Lecture 31: FRI 06 NOV
Induction and Inductance III

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Mutual Induction 890

Fender Stratocaster
Solenoid Pickup
Faraday's Experiments

In a series of experiments, Michael Faraday in England and Joseph Henry in the U.S. were able to generate electric currents without the use of batteries.

The circuit shown in the figure consists of a wire loop connected to a sensitive ammeter (known as a "galvanometer"). If we approach the loop with a permanent magnet we see a current being registered by the galvanometer.

1. A current appears only if there is relative motion between the magnet and the loop.
2. Faster motion results in a larger current.
3. If we reverse the direction of motion or the polarity of the magnet, the current reverses sign and flows in the opposite direction.

The current generated is known as "induced current"; the emf that appears is known as "induced emf"; the whole effect is called "induction."
In the figure we show a second type of experiment in which current is induced in loop 2 when the switch S in loop 1 is either closed or opened. When the current in loop 1 is constant no induced current is observed in loop 2. The conclusion is that the magnetic field in an induction experiment can be generated either by a permanent magnet or by an electric current in a coil.

Faraday summarized the results of his experiments in what is known as "Faraday's law of induction."

An emf is induced in a loop when the number of magnetic field lines that pass through the loop is changing.
Lenz’s Law

• The Loop Current Produces a B Field that Opposes the CHANGE in the bar magnet field.

• Upper Drawing: B Field from Magnet is INCREASING so Loop Current is Clockwise and Produces an Opposing B Field that Tries to CANCEL the INCREASING Magnet Field

• Lower Drawing: B Field from Magnet is DECREASING so Loop Current is Counterclockwise and Tries to BOOST the Decreasing Magnet Field.
**Self - Induction**

In the picture to the right we already have seen how a change in the current of loop 1 results in a change in the flux through loop 2, and thus creates an induced emf in loop 2.

If we change the current through an inductor this causes a change in the magnetic flux \( \Phi_B = Li \) through the inductor according to the equation \( \frac{d\Phi_B}{dt} = L \frac{di}{dt} \). Using Faraday's law we can determine the resulting emf known as

**self - induced** emf: \( \mathcal{E} = -\frac{d\Phi_B}{dt} = -L \frac{di}{dt} \).

**SI unit for \( L \):** the henry (symbol: H)

An inductor has inductance \( L = 1 \text{ H} \) if a current change of 1 A/s results in a self-induced emf of 1 V.
The RL circuit

- Set up a single loop series circuit with a battery, a resistor, a solenoid and a switch.
- Describe what happens when the switch is closed.
- Key processes to understand:
  - What happens JUST AFTER the switch is closed?
  - What happens a LONG TIME after switch has been closed?
  - What happens in between?

Key insights:
- You cannot change the CURRENT in an inductor instantaneously!
- If you wait long enough, the current in an RL circuit stops changing!

At $t = 0$, a capacitor acts like a solid wire and inductor acts like break in the wire.
At $t = \infty$ a capacitor acts like a break in the wire and inductor acts like a solid wire.
30.8: Self-Induction:

An induced emf $\mathcal{E}_L$ appears in any coil in which the current is changing.

This process (see Fig. 30-13) is called **self-induction**, and the emf that appears is called a **self-induced emf**. It obeys Faraday’s law of induction just as other induced emfs do.

\[
N \Phi_B = Li.
\]

\[
\mathcal{E}_L = - \frac{d(N \Phi_B)}{dt}.
\]

**Fig. 30-13** If the current in a coil is changed by varying the contact position on a variable resistor, a self-induced emf $\mathcal{E}_L$ will appear in the coil while the current is changing.

\[
\mathcal{E}_L = -L \frac{di}{dt} \quad \text{(self-induced emf).}
\]
Initially, an inductor acts to oppose changes in the current through it. A long time later, it acts like ordinary connecting wire.

**Fig. 30-15** An *RL* circuit. When switch S is closed on *a*, the current rises and approaches a limiting value $\frac{\mathcal{E}}{R}$.

Switch at *a* is fluxing up the inductor $L$:
$L$ acts like a break in circuit at $t = 0$.
Hence $i(0) = 0$

Outer Loop Diffy-Q Variables Seperable:

$$\mathcal{E} - iR - L \frac{di}{dt} = 0$$

Solve with initial condition: $i(0) = 0$

$$i(t) = \frac{\mathcal{E}}{R} \left(1 - e^{-t/\tau_{LR}}\right)$$

Now flip switch to *b*.
This defluxes the inductor $L$:
$L$ acts likes battery with $-\mathcal{E}$

Inner Loop Diffy-Q Current Reverses:

$$-iR - L \frac{di}{dt} = 0$$

Solve with initial condition: $i(0) = \frac{\mathcal{E}}{R}$

$$i(t) = \frac{\mathcal{E}}{R} e^{-t/\tau_{LR}}$$
Initially, an inductor acts to oppose changes in the current through it. A long time later, it acts like ordinary connecting wire.

