where Eq. (5b) is obtained by integrating Eq. (1) over an infinitesimal interval $(-\epsilon, \epsilon)$ and then allowing $\epsilon \to 0^+$. The boundary conditions at all the other $\delta$ functions follow from periodicity.

Using Eq. (4) andBloch’s theorem, we may write the solutions in regions I and II as

$$\Psi_I(x) = Ae^{iKx} + Be^{-iKx},\quad (6a)$$

$$\Psi_{II}(x) = e^{-ikd} \left\{ A e^{iK(x + d)} + Be^{-iK(x + d)} \right\},\quad (6b)$$

where $K = \omega/c$ is determined by the frequency in the bulk medium and $k$ is the wave number corresponding to waves propagating through the periodic material. Inserting the wave functions, Eqs. (6), into Eqs. (5) for the boundary conditions we obtain the following matrix equation for the coefficients $A$ and $B$,

$$\hat{M} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\hat{M} = \begin{pmatrix} 1 - e^{ikd} & 1 - e^{-ikd} \\ iK(1 - e^{-ikd}) + gdK^2 & iK(1 - e^{-ikd}) + gdK^2 \end{pmatrix},$$

and where we have defined $k \equiv k \pm K$. An equation such as Eq. (7) has an infinite number of nontrivial solutions if and only if the determinant $|\hat{M}| = 0$, which yields the following dispersion relation between the lattice propagation wave number $k$ and the bulk frequency $\omega = Kc$, where $c$ is the velocity in the bulk medium:

$$\cos kd = \cos Kd - (g/2) Kd \sin Kd \equiv f_g(Kd).$$

It is clear from the condition $|\cos kd| < 1$ that this relation can only be satisfied for $|f_g(Kd)| < 1$. For frequency values of $Kd$ where this condition holds, propagation through the lattice is allowed; for a range of $Kd$ where it is violated, we have a frequency band gap and sound transmission is proscribed. These results are indicated schematically in Fig. 2 for a particular value of the $\delta$ function strength $g = 2$.

Although this proof of the existence of sonic band gaps is one dimensional, the scalar nature of the sound field is closely paralleled by the scalar (spinless) Schrödinger electron wave function that experiences energy band structure in many three-dimensional periodic potentials. It would seem reasonable that a fluid with a density variation that was, say, face-centered cubic could be constructed that would have complete sonic band gaps for all directions $k$. If so, sound propagation in the band gap frequencies would be blocked in all $4\pi$ steradians of solid angle. Such a material could be as useful in acoustics as the photonic analogue is in optics. It should be a simple matter to adapt electronic band calculations to the acoustic case—in view of the scalar nature of the two fields—in order to search for a three-dimensional sonic band gap. For the scalar electromagnetic wave equation, for example, a complete band gap occurs in a face-centered-cubic arrangement.

II. POWER OUTPUT OF A MONOPOLE IN A SONIC BAND STRUCTURE

In order to probe the sonic band structure computed in Sec. I, we consider the steady-state power output of a point source in an infinitely long thin tube of fluid in which sharp increases of fluid density occur periodically with period $d$. The effect of the periodic density increases on the Helmholtz equation is modeled by a periodic array of $\delta$ functions, $\delta(x - nd)$, $n = 0, \pm 1, \pm 2, \ldots$. In Sec. I, we consider the power radiated by a harmonically oscillating monopole sound source located between $x = 0$ and $x = d$. We represent the monopole radiator here by a longitudinally oscillating cylinder aligned with the tube and with position $x_0$. For certain frequencies $\omega$ of the oscillator the emitted radiation will lie in a forbidden frequency band gap and no net power will be radiated. Emissions at passband frequencies can be either enhanced or inhibited—depending on the location of the radiator in the tube.

FIG. 1. A one-dimensional model of a sonic band structure is considered in an infinitely long thin tube of fluid in which sharp increases of fluid density occur periodically with period $d$. The effect of the periodic density increases on the Helmholtz equation is modeled by a periodic array of $\delta$ functions, $\delta(x - nd)$, $n = 0, \pm 1, \pm 2, \ldots$. In Sec. I, we consider the power radiated by a harmonically oscillating monopole sound source located between $x = 0$ and $x = d$. We represent the monopole radiator here by a longitudinally oscillating cylinder aligned with the tube and with position $x_0$. For certain frequencies $\omega$ of the oscillator the emitted radiation will lie in a forbidden frequency band gap and no net power will be radiated. Emissions at passband frequencies can be either enhanced or inhibited—depending on the location of the radiator in the tube.
FIG. 2. We plot here as a solid curve, the right-hand side of the dispersion relation (9), denoted by $f(\alpha)$ where $\alpha = \omega d / c = K_d$. We take a $g$ function strength of $g = 2$. The mode frequency $\omega$ is related to the corresponding wave number $k$ through Eq. (9). We also plot the two lines $f(\alpha) = \pm 1$ as dashed lines. From the condition $|f(\alpha)| < 1$, we can see that sonic band gaps will occur when this inequality is violated and sonic passbands occur when it is obeyed. If we plot the two step functions $1 \pm (f(\alpha))$ as solid lines then we see enclosed by solid rectangles the first three passband regions, each of which begins at $\alpha = n\pi, n = 0, 1, 2$. The condition $\alpha = n\pi$ implies $d = n(2/\pi)$ and hence occurs when an integer number of half-wavelengths of a normal-mode wave function fits across one period interval of length $d$.

