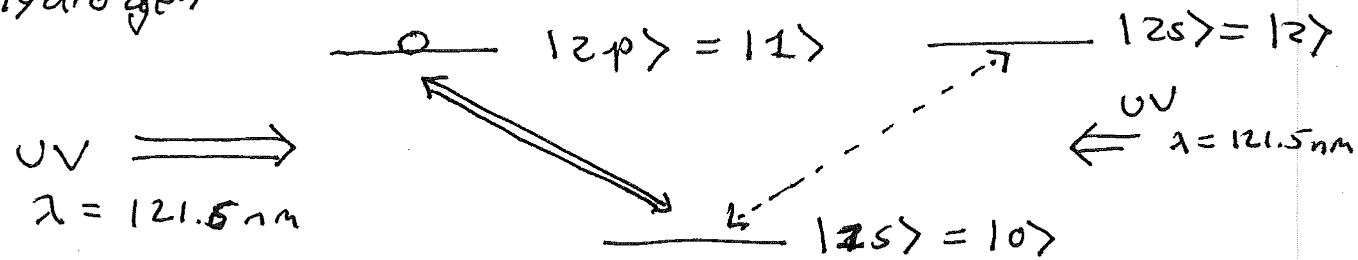
8.3 Quantum Trajectory

The Optical Bloch Eqs. describe behavior of a collection or "ensemble" of atoms. For example when we say a two-level atom suffers 50% loss we mean that in an ensemble 50% of the atoms are missing an electron from the two levels under consideration. One atom cannot be missing 1/2 an electron. A measurement made on a single atom shows either the electron is there (50%) or not (50%). Quantum trajectory approach was developed to ~~solve~~ study loss and decoherence in individual quantum systems

Quantum Jumps

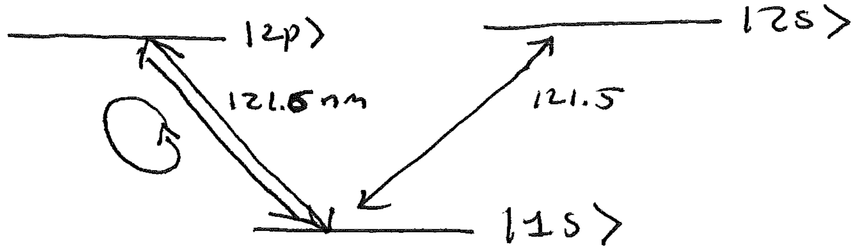
Consider a three-level atom isolated in empty space. For concreteness let's consider hydrogen



The transition $|2p\rangle = |\overset{n\,l\,m}{211}\rangle \rightarrow |1s\rangle = |\overset{n\,l\,m}{100}\rangle$ is dipole allowed $\Delta l = 1 \quad \Delta m = \pm 1$ and so an atom in $|2p\rangle \rightarrow |1s\rangle$ by spontaneous emission in 1 ns.

However $|2s\rangle = |200\rangle \to |1s\rangle = |100\rangle$ is not dipole allowed and must take place by a metastable 2-photon transition. Life time of $|2s\rangle$ is 1s instead of 1ns!

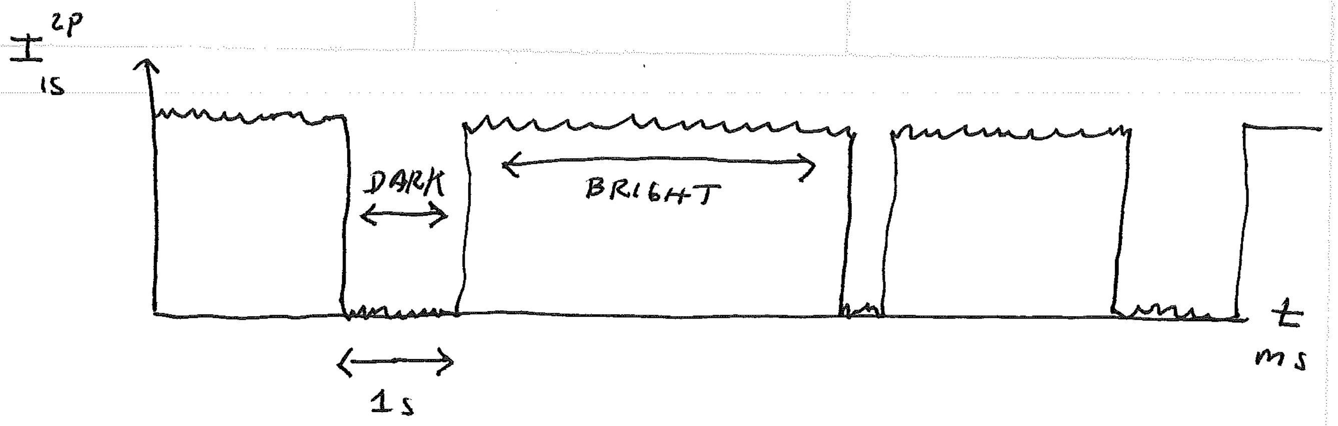
I now tune my $\overbrace{\text{UV}}^{\text{strong}}$ laser 121.5 nm to the $|1s\rangle \to |2p\rangle$ transition



The electron will cycle between $|1s\rangle \to |2p\rangle$ and also emit 121.6 nm photons by spontaneous emission or stimulated emission — typically 1 photon / ns $\Rightarrow$ 10^6 photons / sec. This can be seen with a sensitive CCT camera!

Now we place a second weak UV beam tuned to 121.5 nm the 1s → 2s.

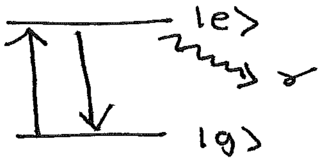
Every once in a while the electron will jump $|1s\rangle \to |2s\rangle$ and stay there for a second. The emission on the $|1s\rangle \to |2p\rangle$ will stop.



Hence in such a set up we can see the atom blink on and off. Hence we can see quantum jumps!

