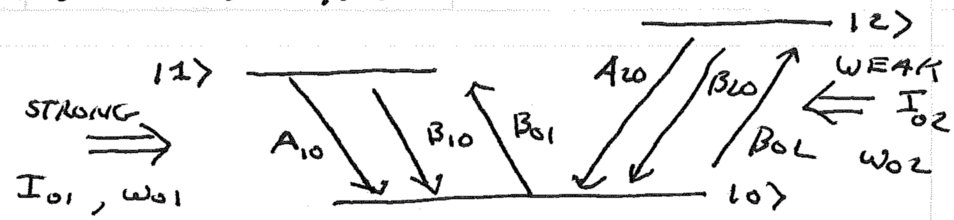
8.4 Closed Three Level system

A sp. E

B st. A/E



For full three-level system we must solve 3×3 density matrix equ.

Assume $|\psi(0)\rangle$ a pure state

$$|\psi\rangle = C_0|0\rangle + C_1|1\rangle + C_2|2\rangle$$

$$\hat{\rho}_{PURE} = |\psi\rangle\langle\psi| = \begin{array}{cc} \begin{array}{ccc} |0\rangle & |1\rangle & |2\rangle \end{array} & \\ \begin{bmatrix} C_0^*C_0 & C_1^*C_0 & C_2^*C_0 \\ C_0^*C_1 & C_1^*C_1 & C_2^*C_1 \\ C_0^*C_2 & C_1^*C_2 & C_2^*C_2 \end{bmatrix} & \begin{array}{c} \langle 0| \\ \langle 1| \\ \langle 2| \end{array} \end{array}$$

$$= \begin{bmatrix} \rho_{00} & \rho_{10} & \rho_{20} \\ \rho_{01} & \rho_{11} & \rho_{21} \\ \rho_{02} & \rho_{12} & \rho_{22} \end{bmatrix}$$

where

$$\rho_{ii} = |C_i|^2 = P_i \quad \text{and} \quad \sum_{i=0}^{2} \rho_{ii} = 1$$

cons of probability

ρ_{ii} is called electron population in level $|i\rangle$

$\forall i \neq j$ (off diagonal) $\rho_{ij} = \rho_{ji}^*$ <u>coherence terms</u>

Let us assume $A_{10} = A_{20} = 0$

How do we obtain 3-level Rabi solution?

TDSE: $$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle$$

$$-i\hbar \frac{d}{dt}\langle\psi| = \langle\psi|\hat{H}$$

Hence

$$i\hbar \frac{d\hat{\rho}}{dt} = i\hbar \left[|\dot{\psi}\rangle\langle\psi| + |\psi\rangle\langle\dot{\psi}| \right]$$

$$= \hat{H}\, |\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| \hat{H}$$

$$= [\hat{H}, \hat{\rho}]$$

Even without loss / dissipation $A_{10} = A_{20} = 0$ this is NINE coupled first order diffy-Q. even assuming field is classical. Adding loss

$$\boxed{i\hbar \frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}] + \hat{\mathcal{L}}[\hat{\rho}]}$$

JUST MAKES IT WORSE – MUST BE DONE NUMERICALLY. However there is another approach!

Assume $A_{10} = A_1 \neq 0$ and $A_{20} = A_2 \neq 0$

$B_{10} = B_{01} = B_1$ Reciprocity

$B_{20} = B_{02} = B_2$ Reciprocity

For long times / after many jumps / all off diagonal coherence terms will vanish

$t \to 0 \Rightarrow \rho_{ij} = 0 \quad i \neq j$

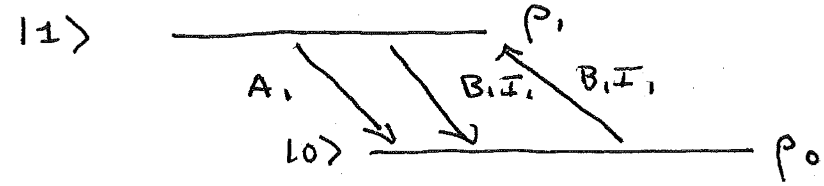
Hence we can write down rate equations for only $\rho_{11} = \rho_1$

