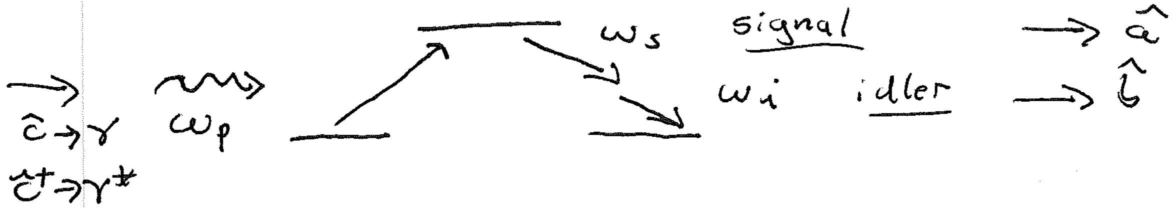
7.7 Two Mode Squeezed Vacuum

In χ_2 parametric downconversion: the general case



$$|\omega_p\rangle \rightarrow |\omega_s\rangle|\omega_i\rangle$$

We do not have to have $\omega_s = \omega_i$

polarization and direction can be different

$$|1\rangle_{\omega_p, \vec{k}_p, s_p} \longrightarrow |1\rangle_{\omega_s \vec{k}_s s_s} \quad |1\rangle_{\omega_i \vec{k}_i s_i}$$

↑ frequency ↑ wavenumber ↑ polarization

They are all connected

Freq.	$\hbar\omega_p = \hbar\omega_s + \hbar\omega_i$	Cons. Erg
Wavenumber	$\hbar\vec{k}_p = \hbar\vec{k}_s + \hbar\vec{k}_i$	Cons. mom.
polarization	$\hbar\vec{e}_p = \hbar\vec{e}_s + \hbar\vec{e}_i$	Cons. ∡ mom.

so in general signal and idler in <u>different</u> modes labled by $\omega_i \neq \omega_s$; $\vec{k}_i \neq \vec{k}_s$; $\vec{e}_i \neq \vec{e}_s$

To model this we associate $\hat{a}$ with one mode and $\hat{b}$ with the other $\Rightarrow [\hat{a}, \hat{b}^+] = 0$

The hamiltonian is

γ = pump field

$$\hat{H}_i = \hbar K [\gamma \hat{a}\hat{b} - \gamma^* \hat{a}^+ \hat{b}^+]$$

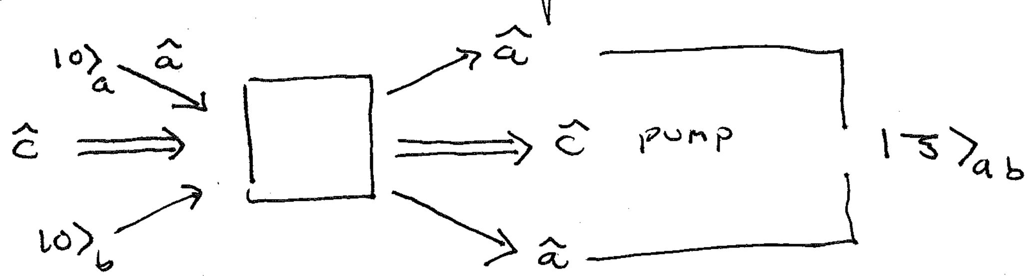
$$\nearrow\nearrow \qquad \searrow\searrow$$

This implies a two mode evolution

$$\hat{U} = e^{-i\varphi \hat{H}_{inT}}$$

$$= \exp\left[\xi^* \hat{a}\hat{b} - \xi \hat{a}^\dagger \hat{b}^\dagger\right] = \hat{S}^{ab}_{\xi}$$

where $\xi = K\gamma t = re^{i\theta}$ is the squeeze operator which inherits the phase of the pump.



where $|0\rangle_a |0\rangle_b \Rightarrow \hat{S}^{ab}_{\xi} |0\rangle_a |0\rangle_b = |\xi\rangle_{ab}$