The unitary evolution of the three-level system in the two laser beams expierences "jumps" due to random spontaneous emission events. Recall our theory Ch. 4.3 of atom interacting with a quantized field

$$P_{g\to e} \propto \underset{\substack{\uparrow\\ \#\ \text{photons}}}{n}$$ stimulated absorption



$$P_{e\to g} \propto \underset{\substack{\uparrow\\ \#\text{photons}\\ \text{stimulated}\\ \text{emission}}}{n} + \underset{\substack{\uparrow\ \text{vaccum}\\ \text{spontaneous}\\ \text{emission.}}}{1}$$

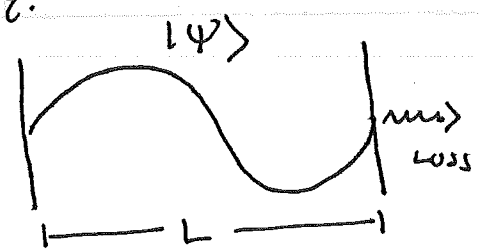
$$\omega_0 = \omega_e - \omega_g$$

The probability of spon. Emission

$$P_{eg}^{SpE} = |\vec{d}_{ab} \cdot \vec{\mathcal{E}}_0|^2$$

where $\vec{\mathcal{E}}_0 = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2}\vec{e}$ is vacuum electric field

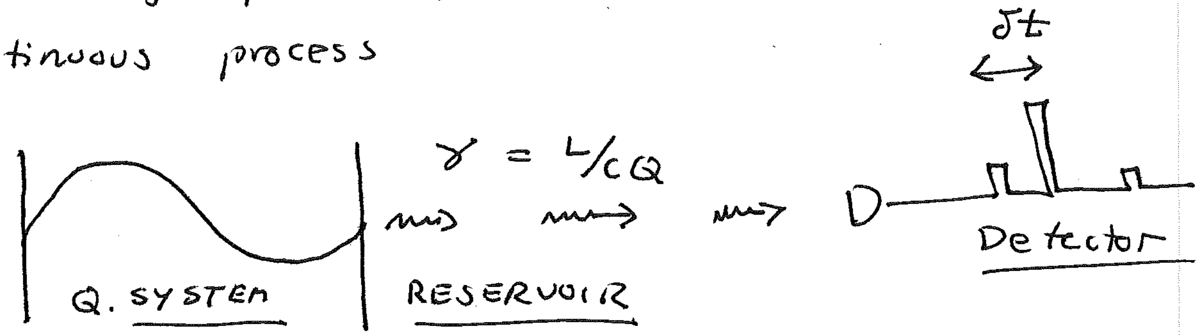
8.3 Quantum Jumps and Master Eq.



Suppose we have a high-Q cavity with a coherent or other state $|\Psi\rangle$ in the cavity

Periodically the cavity will emit a photon through the wall at a rate $\boxed{\gamma = \frac{L}{cQ}}$

At the single photon level this is not a continuous process



Eventually $|\Psi\rangle \to |0\rangle$ and only vaccum is left. If $|\Psi(0)\rangle$ is a pure state it will be a mixed state $t > 0$

1. $$\underset{\text{prob of loss}}{\underset{\uparrow}{\Delta P}} = \underset{\text{rate of loss}}{\underset{\uparrow}{\gamma}} \langle\Psi|\underset{\text{\# photons in cavity}}{\underset{\uparrow}{\hat n}}|\Psi\rangle \underset{\text{short time interval}}{\underset{\uparrow}{\delta t}}$$

2. choose a <u>random</u> number $0 \le r \le 1$

3. <u>If</u> $\Delta P > r$ ~~$r < \Delta P$~~ JUMP CONDITION (emission)

$$\Rightarrow \quad |\Psi\rangle \to |\Psi_{emit}\rangle = \frac{\overset{\substack{\text{photon anihilation}\\ \downarrow}}{\hat a}|\Psi\rangle}{\sqrt{\langle\Psi|\hat n|\Psi\rangle}} \quad \text{\& renormalize}$$

4. If $\Delta P < r$ (no emission) <u>no</u> jump condition

$$|\psi\rangle \rightarrow |\psi_{no}\rangle = \frac{e^{-i\text{ɏ}\,\delta t\,\hat{H}_{eff}}\;|\psi\rangle}{\sqrt{\langle\psi|e^{+i\text{ɏ}\,\delta t\,\hat{H}^{+}_{eff}}\;e^{-i\text{ɏ}\,\delta t\,\hat{H}_{eff}}|\psi\rangle}} \longleftarrow \text{pseudo unitary}$$

where $\hat{H}_{eff} = \hat{H}_0 - i\frac{\hbar\gamma}{2}\,a^{+}a = \hbar\omega\, a^{+}a - i\frac{\hbar}{2}\gamma\, a^{+}a$

$$= \hbar\underbrace{(\omega - i\gamma/2)}\hat{a}^{+}\hat{a} \equiv \hbar\underbrace{\Gamma}\, a^{+}a$$

$\hat{H}^{+}_{eff} \neq \hat{H}_{eff}$ <u>non</u>-hermitian $\omega \rightarrow \omega - i\gamma/2$

gives correct Lorentzian line shape.

$[\hat{H}^{+}_{eff}, \hat{H}_{eff}] = 0$ and $\hat{H}^{+}_{eff} - \hat{H}_{eff} = i\hbar\gamma\, a^{+}a$

$$\Rightarrow \exp[i\text{ɏ}\,\delta t\,\hat{H}^{+}_{eff}]\exp[-i\text{ɏ}\,\delta t\,\hat{H}_{eff}] = \exp[-\gamma\,\delta t\,\hat{a}^{+}\hat{a}]$$

If $\delta t << \gamma = \frac{L}{cQ}$ $\approx \boxed{1 - \gamma\,\delta t\,\hat{a}^{+}\hat{a}}$