$\rho_{22} = \rho_2$

$\rho_{33} = \rho_3$

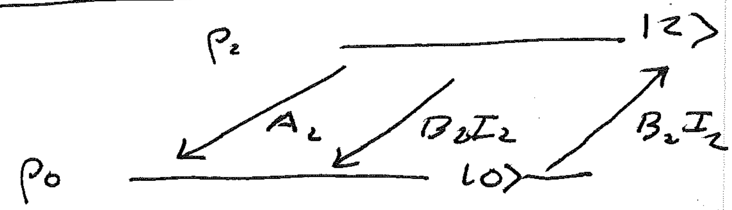
We assume system is <u>closed</u> so population is conserved $\rho_{11}(t) + \rho_{22}(t) + \rho_{33}(t) = 1$ $\forall t$: That is electron is always in one of three levels. We now just á la Einstein compute the rates.

$$\boxed{\dot{\rho}_1 = -[A_1 + B_1 I_1]\rho_1 + B_1 I_1 \rho_0}$$



spontaneous emission prob $A_1\rho_1$ is proportional to Einstein A and prob. $P_1 = \rho_1$ that electron is <u>in</u> level $|1\rangle$. similarly stimulated & $B_1 I_1$

$$\boxed{\dot{\rho}_2 = -[A_2 + B_2 I_2]\rho_2 + B_2 I_2 \rho_0}$$



Finally cons. prob.

$$\boxed{\dot{\rho}_{00} = -\underbrace{[B_1 I_1 + B_2 I_2]}_{\text{St. Abs.}}\rho_{00} + [\underset{\text{Sp.E}}{A_1} + \underset{\text{St.E}}{B_1 I_1}]\rho_{11} + [\underset{\text{Sp E}}{A_2} + \underset{\text{St. E}}{B_2 I_2}]\rho_{22}}$$

Let $A_i = \alpha_i$ and fundamental

$B_i I_i = \beta_i$ rate equations are

$$\dot\rho_{00} = -[\beta_1+\beta_2]\rho_{00} + [\alpha_1+\beta_1]\rho_1 + [\alpha_2+\beta_2]\rho_2$$

$$\dot\rho_1 = -[\alpha_1+\beta_1]\rho_1 + \beta_1\rho_0$$

$$\dot\rho_2 = -[\alpha_2+\beta_2]\rho_2 + \beta_2\rho_0$$

$$1 = \rho_1+\rho_2+\rho_3$$

First consider <u>steady state</u> solutions

$\dot\rho_i = 0$ for $t \to \infty$

$$\boxed{\begin{aligned} 0 &= -[\beta_1+\beta_2]\rho_0 + [\alpha_1+\beta_1]\rho_1 + [\alpha_2+\beta_2]\rho_2 \\ 0 &= -[\alpha_1+\beta_1]\rho_1 + \beta_1\rho_0 \\ 0 &= -[\alpha_2+\beta_2]\rho_2 + \beta_2\rho_0 \\ 1 &= \rho_0+\rho_1+\rho_2 \end{aligned}}$$

These can be solved exactly!

But first consider some limiting cases

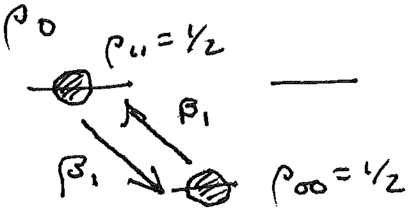
(I) $\beta_2 = \alpha_2 \simeq 0$ That is level two is decoupled I_2 very weak or off

Also $\alpha_1 \approx 0$ $\alpha_1 \lll \beta_1$ so spont. E. neg.

$$0 = -\beta_1\rho_0 + \beta_1\rho_1 \Rightarrow \rho_1 = \rho_0$$

$$\cancel{0 = -\beta_1\rho_1 + \beta}$$

$$1 = \rho_0 + \rho_1 + \underbrace{\cancel{\rho_2}}_{\to 0}$$ if population initially in ρ_0



$\Rightarrow \rho_0 = \rho_1 = 1/2$; $\rho_2 = 0$

Detailed Balance

(II) Assume: $\alpha_1 \cong \alpha_2 \approx 0$

or more realistically $\alpha_1 \lll \beta_1$ & $\alpha_2 \lll \beta_2$

In this case $\rho_0(0) = 1$; $\rho_1(0) = 0$; $\rho_2(0) = 0$

initial conditions.