A monopole radiator in the tube of fluid with a periodically varying density, Fig. 1. A basic tenet of field theory states that the energy stored in a field originates in the work done by the source against the field—here a field that originated from the source itself at earlier times. The idea is to use the Green’s function appropriate for the geometry of the periodic medium to propagate the field from early to current times. The total energy $H(t)$ emitted by a localized source density $\rho(r,t)$ up to time $t$, assuming the source was quiescent at $t = -\infty$, can be written as

$$H(t) = \sigma \int_V \int_{-\infty}^{t} \int_{-\infty}^{t} \rho(r,t') G(r,t';r',t'') \rho(r',t'') \, dr' \, dt' \, dr''$$

or

$$H(t) = \frac{1}{2} \sigma c^2 \int d^n k \left[ \int dV \int_{-\infty}^{t} \int_{-\infty}^{t} \rho(r,t') \varphi_k(r) e^{-i\omega t'} \right]^2$$

Here, $\sigma$ is the mean fluid density in the region between the $\delta$ functions, $G$ is the partial derivative of the retarded propagator $G(r,t';r',t'' \ast)$ with respect to $t'$, the volume of integration is $V$ with volume element $dV$, $n$ is the spatial dimension of the system, and the mode frequency $\omega = \omega(k)$ is a function of $k$ via the dispersion relation. In arriving at Eq. (10b), we have made use of the normal-mode expansion,

$$G(r,t';r',t'' \ast) = \int d^n k \varphi_k^* (r') \varphi_k (r) \cos(\omega(t' - t''),)$$

that allows Eq. (10a) to factorize neatly into Eq. (10b), if we rewrite the factor $\cos[\omega(t - t'')] = Re\{\exp[-i\omega(t - t'')]\}$ and then realize that the real part prescription is not needed upon taking the modulus squared. We see now how the normal-mode functions $\varphi_k(x)$—that depend only on the geometry, boundary conditions, and the inhomogeneous medium—will determine the energy output from the source density $\rho$ up to time $t$. The source will be immersed in sound waves that were emitted at earlier times and have undergone multiple reflections off the lattice—these reflections will influence the radiation rate and are completely accounted for by the Green’s function.

In order to apply Eqs. (10) to a monopole sound source in the one-dimensional lattice, we need the lattice Bloch wave functions $\varphi_k(\omega)$. We will work in the region 1 where $x \in (0,d)$ and then invoke periodicity to extend the result throughout the tube of periodically varying density. Hence, we may take the form of Eq. (6a) for our eigenfunctions. Nontrivial solutions for the coefficients $A$ and $B$ can be obtained from Eq. (7) so long as the dispersion relation (9) holds. Since there are infinitely many solutions if Eqs. (4) and (9) maintain, we are free to choose one of $A$ or $B$ (say $A$) arbitrarily as a nonzero free parameter. We take

$$A = \frac{(1 - e^{-\omega t})}{\sqrt{8\pi (\sin^2 \gamma + \sin^2 \gamma')},}$$

and that implies

$$B = \frac{(1 - e^{-i\omega t})}{\sqrt{8\pi (\sin^2 \gamma + \sin^2 \gamma')},}$$

where we have defined the unitless quantities $\alpha = K_d = \omega d / c, \beta = k d$, and $\gamma = \beta + \omega$. The dispersion relation (9) now reads,

$$\cos \beta = \cos \alpha - g \alpha \sin \alpha.$$

This choice of $A$ properly normalizes the normal-mode wave functions,

$$\varphi_k(\xi) = Ae^{i\alpha \xi} + Be^{-i\alpha \xi},$$

where $(\xi \equiv x/d)$ such that the usual completeness relation holds, namely,

$$\int_{-\infty}^{\infty} d\xi \varphi_k(\xi) \varphi_k^*(\xi) = \delta(k' - k),$$

for all $k$ and $k'$ that are in the frequency passbands allowed by the dispersion relation (13). In other words, if $k$ is in a band gap then $A \equiv B \equiv 0$ and only the trivial solution $\varphi_k(x) \equiv 0$ is permitted for Eq. (14).

