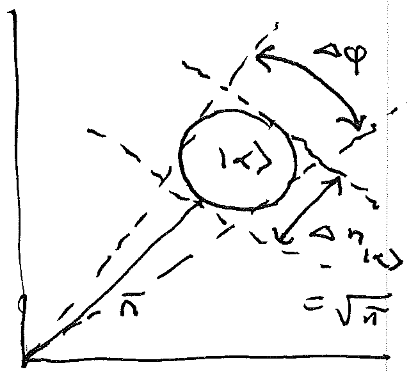
## 7.4 Amplitude (number) SQUEEZING

For a coherent state $|\alpha\rangle$ we have $|\alpha|^2 = \bar{n}$ and

$\Delta n_{|\alpha\rangle} = \sqrt{\bar{n}}$




Since $|\alpha\rangle$ is a M.U.S.

we concluded $\Delta n \cdot \Delta\varphi = 1$

$\Rightarrow \quad \Delta\varphi = \frac{1}{\sqrt{\bar{n}}}$

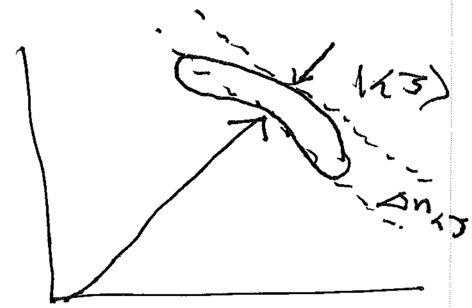
This can be seen graphically via

$$\Delta n = \bar{n}\, \Delta\varphi \Rightarrow \sqrt{\bar{n}} = \bar{n}\, \Delta\varphi \Rightarrow \boxed{\Delta\varphi = \frac{1}{\sqrt{\bar{n}}}}$$

shot noise

we'll call a state $|\alpha, \xi\rangle$ number squeezed iff

$$\Delta n_{|\alpha,\xi\rangle} < \Delta n_{|\alpha\rangle}$$

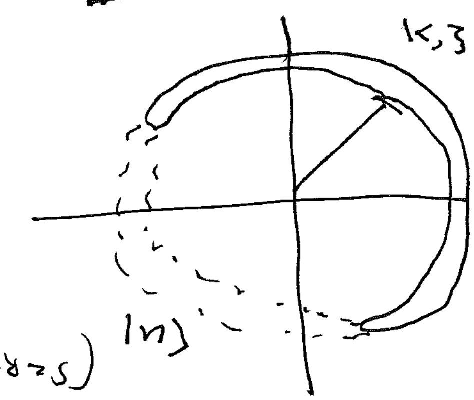



since $|\alpha\xi\rangle$ is a M.U.S we have

$\Delta\varphi_{|\alpha\xi\rangle} > \frac{1}{\sqrt{\bar{n}}}$ so increased phase noise. As we have

$\xi = re^{i\theta} \qquad r \to \infty$

$\Rightarrow \quad |\alpha, \xi\rangle \to |n\rangle$ number state. (zero number fluctuations)




Let's quantify these ideas.

For any state $|\psi\rangle$

$$\Delta n^2 \equiv \langle\psi|\hat{n}^2|\psi\rangle - \langle\psi|\hat{n}|\psi\rangle^2$$

$$= \langle a^\dagger a\, a^\dagger a\rangle - \langle a^\dagger a\rangle^2 \qquad aa^\dagger = 1 + a^\dagger a$$

$$= \langle a^\dagger (a^\dagger a + 1) a\rangle - \langle a^\dagger a\rangle^2$$

$$= \langle a^\dagger a\rangle + \langle a^\dagger a^\dagger a a\rangle - \langle a^\dagger a\rangle^2$$

$$= \langle \hat{n}\rangle + \langle a^\dagger a^\dagger a a\rangle - \langle \hat{n}\rangle^2$$

If $|\psi\rangle$ = pure state then $\hat{\rho}_{|\psi\rangle} \equiv |\psi\rangle\langle\psi|$

and using 3.96 we have

$$P_{|\psi\rangle}(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\psi\rangle\langle\psi|u\rangle\, e^{u^*\alpha - u\alpha^*}\, d^2u$$