This is a two-mode squeezed state

The squeezing is distributed across the two modes. Note $|\xi\rangle_{ab} \neq |\xi\rangle_a |\xi\rangle_b$

and hence entanglement can occur; that is quantum correlations between modes. The two-mode quadratures can be defined

$$\hat{X}_1^{ab} \equiv \frac{1}{2^{3/2}}\left[(\hat{a}+\hat{a}^\dagger) + (\hat{b}+\hat{b}^\dagger)\right] \approx \left[\hat{X}_1^a + \hat{X}_1^b\right]\frac{1}{\sqrt{2}}$$

$$\hat{X}_2^{ab} \equiv \frac{1}{2^{3/2} i}\left[(\hat{a}-\hat{a}^\dagger) + (\hat{b}-\hat{b}^\dagger)\right] \approx \left[\hat{X}_2^a + \hat{X}_2^b\right]\frac{1}{\sqrt{2}}$$

Recall $[X_1^a, X_2^a] = [X_1^b, X_2^b] = \frac{i}{2}$

$\Rightarrow \quad [\tilde{X}_1^{ab}, \hat{X}_2^{ab}] = \frac{1}{2}[X_1^a + X_1^b, X_2^a + X_2^b]$

$= \frac{1}{2}\{ [X_1^a, X_2^a] + \cancel{[X_1^a, X_2^b]}^{\to 0} + \cancel{[X_1^b, X_2^a]}^{\to 0} + [X_1^b, X_2^b]$

$= \frac{1}{2}\{ \frac{i}{2} + \frac{i}{2} \} = i/2$

So $[X_i^{ab}]$ obey same commutator as X_i^a or X_i^b

The two-mode squeeze xfrm is

$$\boxed{\begin{aligned} \hat{S}^+_{ab}(\zeta)\, \hat{a}\, \hat{S}_{ab}(\zeta) &= \hat{a}\cosh r - e^{i\theta}\hat{b}^+ \sinh r \\ \hat{S}^+_{ab}(\zeta)\, \hat{b}\, \hat{S}_{ab}(\zeta) &= \hat{b}\cosh r - e^{i\theta}\hat{a}^+ \sinh r \end{aligned}}$$

Proof: since $[a, b] = 0$ this is fairly easy with CBH.

$$e^{i\lambda\hat{X}}\hat{Y}e^{-i\lambda\hat{X}} = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!}[\hat{X}, \hat{Y}]^{(n)}$$

First $\hat{X} = [\zeta^* \hat{a}\hat{b} - \zeta\hat{a}^+\hat{b}^+]^+ = -\hat{X}$ so let $\underline{\lambda = i}$

$$\hat{S}^+_{ab}(\zeta)\, \hat{a}\, \hat{S}_{ab}(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}[\hat{X}, \hat{a}]^{(n)}$$

but $[\hat{X}, \hat{a}]^{(1)} = \zeta^*[\hat{a}, \hat{a}]\hat{b} - \zeta[\hat{a}, \hat{a}^+]b^+ = -\zeta\hat{b}^+$

$[X, [X, a]] = (\zeta^* a)(-\zeta) = -r a$

even terms $\Rightarrow \hat{a}\sum_{n\,\text{even}} \frac{r^n}{n!} = \hat{a}\cosh r$

odd terms $\Rightarrow -e^{i\theta}\hat{b}^+ \sum_{n\,\text{odd}} \frac{r^n}{n!} = -e^{i\theta}\hat{b}^+\sinh r$

$$\Rightarrow \quad \hat{S}^{+} a \hat{S} = \hat{a}\cosh r - e^{i\theta}\hat{b}^{+}\sinh r$$

Note since $[a,b]=0$ $\quad \hat{S}_{ab} = \hat{S}_{ba}$

Hence we may just make the substitution:

$$\hat{a} \rightarrow \hat{b} \;;\quad \hat{b}^{+} \rightarrow \hat{a}^{+}$$

$$\Rightarrow \quad \hat{S}^{+}\hat{b}\hat{S} = \hat{b}\cosh r - e^{i\theta}\hat{a}^{+}\sinh r \quad \therefore$$

Using these transforms we can compute the quadratures ΔX_1^2 ; ΔX_2^2

Note

$$\langle \xi | \hat{X}_1 | \xi \rangle$$

$$= \langle 0 | S^{+}\hat{X}_1 S | 0 \rangle$$

$$= \frac{1}{2^{3/2}} \langle 0 | \hat{S}^{+} [\, a + a^{+} + b + b^{+}] \hat{S} | 0 \rangle$$

$$= \frac{1}{2^{3/2}} \langle 0 | S^{+} a S + S^{+} a^{+} S + S^{+} b S + S^{+} b^{+} S | 0 \rangle$$

$$= \frac{1}{2^{3/2}} \langle 0 | \{ a, a^{+}, b^{+}, b \} | 0 \rangle \quad \text{always zero}$$

$$= 0 \quad \text{By inspection}$$

Same for $\langle \xi | \hat{X}_2^{ab} | \xi \rangle = 0$

Hence $\langle \hat{X}_i^{ab} \rangle^2 \equiv 0$

$$\Rightarrow \quad \boxed{\Delta X_i^{ab\,2} = \underline{\underline{\langle \hat{X}_i^{ab\,2} \rangle}}} - \cancel{\langle \hat{X}_i \rangle^2}^{\,0}$$

Trick!