$\equiv 1$

$$\Rightarrow |\psi_{no}\rangle \cong \frac{e^{-i\text{ɏ}\,\delta t\,\hat{H}_{eff}}\;|\psi\rangle}{\sqrt{\langle\psi|\,1 - \gamma\,\delta t\,\hat{n}\,|\psi\rangle}}$$

$$\approx \frac{(1 - i\text{ɏ}\,\delta t\,\hat{H}_{eff})|\psi\rangle}{\sqrt{1 - \Delta P}}$$

$$\approx \frac{[1 - i\Gamma\,\hat{n}\,\delta t]|\psi\rangle}{\sqrt{1 - \Delta P}} = \frac{[1 - i\text{ɏ}\,\delta t\,\hat{H}_0 - \frac{\gamma}{2}\delta t\,\hat{n}]|\psi\rangle}{\sqrt{1 - \Delta P}}$$

where $\Gamma \equiv \omega - i\gamma/2$

$\hat{n} \equiv \hat{a}^{+}\hat{a}$

The desity of $\hat{\rho}(t)$ becomes

$$\hat{\rho}(t) = |\psi\rangle\langle\psi| \rightarrow |\psi(\delta t)\rangle\langle\psi(\delta t)| = \overset{t+}{\rho(\delta t)}$$

$$= \underbrace{\Delta p\, |\psi_{em}\rangle\langle\psi_{em}|}_{\text{JUMP}} + \underbrace{(1-\Delta p)\, |\psi_{no}\rangle\langle\psi_{no}|}_{\text{NO JUMP}}$$

Note

$$\Delta p\, |\psi_{em}\rangle\langle\psi_{em}| = \left(\frac{\hat{a}\,|\psi\rangle\langle\psi|\hat{a}^{+}}{\langle\hat{n}\rangle}\right)\Delta p$$

$$= \frac{\hat{a}\,|\psi\rangle\langle\psi|\hat{a}^{+}}{\cancel{\langle\hat{n}\rangle}}\left[\gamma \cancel{\langle\hat{n}\rangle}\,\delta t\right]$$

$$= \boxed{\gamma\,(\hat{a}\,\underbrace{|\psi\rangle\langle\psi|}_{\hat{\rho}}\,\hat{a}^{+})\,\delta t} = \boxed{\gamma\,\delta t\;\hat{a}\hat{\rho}\hat{a}^{+}}$$

$$(1-\Delta p)\,|\psi_{no}\rangle\langle\psi_{on}|$$

$$= \frac{(1 - iy\,\hat{H}_0\,\delta t - \frac{\gamma}{2}\,\delta t\,\hat{n})\overbrace{|\psi\rangle\langle\psi|}^{\hat{\rho}}(1 + iy\,\delta t\,\hat{H}_0 - \frac{\gamma}{2}\,\delta t\,\hat{n})\,\cancel{(1-\Delta p)}}{\cancel{(1-\Delta p)}}$$

$$= \rho + 1\,\hat{\rho}\,[\,i y\,\delta t\,H_0 - \tfrac{\gamma}{2}\,\delta t\,\hat{n}\,]$$

$$+ [\,-iy\,\delta t\,H_0 - \tfrac{\gamma}{2}\,\delta t\,\hat{n}\,]\,\hat{\rho}\cdot 1 + \cancel{O(\delta t^2)} \rightarrow 0$$

$$= \rho(t) - iy\,\delta t\,[\hat{H}_0, \hat{\rho}_t] - \tfrac{\gamma}{2}\,\delta t\,[\hat{n}\hat{\rho}_t + \hat{\rho}_t\hat{n}]$$

$$\Rightarrow \frac{\rho(t+\delta t) - \rho(t)}{\delta t} = -iy\,[\hat{H}_0, \hat{\rho}] + \frac{\gamma}{2}\,[2\hat{a}\hat{\rho}\hat{a}^{+} - \hat{n}\hat{\rho} - \hat{\rho}\hat{n}]$$

$$\Rightarrow \boxed{i\hbar\,\frac{d\hat{\rho}(t)}{dt} = \underbrace{[\hat{H}_0, \hat{\rho}]}_{\text{Schrödinger}} + \underbrace{\frac{\gamma}{2}\,[2\hat{a}\hat{\rho}\hat{a}^{+} - \hat{n}\hat{\rho} - \hat{\rho}\hat{n}]}_{\text{Louiville / Lass}}}$$

MASTER EQ.

The Super Operator

$$\hat{\mathbb{L}}_{\hat a}(\hat\rho) = \frac{\gamma}{2}\left[2\hat a\hat\rho\hat a^{+} - \hat n\hat\rho - \hat\rho\hat n\right]$$

is called the Louiville - Loss operator

In general the master equation has to be integrated numerically.

However for $\delta t \ll \gamma$ [no jump]

let $|\psi\rangle = |\alpha\rangle$ and $\hat H_{eff} = \hbar(\omega - i\gamma/2)a^{+}a$

$$|\alpha_{no}\rangle = \frac{e^{-i\hbar\,\delta t\,\hat H_{eff}}|\alpha\rangle}{\sqrt{\langle\alpha|e^{-\gamma\delta t\hat n}|\alpha\rangle}} = \frac{e^{-i(\omega-i\gamma/2)\hat n\,dt}|\alpha\rangle}{\sqrt{\langle\alpha|\,e^{-\gamma\delta t\hat n}|\alpha\rangle}}$$

using $e^{i\lambda\hat n}|\alpha\rangle = |\alpha e^{i\lambda}\rangle$

$$|\alpha_{no}\rangle = \frac{|\alpha e^{-i\Gamma dt}\rangle}{\sqrt{\langle\alpha|e^{-\gamma\delta t}\alpha\rangle}} \approx 1 \quad \text{if} \quad \delta t \ll \gamma$$

$$\approx |\alpha \underset{\substack{\uparrow\\ \text{normal}\\ \text{evolution}}}{e^{-i\omega dt}} \underset{\substack{\uparrow\\ \text{loss}}}{e^{-\frac{\gamma}{2}dt}}\rangle$$

so effect is dissaptive exponential decay!