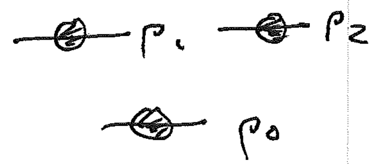
$$0 = -[\beta_1 + \beta_2]\rho_0 + \beta_1\rho_1 + \beta_2\rho_2$$

$$0 = -\beta_1\rho_1 + \beta_1\rho_0 \quad \Rightarrow \quad \rho_0 = \rho_1$$

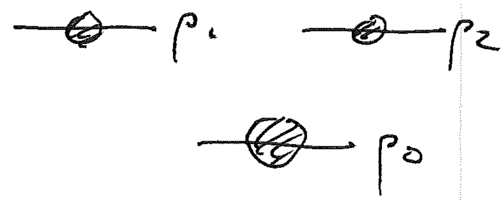
$$0 = -\beta_2\rho_2 + \beta_2\rho_0 \quad \Rightarrow \quad \rho_0 = \rho_2 = \rho_1$$

$$1 = \rho_0 + \rho_1 + \rho_2$$

$$\Rightarrow \quad t \to \infty \qquad \boxed{\rho_0 = \rho_1 = \rho_2 = 1/3}$$



Again detailed balance places population equally. Spontaneous Emission α_1, α_2 spoils detailed balance! Hence $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$ we expect



$$\rho_0(\infty) > 1/3$$

$$\rho_1(\infty) < 1/3$$

$$\rho_2(\infty) < 1/3$$

as electron comes down faster than it goes up.

Astoundingly if you plug $t \to \infty$ equations into matematical it produces an exact solution

$$\boxed{\begin{aligned} \rho_1(\infty) &= \frac{(\alpha_2+\beta_2)\beta_1}{\alpha_1(\alpha_2+2\beta_2)+\beta_1(2\alpha_2+3\beta_2)} \\ \rho_2(\infty) &= \frac{(\alpha_1+\beta_1)\beta_2}{\alpha_2(\alpha_1+2\beta_1)+\beta_2(2\alpha_1+3\beta_1)} \\ \rho_{00}(\infty) &= 1-\rho_1^{(\infty)}-\rho_2(\infty) \end{aligned}}$$

which agrees with book. This makes no assumptions about relative sizes of α_i β_i as indicated that $\rho_1 \leftrightarrow \rho_2$ under interchange of $1 \leftrightarrow 2$.

Taking $\alpha_2 \approx \beta_2 \approx 0$ gives $\rho_0 = \rho_1 = 1/2$; $\rho_2 = 0$

Let's do this carefully

(I) $\alpha_1 = \alpha_2 = 0$

$$\rho_1(\infty) = \frac{\beta_1\beta_2}{3\beta_1\beta_2} = \frac{1}{3}$$

$$\rho_2(\infty) = \frac{\beta_1\beta_2}{3\beta_1\beta_2} = \frac{1}{3}$$

$\rightarrow \rho_0(\infty) = 1/3$ Independent of $\beta_1 >> \beta_2$ or $\beta_1 << \beta_2$

(II) $\beta_2 = 0$ $\alpha_1 = 0$

$$\rho_1(\infty) = \frac{\alpha_2\beta_1}{2\alpha_2\beta_1} = \frac{1}{2}$$

$$\rho_2(\infty) = 0$$

$\rightarrow \rho_0(\infty) = 1/2$

Quantum Jump Condition III

STRONG I_1 pump $\Rightarrow$ $\beta_1 >>> \alpha_1$ we can take $\alpha_1 \simeq 0$ SATURATION OFF $D \Leftrightarrow 1$

$$\rho_1(\infty) = \frac{(\alpha_2+\beta_2)\beta_1}{(2\alpha_2+3\beta_2)\beta_1} = \boxed{\frac{\alpha_2+\beta_2}{2\alpha_2+3\beta_2}}$$