Let $\hat{c} \equiv \frac{\hat{a}+\hat{b}}{\sqrt{2}}$ $\quad \hat{c}^+ = \frac{\hat{a}^+ + \hat{b}^+}{\sqrt{2}}$ Hence

$$\hat{X}_1^{ab} = \frac{1}{2}[c+c^+] = \left[\frac{c+c^+}{2}\right] \equiv \hat{X}_1^c$$

$$\hat{X}_2^{ab} = \frac{1}{2i}[c-c^+] = \left[\frac{c-c^+}{2i}\right] \equiv \hat{X}_2^c$$

we may treat "c" as fictitious single mode

Hence $\boxed{\Delta^2 X_i^{ab} = \Delta^2 X_i^c}$ since $[a,b]$ commute!

$[c,c^+] = \frac{1}{2}([a,a^+]+[b,b^+]) = i$

However $\hat{S}^+ \hat{c}\, \hat{S} = (a\cosh r - e^{i\theta} b^+ \sinh r) + (b\cosh r - e^{i\theta} a^+ \sinh r)$

$$\boxed{\begin{aligned} S^+ \hat{c}\, \hat{S} &= \hat{c}\cosh r - e^{i\theta}\hat{c}^+ \sinh r \\ \hat{S}^+ \hat{c}^+ \hat{S} &= \hat{c}^+ \cosh r - \hat{c}\, e^{-i\theta} \sinh r \end{aligned}}$$

But these are single mode squeeze xfrms; Eq. 7.12 with the identification $\hat{c} \leftrightarrow \hat{a}$. Hence we may lift Eq. 7.16 & 7.17 for single mode case

$$\Delta^2 X_i^{ab} \equiv \Delta^2 X_i^c = \frac{1}{4}\left\{\cosh^2 r + \sinh^2 r \mp 2\sinh r \cosh r \cos\theta\right\}$$

$$= \boxed{\frac{1}{4}\left[\cosh(2r) \mp \sinh 2r \cos\theta\right]}$$

Let $\Theta = 0 \Rightarrow$

$$\Delta X_i^2 = \frac{1}{4}\left[\cosh 2r \mp \sinh 2r\right]$$

$$= \frac{1}{4}\left[\left(\frac{e^{2r}+e^{-2r}}{2}\right) \mp \left(\frac{e^{2r}-e^{-2r}}{2}\right)\right]$$

$$= \frac{1}{4}\begin{cases} e^{-2r} & < 1/4 \\ e^{2r} & > 1/4 \end{cases}$$

So there is squeezing in X_1

The decomposition into numberstates now has form

$$\boxed{|\xi\rangle_{ab} = \sum_{nm} C_{nm}\, |n\rangle_a |m\rangle_b}$$

Hence there is entanglement in number!

Note $\hat{a}|00\rangle = \hat{a}|0\rangle_a |0\rangle_b = 0$

$\hat{b}|00\rangle = |0\rangle_a \hat{b}|0\rangle_b = 0$

IN SIMILAR PROCEEDURE TO SINGLE MODE

$$\hat{S}_\xi^{+} a S_\xi \xrightarrow[\xi \to \xi^*]{} \begin{array}{l} \hat{S}\hat{a}\hat{S}^{+} = \hat{a}\cosh r + e^{i\Theta}\hat{b}^{+}\sinh r \\ \hat{S}\hat{b}\hat{S}^{+} = \hat{b}\cosh r + e^{i\Theta}\hat{a}^{+}\sinh r \end{array}$$

Hence $\hat{a}|0\rangle = 0$

$$\Rightarrow S[\hat{a} S^{+} S |0\rangle] = 0$$

$$\Rightarrow (\hat{S}\hat{a}\hat{S}^{+})|\xi\rangle = 0 \quad ; \quad (\hat{S}\hat{b}\hat{S}^{+})|\xi\rangle = 0$$