General solution Let $|\psi(0)\rangle = |\alpha\rangle = \sum_n C_n(0)|n\rangle$

$$C_n(0) = e^{-\bar{n}/2}\frac{\alpha^n}{\sqrt{n!}}$$

$$\rho_0 = |\alpha\rangle\langle\alpha| = \cancel{e^{-\bar{n}}} \sum_{nm} \underbrace{C_n^*(0)\, C_m(0)}_{C_{nm}(0)} |n\rangle = e^{-\bar{n}} \sum_{nm} \frac{\alpha^{*n}\alpha^m}{\sqrt{n!m!}} |n\rangle\langle m|$$

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \sum_{nm} C_{nm}(t)\,|n\rangle\langle m|$$

$$\dot{\rho}(t) = \sum_{nm} \dot{c}_{nm}(t)\,|n\rangle\langle m|$$

$$[H_0, \rho] = \hbar\omega \sum_{nm} c_{nm}(t)\,[\hat{n}, |n\rangle\langle m|]$$

$$= \hbar\omega \sum_{nm} c_{nm}(t)\,[n-m]\,|n\rangle\langle m|$$

$$a\rho a^+ = \sum_{nm} c_{nm}(t)\,[\,\underbrace{a|n\rangle}\underbrace{\langle m|a^+}$$

$$= \sum_{nm} c_{nm}(t)\,|n-1\rangle\langle m+1|$$

$$\hat{n}\rho = \sum_{nm} c_{nm}(t)\, n\, |n\rangle\langle m|$$

$$\hat{\rho}\hat{n} = \sum_{nm} c_{nm}(t)\, m\, |n\rangle\langle m|$$

so $i\hbar\,\dot{\rho} = [\hat{H}_0, \hat{\rho}] + \hat{\mathbb{L}}(\rho)$

gives doubly infinite coupled set of linear diffy-Qs.

COHERENT STATE

Recall $e^{\lambda\hat{n}}|\alpha\rangle = |e^{\lambda}\alpha\rangle \quad \forall\ \lambda \in \mathbb{C} \quad |\alpha\rangle \in \mathcal{H}$

Hence no jump

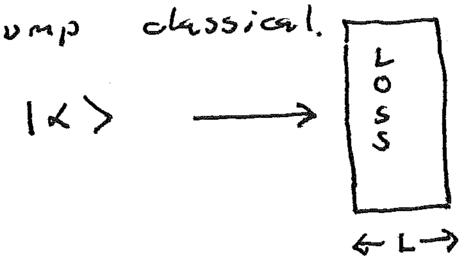
$$|\alpha\rangle \to \frac{e^{-\frac{\gamma}{2}\delta t\hat{n}}|\alpha\rangle}{\underset{\overset{\swarrow}{1/2}}{\langle\alpha|\exp}[-\gamma\delta t\underset{\overset{\searrow}{1/2}}{\hat{n}}]|\alpha\rangle} = \frac{|e^{-\frac{\gamma}{2}\delta t}\alpha\rangle}{\langle e^{-\frac{\gamma}{2}\delta t}\alpha|e^{-\frac{\gamma}{2}\delta t}\alpha\rangle}$$

$$= |e^{-\frac{\gamma}{2}\delta t}\alpha\rangle \qquad \bar{n} = |\alpha|^2$$

$$\bar{n} \to e^{-\gamma\delta t}\bar{n}$$

so coherent state remains coherent – just shrinks!. This is true in general t for $\bar{n} \gg 1$

no jump classical.



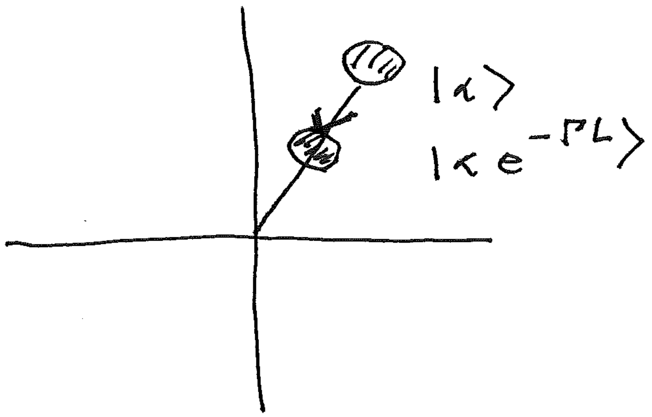
$$|\alpha e^{-\frac{\gamma}{2}T}\rangle = |\alpha e^{-\frac{\gamma}{2c}L}\rangle$$

$$= |\alpha e^{-\Gamma L}\rangle$$

$T = \frac{L}{c}$

where Γ is loss per unit length.

Beer's Law For Absorbtion



JUMPING CATS

Recall

$$\left|\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\rangle = N^{e}_{o}\left[|\alpha\rangle \pm |\bar{\alpha}\rangle\right]$$

are the even and odd cats.

$\hat{a}^2|\text{CAT}\rangle = \alpha^2|\text{CAT}\rangle$ so CAT is

IMMUNE TO <u>TWO</u> <u>JUMPS</u>.

IGNORING Normalization

$$\hat{a}|\text{even}\rangle = |\text{odd}\rangle$$

$$\hat{a}|\text{odd}\rangle = |\text{even}\rangle$$

Hence <u>jump</u> causes cats to switch from even to odd.

No jump

$$e^{-\frac{\gamma}{2}\delta t\,\hat{n}}|\text{CAT}\rangle = N\left[|\alpha e^{-\frac{\gamma}{2}\delta t}\rangle \pm |\bar{\alpha} e^{-\frac{\gamma}{2}\delta t}\rangle\right]$$

causes cat to shrink.