IND OF β_1!

$$\rho_2(\infty) = \frac{\beta_1\beta_2}{2\alpha_2\beta_1 + 3\beta_2\beta_1} = \boxed{\frac{\beta_2}{2\alpha_2+3\beta_2}}$$

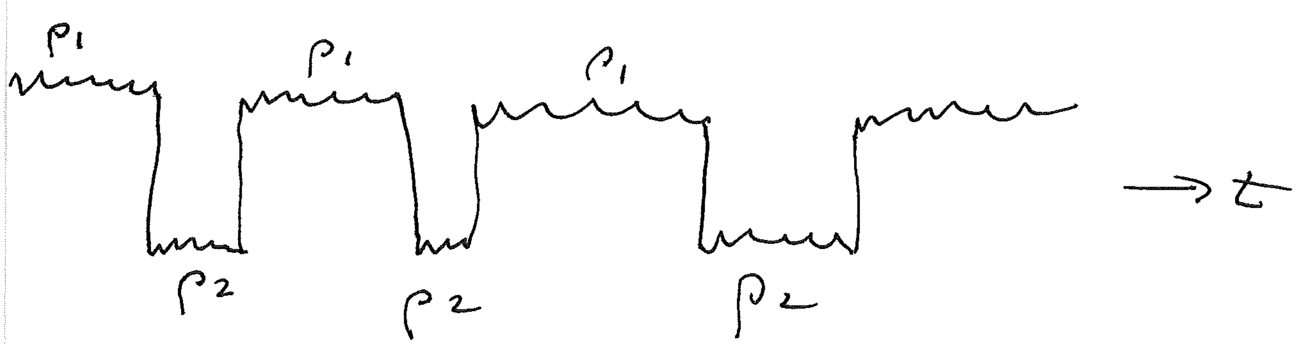
which agrees with book. Note this holds $\forall\ \beta_2$ and so even if $\beta_2 << \beta_1$ (weak I_2 pump) there is probability to find electron in $|2\rangle$

We can further assume $\beta_2 >> \alpha_2 \Rightarrow \gamma_2 = \frac{\alpha_2}{\beta_2} << 1$

$$\rho_1(\infty) = \frac{\gamma_2+1}{2\gamma_2+3} \cong \frac{1}{3} + \frac{1}{9}\gamma_2 \gtrsim \frac{1}{3}$$

$$\rho_2(\infty) = \frac{1}{2\gamma_2+3} \cong \frac{1}{3} - \frac{2}{9}\gamma_2 \lesssim \frac{1}{3}$$

So even when I_1 pump is very strong the prob of finding in $|2\rangle$ is <u>very</u> <u>high</u>



For $0 < t < \infty$ we must solve

$$\dot{\rho}_1 = -[\alpha_1 + \beta_1]\rho_1 + \beta_1(1-\rho_1-\rho_2)$$
$$\dot{\rho}_2 = -[\alpha_2 + \beta_2]\rho_2 + \beta_2(1-\rho_1-\rho_2)$$

where we have eliminated $\rho_0 = 1-\rho_1-\rho_2$

$$\boxed{\begin{aligned}\dot{\rho}_1 &= -[\alpha_1+2\beta_1]\rho_1 + \beta_1 - \beta_1\rho_2 \\ \dot{\rho}_2 &= -[\alpha_2+2\beta_2]\rho_2 + \beta_2 - \beta_2\rho_1\end{aligned}}$$

with initial condition $\rho_1(0) = \rho_2(0) = 0$ $\rho_0(0) = 1$

Mathematica gives: A very Large output!