Let $\mu \equiv \cosh r$ $\quad \nu = e^{i\theta} \sinh r$

$\Rightarrow \quad [\mu \hat{a} + \nu \hat{b}^{+}] |\xi\rangle = 0 \quad$ <u>and</u> $\quad [\mu \hat{b} + \nu \hat{a}^{+}] |\xi\rangle = 0$

$\Leftrightarrow \quad |\xi\rangle_{ab} \equiv |\xi\rangle_{ba} \quad \forall \quad n$

$$\Leftrightarrow \quad \boxed{|\xi\rangle = \sum_n C_n |n\rangle_a |n\rangle_b}$$

$$(\mu a + \nu b^{+})|\xi\rangle = \sum_{n=\cancel{0}}^{\infty} C_n \mu \sqrt{n} \, |\overbrace{n-1}^{n'}\rangle_a |n\rangle_b \quad \leftarrow \text{resum}$$

$$+ \sum_{n=0}^{\infty} C_n \nu \sqrt{n+1} \, |n\rangle_a |\overbrace{n+1}^{n''}\rangle_b \quad \leftarrow \text{don't}$$

$$= \sum_{n=0}^{\infty} C_{n+1} \mu \sqrt{n+1} \, |n\rangle |n+1\rangle + \sum_{n=0}^{\infty} C_n \nu \sqrt{n+1} \, |n\rangle |n+1\rangle = 0$$

$$\Rightarrow \quad \boxed{C_{n+1} = -\frac{\nu}{\mu} C_n}$$

$$\Rightarrow \quad C_1 = -\frac{\nu}{\mu} C_0$$

$$C_2 = -\frac{\nu}{\mu} C_1 = \left(-\frac{\nu}{\mu}\right)^2 C_0$$

$$\vdots$$

$$C_n = \left(-\frac{\nu}{\mu}\right)^n C_0$$

$$= \left[-\frac{e^{i\theta} \sinh r}{\cosh r}\right]^n C_0$$

$$= \boxed{(-1)^n e^{in\theta} \tanh^n r \; C_0}$$

To find C_0

$$\boxed{|\xi\rangle_{ab} = C_0 \sum_n (-1)^n e^{in\theta} \tanh^n r \, |n\rangle |n\rangle}$$

Demand

$$1 = {}_{ab}\langle \xi | \xi \rangle_{ab} = |c_0|^2 \sum_n (\tanh^2 r)^n \quad (0 < \tanh r < 1)$$

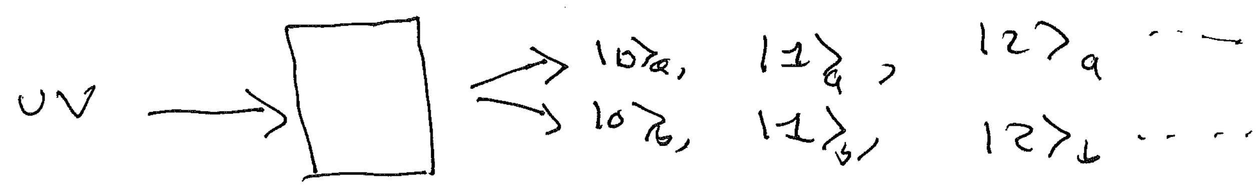
$$= |c_0|^2 \frac{1}{1 - \tanh^2 r}$$

$$= |c_0|^2 \frac{1}{\operatorname{sech}^2 r} = |c_0|^2 \cosh^2 r$$

WLOG

$$c_0 = \frac{1}{\cosh r} = \operatorname{sech} r$$

$$\boxed{|\xi\rangle_{ab} = \operatorname{sech} r \sum_n (-1)^n e^{in\theta} \tanh^n r \, |n\rangle_a |n\rangle_b}$$



Spontaneous Parametric Downconversion

$r \ll 1 \Rightarrow \operatorname{sech} r \cong 1 \quad \tanh r \approx r$

neglect

$$|\xi\rangle^{ab}_{r \ll 1} \approx \underbrace{|0\rangle_a |0\rangle_b}_{\text{MOSTLY VACUUM}} + e^{i(\theta+\pi)} r |1\rangle_a |1\rangle_b + \cancel{\mathcal{O}}(r^2)$$

SMALL # $|1\rangle|1\rangle$

NEG. $|2\rangle|2\rangle$

The two modes $|n\rangle_a |n\rangle_b$ are strongly corollated in number. Let's ask suppose I detect "m" in mode "a" what will I detect in mode b?