Compare Wigner Fcns Eq. 7.137 / 7.138

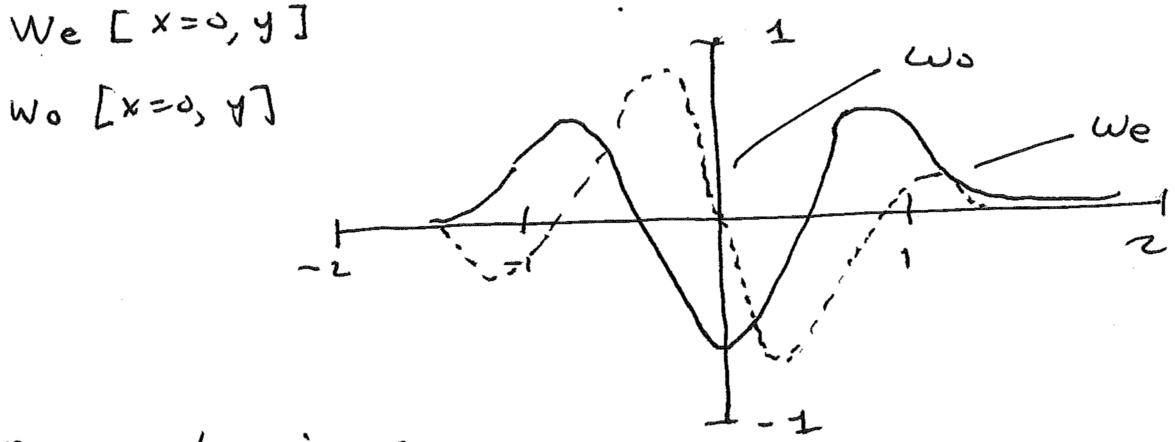
$$W_e(x,y) = \frac{1}{\pi[1-e^{-2\bar{n}}]}\left\{ e^{\overset{\overset{\text{Gaussian at }\alpha}{\downarrow}}{-2[(x-\alpha)^2+y^2]}} \underset{\uparrow}{} + e^{\overset{\overset{\text{Gaussian at }\bar{\alpha}}{\downarrow}}{-2[(x+\alpha)^2+y^2]}} \right.$$

$$-2\exp[-2(x^2+y^2)]\,\underline{\cos}\,4ya$$

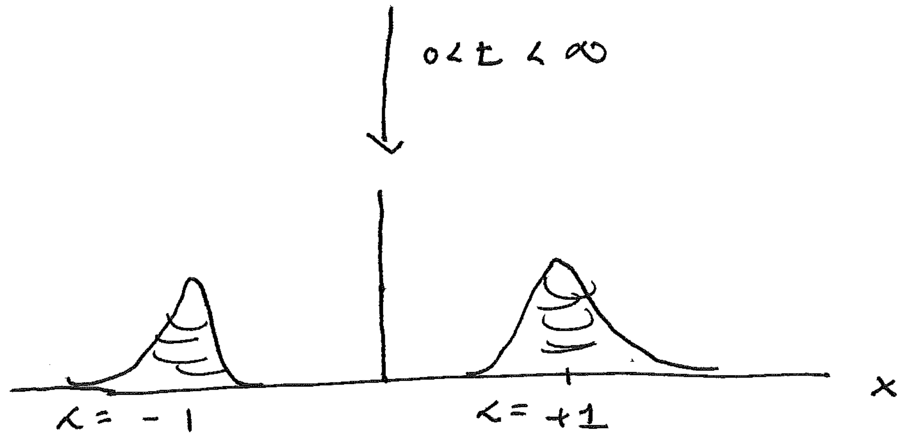
$$W_o(x,y) = \quad '' \qquad '' \qquad ''$$

$$-2\exp[-2(x^2+y^2)]\sin 4ya$$

since $\alpha, \bar{\alpha} \in \mathbb{R}$ we can plot slice at $x = 0$



On each jump
the wiggles tend to cancle out



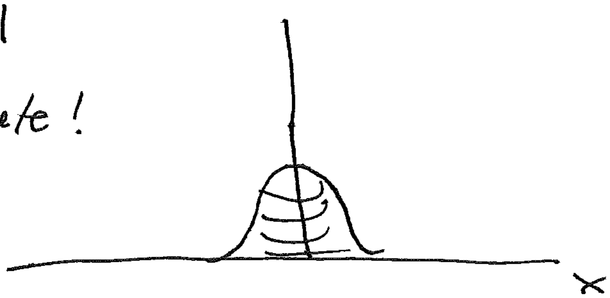
No wiglles ⇒ classical mixture

$$\hat{\rho}_{mix} = |\alpha e^{-\frac{\gamma}{2}t}\rangle\langle\alpha e^{-\frac{\gamma}{2}t}| + |\bar{\alpha} e^{-\frac{\gamma}{2}t}\rangle\langle\bar{\alpha} e^{-\frac{\gamma}{2}t}|$$

no more in tereference. As $t \to \infty$

$$\hat{\rho} \to |0\rangle\langle 0|$$

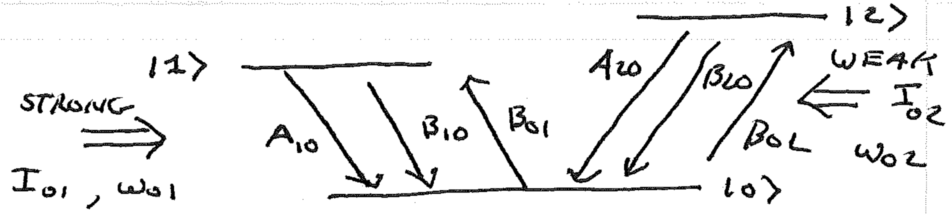
Pure Vacuum state!



8.4 Closed Three Level system

A sp. E

B st. A/E



For full three-level system we must solve 3x3 density matrix equ.

Assume $|\Psi(0)\rangle$ a pure state

$$|\Psi\rangle = C_0|0\rangle + C_1|1\rangle + C_2|2\rangle$$

$$\hat{\rho}_{pure} = |\Psi\rangle\langle\Psi| = \begin{array}{c} \\ \end{array}\left[\begin{array}{ccc} |0\rangle & |1\rangle & |2\rangle \\ c_0^* c_0 & c_1^* c_0 & c_2^* c_0 \\ c_0^* c_1 & c_1^* c_1 & c_2^* c_1 \\ c_0^* c_2 & c_1^* c_2 & c_2^* c_2 \end{array}\right]\begin{array}{c} \\ \langle 0| \\ \langle 1| \\ \langle 2| \end{array}$$

$$= \begin{bmatrix} \rho_{00} & \rho_{10} & \rho_{20} \\ \rho_{01} & \rho_{11} & \rho_{21} \\ \rho_{02} & \rho_{12} & \rho_{22} \end{bmatrix}$$

where

$$\rho_{ii} = |C_i|^2 = P_i \quad \text{and} \quad \sum_{i=0}^{2} \rho_{ii} = 1$$

cons of probability

ρ_{ii} is called electron population in level $|i\rangle$

$\forall i \neq j$ (off diagonal) $\rho_{ij} = \rho_{ji}^*$ <u>coherence terms</u>

Let us assume $A_{10} = A_{20} = 0$

How do we obtain 3-level Rabi solution?