If we suddenly remove the emf from this same circuit, the flux does not immediately fall to zero but approaches zero in an exponential fashion:

\[
i = \frac{E}{R} (1 - e^{-\frac{Rt}{L}}),
\]

\[
i = \frac{E}{R} (1 - e^{-\frac{t}{\tau_L}})
\]

rise of current.

\[
\tau_L = \frac{L}{R} \quad \text{(time constant)}.
\]
30.9: RL Circuits:

The resistor's potential difference turns on. The inductor's potential difference turns off.

**Fig. 30-17** The variation with time of (a) $V_R$, the potential difference across the resistor in the circuit of Fig. 30-16, and (b) $V_L$, the potential difference across the inductor in that circuit. The small triangles represent successive intervals of one inductive time constant $\tau_L = L/R$. The figure is plotted for $R = 2000 \, \Omega$, $L = 4.0 \, \text{H}$, and $\mathcal{E} = 10 \, \text{V}$.

\[
V_R = [i]R = \left[ \frac{\mathcal{E}}{R} \left(1 - e^{-t/\tau_{LR}} \right) \right]R = \mathcal{E} \left(1 - e^{-t/\tau_{LR}} \right)
\]

\[
V_L = L \frac{di}{dt} = L \frac{d}{dt} \left[ \frac{\mathcal{E}}{R} \left(1 - e^{-t/\tau_{LR}} \right) \right]
\]

\[
= L \frac{\mathcal{E}}{R \tau_{LR}} e^{-t/\tau_{LR}} = L \frac{\mathcal{E} R}{R L} e^{-t/\tau_{LR}}
\]

\[
= \mathcal{E} e^{-t/\tau_{LR}}
\]

**Fig. 30-16** The circuit of Fig. 30-15 with the switch closed on $a$. We apply the loop rule for the circuit clockwise, starting at $x$. 
In an RC circuit, while charging, $Q = CV$ and the loop rule mean:

- charge increases from 0 to $CE$
- current decreases from $E/R$ to 0
- voltage across capacitor increases from 0 to $E$

In an RL circuit, while fluxing up (rising current), $E = Ldi/dt$ and the loop rule mean:

- magnetic field increases from 0 to $B$
- current increases from 0 to $E/R$
- voltage across inductor decreases from $-E$ to 0
Immediately after the switch is closed, what is the potential difference across the inductor?

(a) 0 V  
(b) 9 V  
(c) 0.9 V

• Immediately after the switch, current in circuit = 0.
• So, potential difference across the resistor = 0!
• So, the potential difference across the inductor = $\mathcal{E} = 9 \text{ V}$!
Immediately after the switch is closed, what is the current \( i \) through the 10 \( \Omega \) resistor?

(a) 0.375 A
(b) 0.3 A
(c) 0

Long after the switch has been closed, what is the current in the 40 \( \Omega \) resistor?

(a) 0.375 A
(b) 0.3 A
(c) 0.075 A

Immediately after switch is closed, current through inductor = 0. Why???

Hence, current through battery and through 10 \( \Omega \) resistor is
\[
i = \frac{3 \text{ V}}{10 \text{ } \Omega} = 0.3 \text{ A}
\]

Long after switch is closed, potential across inductor = 0. Why???

Hence, current through 40 \( \Omega \) resistor
\[
i = \frac{3 \text{ V}}{10 \text{ } \Omega} = 0.375 \text{ A (Par-V)}
\]
Fluxing Up The Inductor

- How does the current in the circuit change with time?

\[-iR + \mathcal{E} - L \frac{di}{dt} = 0\]

\[i(t) = \frac{\mathcal{E}}{R} \left(1 - e^{-t/\tau} \right)\]

Time constant of RL circuit: \(\tau_{LR} = L/R\)
RL Circuit Movie
Fluxing Down an Inductor

The switch is at \( a \) for a long time, until the inductor is charged. Then, the switch is closed to \( b \).

What is the current in the circuit?

Loop rule around the new circuit walking counter clockwise:

\[
iR + L \frac{di}{dt} = 0
\]

\[
i(t) = \frac{\mathcal{E}}{R} e^{-Rt/L} = \frac{\mathcal{E}}{R} e^{-t/\tau}
\]

\[
\tau_{RL} = \frac{L}{R}
\]
Inductors & Energy

• Recall that **capacitors** store energy in an **electric** field.