$$\underbrace{|m\rangle_a\langle m|}_{\text{projector}}\, |\xi\rangle_{ab} = |m\rangle_a \sum_n c_n \; {}_a\underbrace{\langle m|n\rangle_a}_{\delta_{mn}} |n\rangle_b$$

$$= \underbrace{c_n |m\rangle_a\, |m\rangle_b}_{\text{state after collapse}}$$

renormalize $\Longrightarrow$ $\boxed{\left(|m\rangle_a\langle m|\right)|\xi\rangle_{ab} = |m\rangle_a |m\rangle_b}$

That is if I measure "5" photons in mode "a" I will then see "5" photons in mode "b". <u>Highly corellated!</u>

Consider the difference operator

$$\hat{M}_{ab} \equiv \hat{n}_a - \hat{n}_b = a^+a - b^+b$$

$${}_{ab}\langle\xi|\, \hat{M}_{ab}\, |\xi\rangle_{ab} = \sum_n c_n \left[{}_a\langle n|\hat{n}_a|n\rangle_a \underbrace{\langle n|n\rangle_b}_{1} - \underbrace{{}_a\langle n|n\rangle_a}_{1} \langle n|\hat{n}_b|n\rangle_b \right]$$

$$= \sum_n c_n \left[\, n - n \,\right]$$

$$\equiv 0$$

Again indicates highly corellated in <u>number</u>

$$\langle\xi|\, \hat{n}_a\, |\xi\rangle = \langle 0|\langle 0|\, \hat{S}\, \hat{a}^+ s\, s^+ a\, s\, |0\rangle|0\rangle$$

$$= \langle 0|\langle 0|\left[a^+ c - e^* b\, \nu \right]\left[a\, c - e\, b^+ \nu \right] |0\rangle|0\rangle$$

$$= +|e|^2 \; {}_a\langle 0|0\rangle_a \; \langle 0|\, b\, b^+|0\rangle\, \nu^2$$

$$= \boxed{\sinh^2 r = \langle \hat{n}_b\rangle} \quad \text{by symmetry } a \Leftrightarrow b$$

$$\langle \hat{n}_a^2 \rangle = \langle 0| s^\dagger a^\dagger s s^\dagger a s s^\dagger a^\dagger s s^\dagger a s |0\rangle$$

$$= \langle 0| [a^\dagger c - e^* b r][a c - e b^\dagger r][a^\dagger c - e^* b r][a c - e b^\dagger r] |0\rangle$$

$$= \langle 0| [a^\dagger a c^2 - e a^\dagger b^\dagger c r - e^* b a c r + b b^\dagger r^2] \qquad [a, b] = 0 \text{ etc}$$

$$\times [a^\dagger a c^2 - e a^\dagger b^\dagger c r - e^* b a c r + b b^\dagger r^2] |0\rangle$$

$$= \langle 0| [a^\dagger a c^2 - e a^\dagger b^\dagger c r - e^* a b c r + b b^\dagger r^2]$$

$$\times [a^\dagger a c^2 - e a^\dagger b^\dagger c r - e^* a b c r + b b^\dagger r^2] |0\rangle$$

$$= \langle 0| \begin{bmatrix} a^\dagger a \cancel{a^\dagger a}^{\to 0} c^4 & 0 & 0 & \cancel{a^\dagger a}^{\to 0} b b^\dagger c^2 r^2 \\ 0 & + e^2 \cancel{a^{\dagger 2} b^{\dagger 2}}^{\to 0} c^2 r^2 & + a^\dagger a \cancel{b^\dagger b}^{\to 0} c^2 r^2 & 0 \\ 0 & + b b^\dagger a a^\dagger c^2 r^2 & + e^{*2} \cancel{a^2 b^2}^{\to 0} c^2 r^2 & 0 \\ b b^\dagger \cancel{a^\dagger a}^{\to 0} c^2 r^2 & 0 & 0 & + b b^\dagger b b^\dagger r^4 \end{bmatrix} |0\rangle$$

$$= \langle 0|\langle 0| [(1 + b^\dagger b)(1 + a^\dagger a) c^2 r^2 + (1 + b^\dagger b)(1 + b^\dagger b) r^4] |0\rangle|0\rangle$$

$$= c^2 r^2 + r^4 = \cosh^2 r \sinh^2 r + \underbrace{\sinh^4 r}_{\langle n \rangle^2}$$

$$\Rightarrow \boxed{\Delta^2 n_a = \sinh^2 r \cosh^2 r = \frac{1}{4} \sinh^2(2r) = \Delta^2 n_b}$$