TDSE: $$i\hbar \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle$$

$$-i\hbar \frac{d}{dt}\langle\Psi| = \langle\Psi|\hat{H}$$

Hence

$$i\hbar \frac{d\hat{\rho}}{dt} = i\hbar \left[|\dot{\psi}\rangle\langle\psi| + |\psi\rangle\langle\dot{\psi}| \right]$$

$$= \hat{H} |\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| \hat{H}$$

$$= [\hat{H}, \hat{\rho}]$$

Even without loss / dissipation $A_{10} = A_{20} = 0$ this is <u>NINE</u> coupled first order diffy-Q. even assuming field is classical. Adding loss

$$\boxed{i\hbar \frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}] + \underline{\hat{L}}[\hat{\rho}]}$$

JUST MAKES IT WORSE - MUST BE DONE NUMERICALLY. However there is another approach!

Assume $A_{10} = A_1 \neq 0$ and $A_{20} = A_2 \neq 0$

$B_{10} = B_{01} = B_1$ Reciprocity

$B_{20} = B_{02} = B_2$ Reciprocity

For long times / after many jumps / all off diagonal coherence terms will vanish

$t \to 0 \Rightarrow \rho_{ij} = 0 \quad i \neq j$

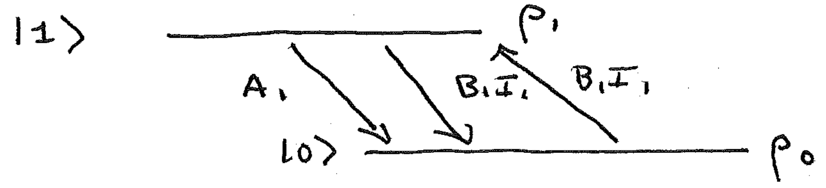
Hence we can write down <u>rate</u> equations for only $\rho_{11} = \rho_1$

$\rho_{22} = \rho_2$

$\rho_{33} = \rho_3$

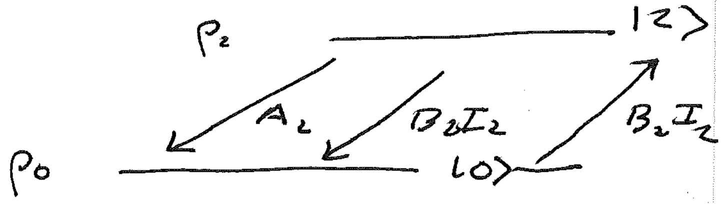
We assume system is <u>closed</u> so population is conserved $\rho_{11}(t) \mp \rho_{22}(t) + \rho_{33}(t) = 1$ $\forall t$: That is electron is always in one of three levels. We now just a' la Einstein compute the rates.

$$\boxed{\dot{\rho}_1 = -[A_1 + B_1 I_1]\rho_1 + B_1 I_1 \rho_0}$$



spontaneos emission prob $A_1\rho_1$ is proportional to Einstein A and prob. $P_1 = p_1$ that electron is <u>in</u> level |1⟩. similarly stimulated & $B_1 I_1$

$$\boxed{\dot{\rho}_2 = -[A_2 + B_2 I_2]\rho_2 + B_2 I_2 \rho_0}$$



Finally cons. prob.

$$\boxed{\dot{\rho}_{00} = -\underbrace{[B_1I_1 + B_2I_2]}_{\text{St. Abs.}}\rho_{00} + [\underset{\text{Sp.E}}{A_1} + \underset{\text{St. E}}{B_1I_1}]\rho_{11} + [\underset{\text{Sp E}}{A_2} + \underset{\text{St. E}}{B_2I_2}]\rho_{22}}$$

Let $A_i = \alpha_i$ and fundamental

$B_i I_i = \beta_i$ rate equations are

$$\dot\rho_{00} = -[\beta_1+\beta_2]\rho_{00} + [\alpha_1+\beta_1]\rho_1 + [\alpha_2+\beta_2]\rho_2$$

$$\dot\rho_1 = -[\alpha_1+\beta_1]\rho_1 + \beta_1\rho_0$$

$$\dot\rho_2 = -[\alpha_2+\beta_2]\rho_2 + \beta_2\rho_0$$

$$1 = \rho_1+\rho_2+\rho_3$$

First consider <u>steady state</u> solutions
$\dot\rho_i = 0$ for $t \to \infty$

$$\boxed{\begin{aligned} 0 &= -[\beta_1+\beta_2]\rho_0 + [\alpha_1+\beta_1]\rho_1 + [\alpha_2+\beta_2]\rho_2 \\ 0 &= -[\alpha_1+\beta_1]\rho_1 + \beta_1\rho_0 \\ 0 &= -[\alpha_2+\beta_2]\rho_2 + \beta_2\rho_0 \\ 1 &= \rho_0+\rho_1+\rho_2 \end{aligned}}$$

These can be solved exactly!
But first consider some limiting cases

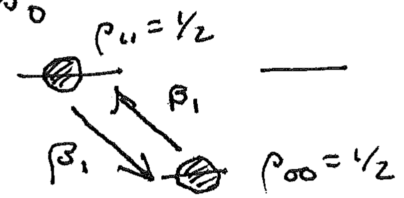
(I) $\beta_2 = \alpha_2 \simeq 0$ That is level two is decoupled I_2 very weak or off
Also $\alpha_1 \approx 0$ $\alpha_1 \lll \beta_1$ so spont. E. neg.

$$0 = -\beta_1\rho_0 + \beta_1\rho_1 \Rightarrow \rho_1 = \rho_0$$

~~$0 = -\beta_1\rho_1 + \beta$~~

$$1 = \rho_0 + \rho_1 + \cancel{\rho_2} \to 0$$ if population initially in ρ_0

$$\Rightarrow \rho_0 = \rho_1 = 1/2 \; ; \; \rho_2 = 0$$
Detailed Balance



(II) Assume: $\alpha_1 \cong \alpha_2 \approx 0$

or more realistically $\alpha_1 <<< \beta_1$ & $\alpha_2 <<< \beta_2$

In this case $\rho_0(0) = 1$; $\rho_1(0) = 0$; $\rho_2(0) = 0$ initial conditions.