For a harmonically oscillating monopole sound source that turns on at $t = 0$, we take

$$\rho(x,t) = q e^{i\omega t} \delta(x - x_0) \theta(t),$$

where $q$ is the monopole strength, $\omega$ its frequency, $x_0 \in (0,d)$ its location, and $\theta(t)$ a unit step function that turns it on at $t = 0$. Inserting Eqs. (16) for the source and (14) for the wave functions, with the definitions (12) for $A$ and $B$, into (10b) for the total energy output, we obtain

$$H(t) = 2q^2 \sigma c^2 \int_{-\infty}^{\infty} dk |\varphi_k(x_0)|^2 \left[ \frac{\sin[(\Omega - \omega)t/2]}{\Omega - \omega} \right]^2.$$
From this we can obtain the power $P(t)$ emitted at time $t$ by derivation, hence

$$P(t) = H'(t) = \pi q^2 \sigma c^2 \int_{-\infty}^{\infty} dk \left| \varphi_k(x_0) \right|^2 \frac{\sin \left[ (\Omega - \omega) t \right]}{\pi (\Omega - \omega)} ,$$

(18)

where the last factor in Eq. (18) is a representation of $\delta(\Omega - \omega)$ in the limit $t \to \infty$.

Since we are precisely interested in this steady-state situation where all initial transients have died away, we take this limit and arrive at

$$P_\text{lattice} = \lim_{t \to \infty} P(t) = \pi q^2 \sigma c \left| \varphi_k(x_0) \right|^2 \delta(\Omega - \omega) ,$$

(19)

where it is assumed throughout that $\omega$ (and hence $\Omega$) is related to the propagation wave number $k$ through the dispersion relation. In the band gap, we have $|q_k(x_0)|^2 \equiv 0$ and no steady-state radiative power flux occurs.

In order to compare the power output in the lattice to the bulk medium (one-dimensional) steady-state power, we take the bulk medium, plane-wave, normal-mode wave functions,

$$\varphi_k^{\text{bulk}}(x) = (1/\sqrt{2\pi}) e^{-ikx} ,$$

(20)

and insert them into the energy equation (10b) using the same source (10). We arrive at

$$P_\text{bulk} = \frac{p}{\omega} = \frac{p_\text{lattice}}{p_{\omega}^{\text{full}}} = 2\pi \int_{-\infty}^{\infty} dk \left| \varphi_k(x_0) \right|^2 \delta(\Omega - \omega) \left[ 1 - \frac{1}{2} \cos 2\alpha \xi - 2 \cos \beta \cos \left[ 2\alpha(\xi - 1/2) \right] + \cos \left[ 2\alpha(\xi - 1) \right] \right]$$

$$\times \sin \alpha + (g/2) \left( \sin \alpha + \alpha \cos \alpha \right) \frac{\sin \beta}{\sin \beta} \left[ 1 - |f_\delta(\alpha)| \right] ,$$

(22)

where $\beta \equiv kd$ is the wave-number parameter, $\alpha \equiv \Omega d/c$ is the monopole frequency parameter, $\xi \equiv x/d$ is the monopole position parameter, and the step function $\delta(x)$ explicitly enforces the band gaps of the dispersion relation Eq. (13) via the function $f_\delta(\alpha)$ defined in Eq. (9). (We have dropped the subscript naught on the monopole position $x_0 \to x$ since no other position now occurs. The parameter $\alpha$ henceforth is used for the monopole frequency.) The $\beta$ and $\alpha$ are related implicitly by Eq. (13) for the dispersion relation.

Physically what has occurred is that in the limit $t \to \infty$ the $\delta(\Omega - \omega)$ has selected out only one normal-mode frequency at $\omega = \Omega$ that controls the steady-state power output of the monopole of frequency $\Omega$. However, this mode propagates with a wave number $k$ that is determined from the dispersion relation. In terms of the unitless parameters: fixing the oscillator frequency with $\alpha \equiv \Omega d/c$ selects out the wave numbers $|k| = \beta/d$ as a function of $\alpha$ through the dispersion relation. Since the dispersion relation (9) or (13) is easily invertible to solve for $\beta$ in terms of $\alpha$ ($k$ in terms of $\Omega$), but not the reverse, we shall consider $p = p(\alpha, \xi)$, Eq. (22), as a function of the unitless monopole frequency parameter $\alpha$ and the position parameter $\xi \in (0,1)$.

For a $\delta$ function strength of $g = 2$, we plot in Fig. 3 a two-dimensional surface $p(\alpha, \xi)$ over the domain $\alpha \in (0,8)$ and $\xi \in (0,1)$. We see that the power output is identically zero in the sonic band gaps where $|f_\delta(\alpha)| > 1$. Inside the sonic bands, $|f_\delta(\alpha)| < 1$, we see suppression and enhancement of the emission rates from the bulk value of $p = 1$. Enhancements by as much as a factor of 30 are seen to occur. The figure illustrates the symmetry of $p(\alpha, \xi)$ about $\xi = 1/2$ that reflects the underlying symmetry of the periodic medium. The overall behavior is qualitatively similar to that of an electromagnetic monopole radiator in a photonic band material. The behavior is also reminiscent of either an electromagnetic or acoustic radiator localized between parallel reflectors for which total suppression and a modest enhancement of the radiation rate can occur, dependent upon the radiator frequency, the reflector spacing, and the location of the radiator between the reflectors.