Mandel Q Test

$$Q_{ab} = \frac{\Delta^2 n - \langle \hat{n} \rangle}{\langle \hat{n} \rangle} = \frac{\frac{1}{4}\sinh^2(2r) - \sinh^2 r}{\sinh^2 r}$$

$$= \frac{\sinh^2 r \cosh^2 r - \sinh^2 r}{\sinh^2 r} = \underbrace{\cosh^2 r}_{\geq 1} - 1 \geq 0 \quad \frac{\text{Super}}{\text{Poissonian}}$$

Variance $\equiv \Delta^2 n_a = \Delta^2 n_b = \frac{1}{4}\sinh^2(r)$

describes fluctuations/corellations <u>in</u> each mode a, b separately.

Co Variance $\equiv \text{Cov}[\hat{n}_a, \hat{n}_b] \equiv \langle \hat{n}_a \hat{n}_b \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle$

describes fluctuations/corellations <u>across</u> both modes a, b.

The Covariance of any two random variables is

$$\text{Cov}[X, Y] = \langle XY \rangle - \langle X \rangle \langle Y \rangle$$ Classical Prob. Theory.

So Covariance of any two observables $\hat{X}, \hat{Y}$

$$\text{Cov}[\hat{X}, \hat{Y}] = \langle \hat{X}\hat{Y} \rangle - \langle \hat{X} \rangle \langle \hat{Y} \rangle$$

measures degree of corellation!

simple example $\hat{X} = \hat{X}, \quad \hat{Y} = \hat{X}$

$$\text{Cov}[\hat{X}, \hat{X}] = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 = \text{Variance}.$$

Note: $\langle \hat{X}\hat{Y} \rangle = \langle \hat{X} \rangle \langle \hat{Y} \rangle \Rightarrow \text{Cov}[X, Y] = 0$

completely un-corellated

Consider ops $\boxed{\hat{M}_\pm \equiv \hat{n}_a \pm \hat{n}_b}$ commute! $[\hat{n}_a, \hat{n}_b] = 0$

$$\begin{aligned}
\Delta^2 M_\pm &= \langle [\hat{n}_a \pm \hat{n}_b][\hat{n}_a \pm \hat{n}_b] \rangle - \langle \hat{n}_a \pm \hat{n}_b \rangle^2 \\
&= \langle [\hat{n}_a^2 \pm 2\hat{n}_a\hat{n}_b + \hat{n}_b^2] \rangle - [\langle \hat{n}_a \rangle \pm \langle \hat{n}_b \rangle]^2 \\
&= \langle \hat{n}_a^2 \rangle \pm 2\langle \hat{n}_a\hat{n}_b \rangle + \langle \hat{n}_b^2 \rangle - [\langle \hat{n}_a \rangle^2 \pm 2\langle \hat{n}_a \rangle \langle \hat{n}_b \rangle + \langle \hat{n}_b \rangle^2] \\
&= \boxed{\Delta^2 n_a + \Delta^2 n_b \pm 2\,\text{cov}[\hat{n}_a, \hat{n}_b]}
\end{aligned}$$

Recall $\langle \hat{M}_- \rangle \equiv 0$ Eq 7.170

but $\hat{M}_- |\xi\rangle = 0 \Rightarrow \hat{M}_-^2 |\xi\rangle = 0$

$$\Rightarrow \langle \xi | \hat{M}_-^2 | \xi \rangle = 0$$

$$\Rightarrow \Delta^2 M_- = 0 = \Delta^2 n_a + \Delta^2 n_b - 2\,\mathrm{cov}[\hat{n}_a, \hat{n}_b]$$

$$\Rightarrow \boxed{\mathrm{cov}[\hat{n}_a, \hat{n}_b] = \frac{1}{2}\left[\Delta^2 n_a + \Delta^2 n_b\right] = \frac{1}{4}\sinh^2(2r)}$$

$r \to 0$: 0 $\qquad$ $r \to \infty$: ∞

So as $r \to \infty$ the correlations grow as $\frac{1}{4} \cdot \frac{1}{4} e^{4r}$ or very fast!