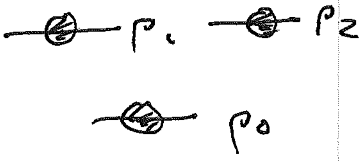
$$0 = -[\beta_1 + \beta_2]\rho_0 + \beta_1\rho_1 + \beta_2\rho_2$$

$$0 = -\beta_1\rho_1 + \beta_1\rho_0 \Rightarrow \rho_0 = \rho_1$$

$$0 = -\beta_2\rho_2 + \beta_2\rho_0 \Rightarrow \rho_0 = \rho_2 = \rho_1$$

$$1 = \rho_0 + \rho_1 + \rho_2$$

$$\Rightarrow t \to \infty \quad \boxed{\rho_0 = \rho_1 = \rho_2 = 1/3}$$

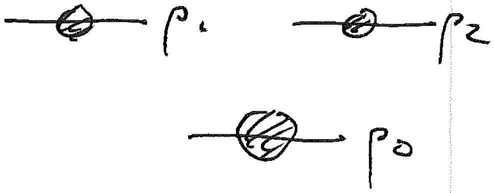


Again detailed balance places population equally. Spontaneous Emission α_1, α_2 <u>spoils</u> detailed balance! Hence $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$ we expect

$$\rho_0(\infty) > 1/3$$

$$\rho_1(\infty) < 1/3$$

$$\rho_2(\infty) < 1/3$$



as electron comes <u>down</u> faster than it goes up.

Astoundingly if you plug $t \to \infty$ equations into matematical it produces an <u>exact</u> solution

$$\boxed{\begin{aligned} \rho_1(\infty) &= \frac{(\alpha_2+\beta_2)\beta_1}{\alpha_1(\alpha_2+2\beta_2)+\beta_1(2\alpha_2+3\beta_2)} \\ \rho_2(\infty) &= \frac{(\alpha_1+\beta_1)\beta_2}{\alpha_2(\alpha_1+2\beta_1)+\beta_2(2\alpha_1+3\beta_1)} \\ \rho_{00}(\infty) &= 1-\rho_1^{(\infty)}-\rho_2(\infty) \end{aligned}}$$

which agrees with book. This makes no assumptions about relative sizes of α_i β_i as indicated that $\rho_1 \leftrightarrow \rho_2$ under interchange of $1 \leftrightarrow 2$.

Taking $\alpha_2 \approx \beta_2 \approx 0$ gives $\rho_0 = \rho_1 = 1/2$; $\rho_2 = 0$

Let's do this carefully

Ⓘ $\alpha_1 = \alpha_2 = 0$

$$\left.\begin{aligned} \rho_1(\infty) &= \frac{\beta_1\beta_2}{3\beta_1\beta_2} = \frac{1}{3} \\ \rho_2(\infty) &= \frac{\beta_1\beta_2}{3\beta_1\beta_2} = \frac{1}{3} \end{aligned}\right] \rightarrow \rho_0(\infty) = 1/3$$

Independent of $\beta_1 >> \beta_2$ or $\beta_1 << \beta_2$

Ⓘⓘ $\beta_2 = 0 \quad \alpha_1 = 0$

$$\left.\begin{aligned} \rho_1(\infty) &= \frac{\alpha_2\beta_1}{2\alpha_2\beta_1} = \frac{1}{2} \\ \rho_2(\infty) &= 0 \end{aligned}\right] \rightarrow \rho_0(\infty) = 1/2$$

Quantum Jump Condition Ⓘ Ⓘ Ⓘ

STRONG I_1 pump $\Rightarrow$ $\beta_1 >>> \alpha_1$ we can take $\alpha_1 \approx 0$ SATURATION OFF $D \Leftrightarrow 1$

$$\rho_1(\infty) = \frac{(\alpha_2+\beta_2)\beta_1}{(2\alpha_2+3\beta_2)\beta_1} = \boxed{\frac{\alpha_2+\beta_2}{2\alpha_2+3\beta_2}}$$

IND OF β_1!

$$\rho_2(\infty) = \frac{\beta_1\beta_2}{2\alpha_2\beta_1+3\beta_2\beta_1} = \boxed{\frac{\beta_2}{2\alpha_2+3\beta_2}}$$

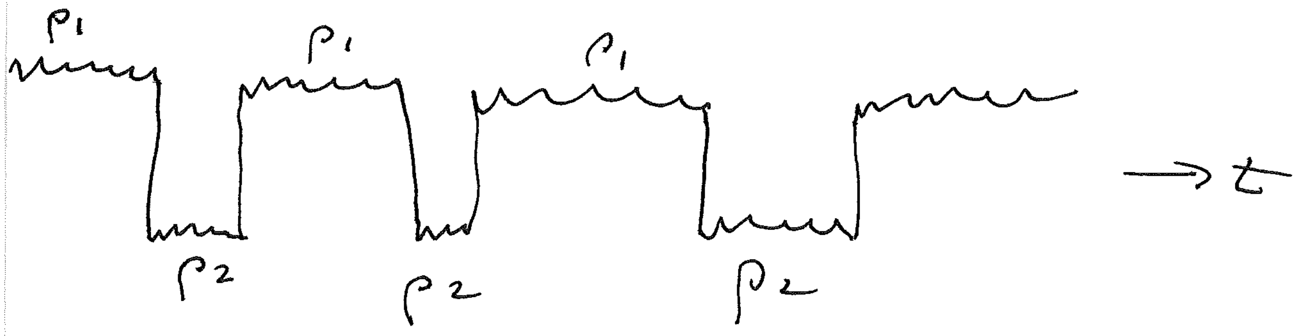
which agrees with book. Note this holds $\forall\ \beta_2$ and so even if $\beta_2 << \beta_1$ (weak I_2 pump) there is probability to find electron in $|2\rangle$