Also since $\Delta n^2$ is now written in normal order $\boxed{\Delta n^2 = \langle a^\dagger a\rangle + \langle a^\dagger a^\dagger a a\rangle - \langle a^\dagger a\rangle^2}$ we may use Eq. 3.106

$$\langle G^N(a, a^\dagger)\rangle = \int P(\alpha)\, G^N(\alpha, \alpha^*)\, d\alpha^2$$

so $\hat{a} \to \alpha$ $\quad \hat{a}^\dagger \to \alpha^*$ and

$$\Delta n^2 = \langle\hat{n}\rangle + \langle a^\dagger a^\dagger a a\rangle - \langle\hat{n}\rangle^2$$

$$\cancel{= \langle\hat{n}\rangle + \langle a^\dagger a^\dagger a a\rangle - \langle a^\dagger a\rangle\langle\hat{n}\rangle}$$

$$\cancel{= \int P(\alpha)\{|\alpha|^2}$$

$$\cancel{= \langle\hat{n}\rangle \quad \int P(\alpha)\left[|\alpha|^4 - |\alpha|^2\langle\hat{n}\rangle\right] d\alpha}$$

$$= \langle\hat{n}\rangle + \langle a^\dagger a^\dagger a a\rangle - \langle a^\dagger a\rangle\langle a^\dagger a\rangle$$

$$= \langle\psi|\{\hat{n} + a^\dagger a^\dagger a a - a^\dagger a\langle a^\dagger a\rangle\}|\psi\rangle$$

$$= \langle\hat{n}\rangle + \int d\alpha\, P(\alpha)\left[|\alpha|^4 - |\alpha|^2\langle\hat{n}\rangle\right]$$

$$\Delta n^2 = \langle \hat{n} \rangle + \langle a^\dagger a^\dagger a a \rangle - \langle a^\dagger a \rangle \langle a^\dagger a \rangle$$

$$= \langle \hat{n} \rangle + \langle a^\dagger a^\dagger a a \rangle - \langle \hat{n} \rangle^2$$ complete square!

$$= \langle \hat{n} \rangle + \langle a^\dagger a^\dagger a a \rangle - 2 \langle \hat{n} \rangle^2 + \langle n \rangle^2$$

$$= \langle \hat{n} \rangle + \langle a^\dagger a^\dagger a a \rangle - 2 \langle \hat{n} \rangle \langle \hat{n} \rangle + \langle \hat{n} \rangle^2$$

$$= \langle \hat{n} \rangle + \int d\alpha P(\alpha) \{ \alpha^* \alpha^* \alpha \alpha - 2 \alpha^* \alpha \langle \hat{n} \rangle + \langle \hat{n} \rangle^2 \}$$

$$= \langle \hat{n} \rangle + \int d\alpha P(\alpha) \{ |\alpha|^4 - 2 |\alpha| \langle \hat{n} \rangle + \langle \hat{n} \rangle^2 \}$$

$$= \boxed{\langle \hat{n} \rangle + \int d\alpha P_\psi(\alpha) [ |\alpha|^2 - \langle \hat{n} \rangle ]^2}$$

where $\langle \hat{n} \rangle = \langle a^\dagger a \rangle = \int d\alpha P_\psi(\alpha) |\alpha|^2$

so

$$\Delta n_\psi = \sqrt{\bar{n}_\psi + \int d\alpha P_\psi(\alpha) \underbrace{[ |\alpha|^2 - \langle \hat{n} \rangle ]^2}_{\geq 0}} .$$

so the only way we can have number squeezing is if

$$\boxed{\begin{array}{l} \Delta n_\psi < \bar{n}_\psi \\ \Rightarrow \quad \exists\, \alpha \in \mathbb{C} \quad \text{s.t.} \quad P(\alpha) < 0 \end{array}}$$

Negative P-function is hallmark of nonclassical state.

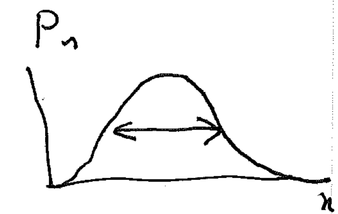
Recall that $|\langle n|\alpha\rangle|^2 = P_n(\alpha) = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$

is a Poisson distribution of width

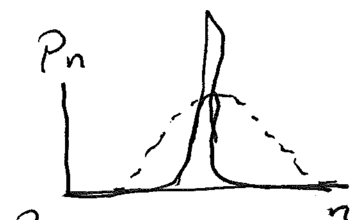
$\Delta n_\alpha = \sqrt{\bar{n}_\alpha}$

So if $\Delta n_\psi < \sqrt{\bar{n}_\psi}$ we say number statistics are sub-Poissonian