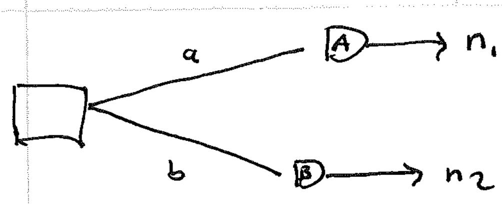
Again from classical prob. theory given two random variables we define the linear corellation func.

$$\boxed{J(x,y) \equiv \frac{\mathrm{cov}[x\,y]}{\Delta x \cdot \Delta y}} \in [0,1]$$

which is just the normalized covariance

$$\boxed{J[\hat{n}_a, \hat{n}_b] = \frac{\mathrm{cov}[\hat{n}_a, \hat{n}_b]}{\Delta n_a \cdot \Delta n_b} \begin{array}{l} = 1 \\ = 0 \end{array} \begin{cases} r > 0 \\ r = 0 \end{cases}}$$

so maximally corellated



what is joint probability a detector in A counts n_1 <u>and</u> detector in B counts n_2?

$$P_{n_1}^{n_2} \equiv \left| {}_a\langle n_1| \, {}_b\langle n_2| \, \xi \rangle_{ab} \right|^2$$

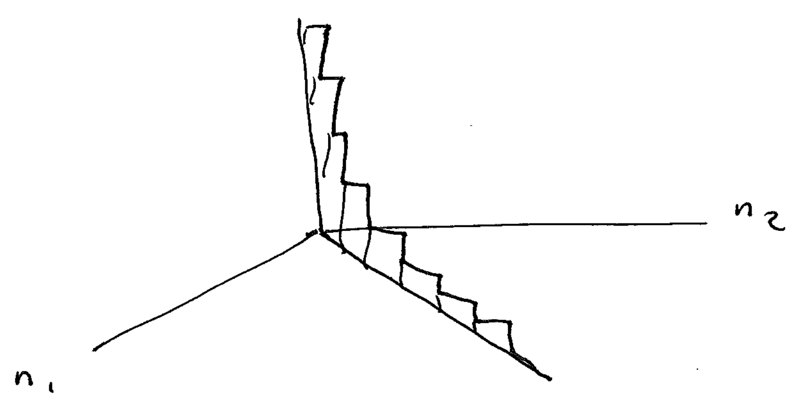
$$= \left| \sum_n C_n \; \overbrace{{}_a\langle n_1 | n \rangle_a}^{\delta_{n_1}^{n}} \; \overbrace{{}_b\langle n_2 | n \rangle_b}^{\delta_{n_2}^{n}} \right|^2$$

$$= \left| C_{n_1} \; {}_b\langle n_2 | n_1 \rangle_b \right|^2$$

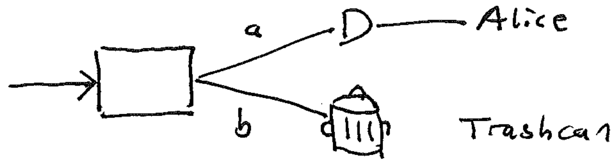
$$= |C_n|^2 \, \delta_{n_1}^{n_2}$$

$$= \boxed{\tanh^{2n}(r) \cdot \text{sech}^2(r) \; \delta_{n_1}^{n_2}}$$

Hence this joint probability is zero unless $n_1 = n_2$ a measurement by alice collapses both modes into $n \rightarrow n_1$ and hence Bob can only measure n_1 as well. Since $0 < \tanh^2(r) < 1$ then $P_{n_1}^{n_2}$ is diagonal



Different set up



Suppose I place a beam dump in mode b — what does Alice see? Alice may not look in trash can (environment) so she must express her ignorance by tracing out mode b. Alice then has a mixed state. Start with pure state density matrix

$$\hat{\rho}_{ab} \equiv |\xi\rangle_{ab}\langle\xi|$$

$$= \sum_{nm} C_n^* C_m \, |n\rangle_a |n\rangle_b \, {}_a\langle m| \, {}_b\langle m|$$

$$\hat{\rho}_a = \mathrm{tr}_b[\hat{\rho}_{ab}]$$

$$= \sum_l {}_b\langle l|\hat{\rho}_{ab}|l\rangle_b$$

$$= \sum_{lmn} C_n^* C_m \, |n\rangle_a \, \overbrace{{}_b\langle l|n\rangle_b}^{\delta_n^l} \, {}_a\langle m| \, \overbrace{{}_b\langle m|l\rangle_b}^{\delta_n^l}$$