However solution is exact! Well have to make some approximations. Let's take $\beta_1 >>> \alpha_1 \approx 0$

as before we may write

$$\frac{1}{\beta_1\beta_2}\dot{\rho}_1 = -\Big[\cancelto{0}{\frac{\alpha_1}{\beta_1\beta_2}} + 2\frac{1}{\beta_2}\Big]\rho_1 + \frac{1}{\beta_2} - \frac{1}{\beta_2}\rho_2$$

$$\frac{1}{\beta_1\beta_2}\dot{\rho}_2 = -\Big[\cancelto{0}{\frac{\alpha_2}{\beta_1\beta_2}} + 2\frac{1}{\beta_1}\Big]\rho_1 + \frac{1}{\beta_1} - \frac{1}{\beta_1}\rho_1$$

$$\Rightarrow \boxed{\begin{aligned}\dot{\rho}_1 &= -2\beta_1\rho_1 + \beta_1 - \beta_1\rho_2 \\ \dot{\rho}_2 &= -2\beta_2\rho_2 + \beta_2 - \beta_2\rho_1\end{aligned}}$$

$$\frac{1}{\beta_2}\dot{\rho}_1 = -2\frac{\beta_1}{\beta_2}\rho_1 + \frac{\beta_1}{\beta_2} - \frac{\beta_1}{\beta_2}\rho_2$$

$$\frac{1}{\beta_2}\dot{\rho}_2 = -2\rho_2 + 1 - \rho_1$$

Inserting $\dot{\rho}_2/\beta_2 \to \frac{1}{\beta_2}\ddot{\rho}_1$ and drop $\frac{\beta_1}{\beta_2} << 1$

$$\frac{1}{\beta_2}\ddot{\rho}_1 = \cancel{-2\frac{\beta_1}{\beta_2}\dot{\rho}_1}^{\,0} + \cancel{\frac{\beta_1}{\beta_2}}^{\,0} - \beta_1\left[-2\rho_2 + 1 - \rho_1\right]$$

This tells us

$$\boxed{\begin{aligned} \dot{\rho}_1 &\cong -\beta_1\rho_2 \\ \dot{\rho}_2 &\cong -2\beta_2\rho_2 + \beta_2 - \beta_2\rho_1 \end{aligned}} \qquad \begin{aligned} \rho_1(0) &= 0 \\ \rho_2(0) &= 0 \end{aligned}$$

After some work in Mathematica I get

$$\rho_1(t) \approx \frac{1}{3}\left[1 + \frac{1}{2}\left(e^{-3\beta_2 t/2} - 3e^{-2\beta_1 t}\right)\right]$$

$$\rho_2(t) \approx \frac{1}{3}\left[1 - e^{-3\beta_2 t}\right]$$

$$\rho_1(0) = \frac{1}{3}\left[1 + \frac{1}{2}(1-3)\right] = \frac{1}{3}\left[1-1\right] = 0 \checkmark$$

$$\rho_2(0) = \frac{1}{3}\left[1-1\right] = 0 \checkmark$$

$$\rho_1(\infty) = 1/3 = \rho_2(\infty) \checkmark$$

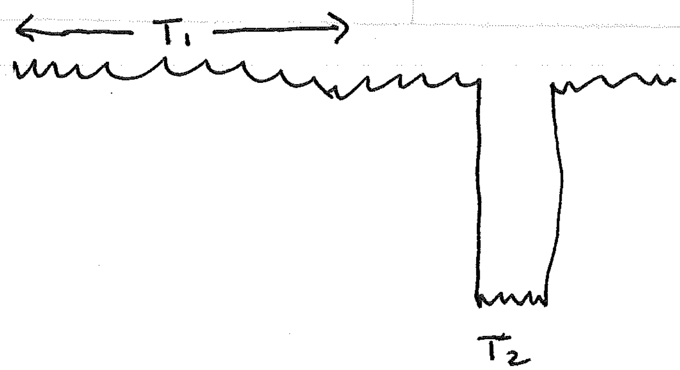
So for $0 < t \ll T_2 = 1/\beta_2$ (JUMP TIME)

$$\rho_1(t \ll T_2) \approx \frac{1}{3}\left[1 + \frac{1}{2}\left[1 - 3e^{-2\beta_1 t}\right]\right]$$

$$= \frac{1}{3}\left[\frac{3}{2} - \frac{3}{2}e^{-2\beta_1 t}\right] \rightarrow 1/2 \qquad 0 \leftrightarrow 1$$

SATURATION

$$\rho_1(t \gg T_2) \approx \frac{1}{3} \qquad \text{Detailed Balance.}$$



Let $\beta_1 = 1$ $\beta_2 = 0.1$