We can further assume $\beta_2 >> \alpha_2 \Rightarrow \gamma_2 = \frac{\alpha_2}{\beta_2} << 1$

$$\rho_1(\infty) = \frac{\gamma_2+1}{2\gamma_2+3} \cong \frac{1}{3} + \frac{1}{9}\gamma_2 \gtrsim \frac{1}{3}$$

$$\rho_2(\infty) = \frac{1}{2\gamma_2+3} \cong \frac{1}{3} - \frac{2}{9}\gamma_2 \lesssim \frac{1}{3}$$

So even when I_1 pump is very strong the prob of finding in $|2\rangle$ is <u>very</u> <u>high</u>



For $0 < t < \infty$ we must solve

$$\dot{\rho}_1 = -[\alpha_1 + \beta_1]\rho_1 + \beta_1(1-\rho_1-\rho_2)$$

$$\dot{\rho}_2 = -[\alpha_2 + \beta_2]\rho_2 + \beta_2(1-\rho_1-\rho_2)$$

where we have eliminated $\rho_0 = 1-\rho_1-\rho_2$

$$\boxed{\begin{aligned}\dot{\rho}_1 &= -[\alpha_1 + 2\beta_1]\rho_1 + \beta_1 - \beta_1\rho_2 \\ \dot{\rho}_2 &= -[\alpha_2 + 2\beta_2]\rho_2 + \beta_2 - \beta_2\rho_1\end{aligned}}$$

with inititial condition $\rho_1(0) = \rho_2(0) = 0$ $\rho_0(0) = 1$

Mathematica gives: A very <u>Large</u> output!

However solution is exact! Well have to make some approximations. Let's take $\beta_1 \gg\gg \alpha_1 \approx 0$

as before we may write

$$\frac{1}{\beta_1\beta_2}\dot{\rho}_1 = -\left[\cancel{\frac{\alpha_1}{\beta_1\beta_2}}^{\to 0} + 2\frac{\text{[illegible]}}{\beta_2}\right]\rho_1 + \frac{1}{\beta_2} - \frac{1}{\beta_2}\rho_2$$

$$\frac{1}{\beta_1\beta_2}\dot{\rho}_2 = -\left[\cancel{\frac{\alpha_2}{\beta_1\beta_2}}^{\to 0} + 2\frac{\text{[illegible]}}{\beta_1}\right]\rho_1 + \frac{1}{\beta_1} - \frac{1}{\beta_1}\rho_1$$

$$\Rightarrow \boxed{\begin{aligned}\dot{\rho}_1 &= -2\beta_1\rho_1 + \beta_1 - \beta_1\rho_2 \\ \dot{\rho}_2 &= -2\beta_2\rho_2 + \beta_2 - \beta_2\rho_1\end{aligned}}$$

$$\frac{1}{\beta_2}\dot{\rho}_1 = -2\frac{\beta_1}{\beta_2}\rho_1 + \frac{\beta_1}{\beta_2} - \frac{\beta_1}{\beta_2}\rho_2$$

$$\frac{1}{\beta_2}\dot{\rho}_2 = -2\rho_2 + 1 - \rho_1$$

Inserting $\dot{\rho}_2/\beta_2 \to \frac{1}{\beta_2}\ddot{\rho}_1$ and drop $\frac{\beta_1}{\beta_2} \ll 1$

$$\frac{1}{\beta_2}\ddot{\rho}_1 = -2\cancelto{0}{\frac{\beta_1}{\beta_2}}\dot{\rho}_1 + \cancelto{0}{\frac{\beta_1}{\beta_2}} - \frac{\beta_1}{[illegible]}\left[-2\rho_2 + 1 - \rho_1\right]$$

This tells us

$$\boxed{\begin{aligned} \dot{\rho}_1 &\cong [illegible] - \beta_1\rho_2 \\ \dot{\rho}_2 &\cong -2\beta_2\rho_2 + \beta_2 - \beta_2\rho_1 \end{aligned}} \qquad \begin{aligned} \rho_1(0) &= 0 \\ \rho_2(0) &= 0 \end{aligned}$$

After some work in Mathematica I get

$$\rho_1(t) \approx \frac{1}{3}\left[1 + \frac{1}{2}\left(e^{-3\beta_2 t/2} - 3e^{-2\beta_1 t}\right)\right]$$

$$\rho_2(t) \approx \frac{1}{3}\left[1 - e^{-3\beta_2 t}\right]$$

$$\rho_1(0) = \frac{1}{3}\left[1 + \frac{1}{2}(1-3)\right] = \frac{1}{3}\left[1-1\right] = 0 \checkmark$$

$$\rho_2(0) = \frac{1}{3}\left[1-1\right] = 0 \checkmark$$

$$\rho_1(\infty) = 1/3 = \rho_2(\infty) \checkmark$$

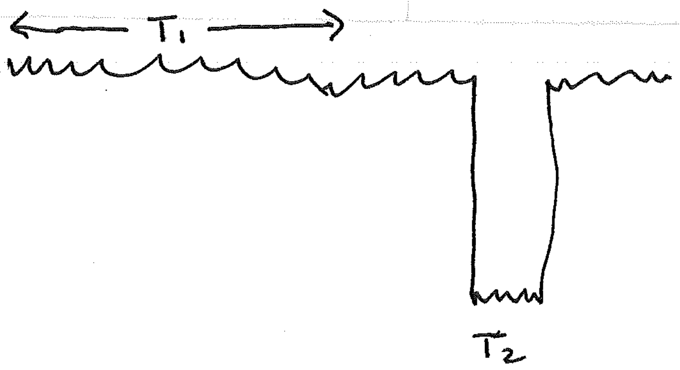
So for $0 < t \ll T_2 = 1/\beta_2$ (JUMP TIME)

$$\rho_1(t \ll T_2) \approx \frac{1}{3}\left[1 + \frac{1}{2}\left[1 - 3e^{-2\beta_1 t}\right]\right]$$

$$= \frac{1}{3}\left[\frac{3}{2} - \frac{3}{2}e^{-2\beta_1 t}\right] \to 1/2 \qquad 0 \leftrightarrow 1$$

SATURATION

$$\rho_1(t \gg T_2) \approx \frac{1}{3}$$ Detailed Balance.



Let $\beta_1 = 1$ $\beta_2 = 0.1$