$$= \boxed{\sum_n |C_n|^2 \, |n\rangle_a}$$ Alice Ignores Bob

By symmetry $a \leftrightarrow b$

$$\boxed{\hat{\rho}_b = \sum_n |C_n|^2 \, |n\rangle_b}$$ Bob ignores Alice

Recall $|C_n|^2 \equiv P_n = \underline{\tanh^{2n} r \cdot \mathrm{sech}^2 r}$

We know $\sum_n P_n = \operatorname{sech}^2 r \dfrac{1}{1-\tanh^2 r} = 1$

So P_n is a normalized probability distribution.

Also Eq. 7.171 $\Rightarrow$ $\langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \sum_n n P_n = \sinh^2 r$

$P_n^{a,b} = \tanh^{2n} r \operatorname{sech}^2 r$ $\qquad \cosh^2 - \sinh^2 = 1$

$$= \frac{\tanh^{2n} r}{\cosh^2 r} = \cancel{\frac{\tanh^{2n} r}{1+\sinh^2 r}} = \frac{[\sinh^2 r]^n}{[\cosh^2 r]^{n+1}} = \frac{[\sinh^2 r]^n}{[1+\sinh^2 r]^{n+1}}$$

$$= \boxed{\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}} \qquad \text{where } \bar{n} = \bar{n}_a = \bar{n}_b$$

Hence Alice E.O. sees a distribution Eq. 2.145 which is a thermal state!

$$\cancel{\bar{n} = \boxed{\sinh^2 r = e^{-\frac{\hbar\omega}{kT}}}} \qquad \bar{n} = \sinh^2 r = \frac{1}{e^x - 1}$$

$x = \dfrac{\hbar\omega}{kT}$ $\quad \Rightarrow \quad e^x - 1 = \cancel{\coth^2 r}\ \operatorname{csch}^2 r$

$\Rightarrow \quad e^x = \cancel{\coth}\ \operatorname{csch}^2 r - 1 = \coth^2 r$

$\Rightarrow \quad x = \ln[\coth^2 r]$

$$\boxed{T = \frac{\hbar\omega}{2k \cancel{\coth^2} \ln[\coth r]}}$$

$r \to 0$: 0 $\qquad$ $r \to \infty$: ∞

So squeezing $\Leftrightarrow$ Temperature Effective.

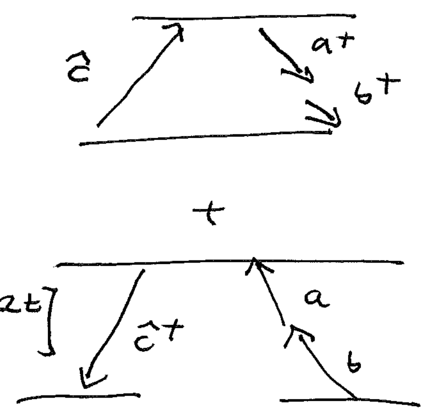
The full hamiltonian is (did this)!

$$\hat{H} = \hbar\omega_a a^+a + \hbar\omega_b b^+b + \hbar\omega_c c^+c + \overbrace{i\hbar\chi_2\left[\hat{a}\hat{b}\hat{c}^+ - \hat{a}^+\hat{b}^+\hat{c}\right]}^{\hat{H}_{INT}}$$

$$\hat{c} \rightarrow \gamma e^{-i\omega_c t}$$

$$\hat{a} \rightarrow \hat{a}\, e^{-i\omega_a t}$$

$$\hat{b} \rightarrow \hat{b}\, e^{-i\omega_b t}$$



$$\Rightarrow \hat{H}_{INT} = i\hbar\chi_2\left[\gamma\hat{a}\hat{b}e^{+i\Omega t} - \gamma^*\hat{a}^+\hat{b}^+e^{-i\Omega t}\right]$$

$$\Omega = \omega_c - \omega_a - \omega_b = 0 \quad \text{cons. Erg.}$$

$$\Rightarrow \hat{H}_{INT} = i\hbar\left[\xi^* ab - \xi a^+b^+\right]$$

$$\xi = \chi_2\gamma$$

$$\Rightarrow \quad U(t) = e^{-\frac{i}{\hbar}\hat{H}_I t} = \hat{S}_{ab}(\zeta)$$

$$\zeta = \xi t = \chi_2\gamma t$$