Examples

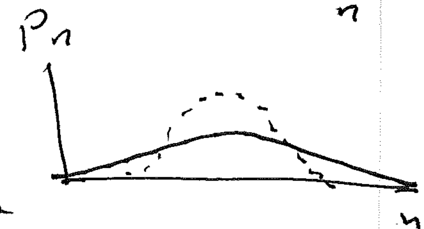



Poissonian: $|\alpha\rangle$ coherent $\Delta n = \sqrt{\bar{n}}$




sub-Poissonian: $|\alpha\xi\rangle$ number squeezed

$\psi - \theta/2 = 0$ $\Delta n < \sqrt{\bar{n}}$




super-Poissonian $|\alpha\xi\rangle$ phase squeezed

$\psi - \theta/2 = \pi/2$ $\Delta n > \sqrt{\bar{n}}$

where $\alpha = |\alpha| e^{i\psi}$ and $\xi = r e^{i\theta}$

Mandel introduced the "Q" parameter to test for super/sub poissonian.

[Not Q function] Q

$$Q_{|\psi\rangle} \equiv \frac{\Delta n_\psi^2 - \langle \hat{n} \rangle_\psi}{\langle \hat{n} \rangle_\psi}$$

For any state $|\psi\rangle$

$Q > 0$ Superpoissonian

$Q = 0$ Poissonian

$-1 \leq Q < 0$ Sub poisson

Example

phase sq. $|\alpha\xi\rangle$

coherent $|\alpha\rangle$

number sq. $|\alpha\xi\rangle$

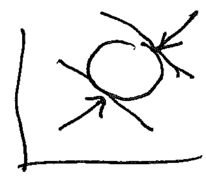
# Mandel Q Parameter Examples

Let $|\psi\rangle = |\alpha\rangle$ coherent state

$$\Delta n^2 = \bar{n} = |\alpha|^2$$

$$Q \equiv \frac{\Delta n_\alpha^2 - \langle \hat{n} \rangle_\alpha}{\langle \hat{n} \rangle} = \frac{\bar{n} - \bar{n}}{\bar{n}} = 0 \checkmark$$

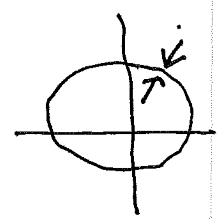
Poissonian



Example Number state $|\psi\rangle = |n\rangle$

$$\Rightarrow \langle \hat{n} \rangle = n \qquad \Delta n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = 0$$

$$Q \equiv \frac{0 - n}{n} = -1$$ sub Poisson



or maximally number squeezed

Example Phase state $|\psi\rangle = \sum_{n=0}^{\infty} e^{in\varphi} |n\rangle$

$$\langle \hat{n} \rangle = \sum_{n=0}^{\infty} n = \infty$$

$$\langle \hat{n} \rangle^2 = \infty^2$$ Not normalizable

Let us truncate sum $N \gg 1$

$$\langle \hat{n} \rangle = \sum_{n=0}^{N} n = N(N+1)/2 \approx \tfrac{1}{2} N^2$$ schaums

$$\langle \hat{n}^2 \rangle = \sum_{n=0}^{N} n^2 = N(N+1)(2N+1)/6 \approx \tfrac{1}{3} N^3$$

$$\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \left[\tfrac{1}{3} N^3\right] - \left[\tfrac{1}{2} N^2\right]^2 = \Delta n^2$$

$$= \tfrac{1}{3} N^3 - \tfrac{1}{4} N^4$$

$$Q = \frac{\Delta n^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle} = \frac{\tfrac{1}{3} N^3 - \tfrac{1}{4} N^4 - \tfrac{1}{2} N^2}{\tfrac{1}{2} N^2} \Rightarrow -\cancel{\infty}\,?$$

Should be super-duper poissonian

Example: squeezed vacuum $|\xi\rangle = \hat{S}_\xi |0\rangle$

we found $\langle a^\dagger a \rangle = \langle \hat{n} \rangle = \bar{n}_\xi = \sinh^2(r)$

$$\langle \hat{n}^2 \rangle = 3\bar{n}_\xi^2 + 2\bar{n}_\xi$$

$$\Rightarrow \quad \Delta n_\xi^2 = 2(\bar{n}_\xi^2 + \bar{n}_\xi)$$

$$\Rightarrow \quad Q = \frac{\Delta n_\xi^2 - \bar{n}_\xi}{\bar{n}_\xi} = \frac{2\bar{n}_\xi^2 + \bar{n}_\xi}{\bar{n}_\xi}$$

$$= 2\bar{n}_\xi + 1 > 0$$

<u>Super Poissonian</u>

Example ~~..~~ displaced squeezed $|\alpha\xi\rangle = \hat{D}_\alpha \hat{S}_\xi |0\rangle$