As $\alpha$ or, equivalently, $\Omega$ increases we will encounter a leading band edge whenever $\omega = n\pi$, which can be seen from the dispersion relation. Hence, the bands correspond very roughly to modes $\omega d/c = n\pi$ with wavelength $l = 2\pi/k$ related to the lattice period $d$ via $d = n(l/2)$. In other words, a passband begins when an integer number of half-wavelengths fit across a single lattice period. (For two parallel one-dimensional reflectors of separation $d$, this condition defines precisely the resonances of such a cavity.) Hence, in Fig. 3 we can see that the radiated power enhancements and suppressions in the bands correspond roughly to the locations of antinodes and nodes, respectively, of the wave function. This is similar to the behavior of a one-dimensional "atom" radiating electromagnetically between one-dimensional mirrors. In fact, the radiation rate at frequency $\omega$ is proportional to the square of the normal-mode wave function at this frequency and at the location of the point radiator—multiplied by the mode density at this frequency. This is a general feature that survives even in higher dimensions and for vector wave fields.

III. SUMMARY AND CONCLUSIONS

We have illustrated that sonic frequency passbands and band gaps can occur in a fluid with a periodically varying medium.
FIG. 3. Here, we plot as a two-dimensional surface the power output \( p \), Eq. (22), of a harmonically oscillating monopole at frequency \( \Omega \), located inside the tube at \( x = (0, d) \). The power \( p(\alpha, \xi) \) is normalized to the free-space output and is a function of the unitless monopole frequency and position parameters \( \alpha \equiv l/\epsilon = Kd \) and \( \xi \equiv x/d \), respectively. The monopole frequency \( \Omega \) is also that of the mode \( k \) that the monopole is radiating into. [The intrinsic monopole frequency \( \Omega \) is related implicitly to \( k \) via the dispersion relations, Eq. (9) or (13).] We see complete quenching of the radiation in the sonic band gaps—appearing as flat “valleys” of height zero running through the surface. Inside the passbands, which always begin at \( \alpha = n\pi \), we see both enhancement (by as much as a factor of 30) and suppression of emission with varying monopole location \( \xi \). Physically, enhanced and suppressed emission rates in the passbands correspond approximately to monopole locations at antinodes or nodes, respectively, of the normal-mode wave function of frequency \( \Omega \). This can be visualized by recalling that in a passband an integer number of half-waves fit across the period interval \( (0, d) \). The peaks and valleys of the power output in the bands correspond to those of the square of the wave function \( |\psi(x)|^2 \), as is evident from Eq. (22) for the function \( p \). The mode density near the band edges can be much larger than in the middle—a phenomenon seen also in solid-state theory. The increased mode density implies an increased radiation rate near the edges as is seen here, especially in the trailing edge of the second band that begins at \( \alpha = \pi \).

density—at least in a simple one-dimensional model. There is every reason to believe that, in analogy to the scalar Schrödinger wave equation for spinless electrons in a periodic lattice, the existence of sonic band structure will persist for a three-dimensional latticelike variation of the density. The three-dimensional normal modes \( q_n(x) \) for such a lattice could be computed using modified solid state energy band methods, and then Eqs. (10) for the energy radiated by a localized source—which are valid in three dimensions—could be used to study the effect of the band structure on radiating sound sources. Just as in electronic and photonic band materials, it seems as if an acoustic material that in three dimensions selectively transmits some frequencies while reflecting others could have many interesting applications. As we showed in Sec. II, for example, it would be possible to embed sound sources in a sonic band medium and quench these sources at band gap frequencies while amplifying their outputs at passband frequencies. The location and width of the bands and gaps are selected by controlling the density variations or pressure variations, the lattice period, and the three-dimensional lattice structure—choosing face-centered cubic for example.

It is possible to extend these ideas from fluids to sound waves in latticelike solids with periodic bulk modulus variations, in which case quantum features might play a role in the propagation of phonons through a density lattice at a microscopic scale. Other effects could be investigated such as the localization of sound in disordered medium—a process that would have direct electronic and optical analogues.

ACKNOWLEDGMENTS

The author would like to thank A. O. Barut, C. M. Bowden, and M. O. Scully for many interesting discussions related to this work. He would also like to acknowledge the National Research Council for financial support.

7 G. Barton, Elements of Green’s Functions and Propagation (Clarendon, Oxford, 1989), Sec. 10.5.3.