$\alpha = |\alpha| e^{i\varphi} \qquad \xi = r e^{i\theta}$

we worked out 89 terms to get

$\bar{n}_{\alpha\xi} = |\alpha|^2 + \sinh^2(r)$ phase Independent

$$\Delta n_{\alpha\xi}^2 = \tfrac{1}{2}\sinh^2(2r) + |\alpha|^2 \left[\cosh(2r) - \sinh(2r)\cos(2\Phi)\right]$$

which is phase dependent: $\Phi = \varphi - \theta/2$

Recall: $2\Phi = 0 \Rightarrow \varphi = \theta/2 \Rightarrow$ number squeezing

$2\Phi = \pi \Rightarrow \varphi - \theta/2 = \pi/2 \Rightarrow$ phase squeeze

In general

$$Q = \frac{\Delta n^2 - \bar{n}}{\bar{n}} = \begin{cases} \xrightarrow{\text{phase sq}} & > 0 \\ \longrightarrow & = 0 \qquad \xi = 0 \\ \xrightarrow[\text{number sq}]{} & < 0 \end{cases}$$

squeezed in phase

Recall

$$\Rightarrow \quad \bar{n}_{\alpha\xi} = |\alpha|^2 + \sinh^2(r) \quad \approx \quad |\alpha|^2 + \tfrac{1}{4} e^{2r}$$

$$\Delta n^2_{\alpha\xi} = \tfrac{1}{2}\sinh^2(2r) + |\alpha|^2 e^{2r}$$

If $r \gg 1$
$$\Delta n^2 \cong \tfrac{1}{2}\left[\frac{e^{2r} - e^{-2r}}{2}\right]^2 + |\alpha|^2 e^{2r}$$

$$\approx \tfrac{1}{8} e^{4r} + |\alpha|^2 e^{2r}$$

$$= \left[\tfrac{1}{8} e^{2r} + |\alpha|^2\right] e^{2r}$$

$$\Rightarrow \quad Q = \frac{\Delta n^2 - \bar{n}}{\bar{n}} = \frac{\left[\tfrac{1}{8} e^{2r} + |\alpha|^2\right] e^{2r}}{\left[\tfrac{1}{4} e^{2r} + |\alpha|^2\right]}$$

$$= \left[\frac{e^{2r} + 8|\alpha|^2}{2e^{2r} + 8|\alpha|^2}\right] e^{2r} > 0$$

superpoisson

squeezed in number $\quad \bar{n} = |\alpha|^2 + \cancel{\sinh^2(r)} \approx |\alpha|^2 + \tfrac{1}{4} e^{2r}$

$$\Delta n^2 = \tfrac{1}{2}\sinh^2(2r) + |\alpha|^2\left[\cosh(2r) - \sinh(2r)\right]$$

$$= \tfrac{1}{2}\sinh^2(2r) + |\alpha|^2\left[\frac{\cancel{e^{2r}} + e^{-2r} - \cancel{e^{2r}} + e^{-2r}}{2}\right]$$

OK
$$= \boxed{\tfrac{1}{2}\sinh^2(2r) + |\alpha|^2 e^{-2r}} \cancel{\approx \tfrac{1}{8} e^{4r} + |\alpha|^2 e^{-2r}}$$

$$\cancel{Q = \frac{\Delta n^2 - \bar{n}}{\bar{n}} \approx \frac{\tfrac{1}{8} e^{4r} + \tfrac{1}{4} e^{2r} + |\alpha|^2\left[1 - e^{-2r}\right]}{|\alpha|^2 + 1/4\, e}}$$

Take $|\alpha| >> 1$ and $r \approx 0 \Rightarrow \sinh r \approx 0$

$\cosh r \approx 1$

$$\bar{n} = |\alpha|^2 + \sinh^2(r) \approx |\alpha|^2 = \bar{n}_\alpha$$

$$\Delta n^2 \approx \cancel{|\alpha|^2 [1 - 0] =} \; 0 + |\alpha|^2 e^{-2r}$$

$$\approx \bar{n}_\alpha e^{-2r}$$

$$\Rightarrow Q = \frac{\Delta n^2 - \bar{n}_{\alpha\xi}}{\bar{n}_{\alpha\xi}} = \frac{\bar{n}_\alpha e^{-2r} - \bar{n}_\alpha}{\bar{n}_\alpha}$$

$$= e^{-2r} - 1 \qquad e^x \cong 1 + x$$

$$\approx 1 - 2r - 1$$

$$= -2r < 0$$ <u>subPoissoin</u>