• **Inductors** store energy in a **magnetic** field.

\[
\mathcal{E} = iR + L \frac{di}{dt}
\]

\[
(i\mathcal{E}) = \left(i^2 R\right) + Li \frac{di}{dt}
\]

\[
(i\mathcal{E}) = \left(i^2 R\right) + \frac{d}{dt}\left(\frac{Li^2}{2}\right)
\]

Power delivered by battery = power dissipated by R + (d/dt) energy stored in L

\[
P = iV = i^2R
\]
Inductors & Energy

\[ U_B = \frac{L i^2}{2} \]

Magnetic Potential Energy \( U_B \) Stored in an Inductor.

\[ P = L i \frac{d i}{d t} \]

Magnetic Power Returned from Defluxing Inductor to Circuit.
Example

• The switch has been in position “a” for a long time.
• It is now moved to position “b” without breaking the circuit.
• What is the total energy dissipated by the resistor until the circuit reaches equilibrium?

- When switch has been in position “a” for long time, current through inductor = \( \frac{9\, \text{V}}{10\, \Omega} \) = 0.9A.
- Energy stored in inductor = \( (0.5)(10\, \text{H})(0.9\, \text{A})^2 \) = 4.05 J
- When inductor de-fluxes through the resistor, all this stored energy is dissipated as heat = 4.05 J.
$E=120\, \text{V}, \, R_1=10\, \Omega, \, R_2=20\, \Omega, \, R_3=30\, \Omega, \, L=3\, \text{H}$.

1. What are $i_1$ and $i_2$ immediately after closing the switch?
2. What are $i_1$ and $i_2$ a long time after closing the switch?
3. What are $i_1$ and $i_2$ immediately after reopening the switch?
4. What are $i_1$ and $i_2$ a long time after reopening the switch?
Energy Density of a Magnetic Field

Consider the solenoid of length $\ell$ and loop area $A$ that has $n$ windings per unit length. The solenoid carries a current $i$ that generates a uniform magnetic field $B = \mu_0 ni$ inside the solenoid. The magnetic field outside the solenoid is approximately zero.

The energy $U_B$ stored by the inductor is equal to $\frac{1}{2} Li^2 = \frac{\mu_0 n^2 A \ell i^2}{2}$.

This energy is stored in the empty space where the magnetic field is present. We define as energy density $u_B = \frac{U_B}{V}$ where $V$ is the volume inside the solenoid. The density $u_B = \frac{\mu_0 n^2 A \ell i^2}{2 A \ell} = \frac{\mu_0 n^2 i^2}{2} = \frac{\mu_0^2 n^2 i^2}{2 \mu_0} = \frac{B^2}{2 \mu_0}$.

This result, even though it was derived for the special case of a uniform magnetic field, holds true in general.

**Units:** $u_B = [\text{J/m}^3]$
Energy Density in E and B Fields

\[ u_E = \frac{\varepsilon_0 E^2}{2} \]

\[ u_B = \frac{B^2}{2\mu_0} \]
The Energy Density of the Earth’s Magnetic Field Protects us from the Solar Wind!
Figure 30-18a shows a circuit that contains three identical resistors with resistance $R = 9.0$ $\Omega$, two identical inductors with inductance $L = 2.0$ mH, and an ideal battery with emf $\mathcal{E} = 18$ V.

(a) What is the current $i$ through the battery just after the switch is closed?

**KEY IDEA**

Just after the switch is closed, the inductor acts to oppose a change in the current through it.

**Calculations:** Because the current through each inductor is zero before the switch is closed, it will also be zero just afterward. Thus, immediately after the switch is closed, the inductors act as broken wires, as indicated in Fig. 30-18b. We then have a single-loop circuit for which the loop rule gives us

$$\mathcal{E} - iR = 0.$$  

Substituting given data, we find that

$$i = \frac{\mathcal{E}}{R} = \frac{18 \text{ V}}{9.0 \text{ $\Omega$}} = 2.0 \text{ A.} \quad \text{(Answer)}$$

(b) What is the current $i$ through the battery long after the switch has been closed?

**KEY IDEA**

Long after the switch has been closed, the currents in the circuit have reached their equilibrium values, and the inductors act as simple connecting wires, as indicated in Fig. 30-18c.

**Fig. 30-18** (a) A multiloop RL circuit with an open switch. (b) The equivalent circuit just after the switch has been closed. (c) The equivalent circuit a long time later. (d) The single-loop circuit that is equivalent to circuit (c).

**Calculations:** We now have a circuit with three identical resistors in parallel; from Eq. 27-23, their equivalent resistance is $R_{eq} = R/3 = (9.0 \text{ $\Omega$})/3 = 3.0 \text{ $\Omega$}$. The equivalent circuit shown in Fig. 30-18d then yields the loop equation $\mathcal{E} - iR_{eq} = 0$, or

$$i = \frac{\mathcal{E}}{R_{eq}} = \frac{18 \text{ V}}{3.0 \text{ $\Omega$}} = 6.0 \text{ A.} \quad \text{(Answer)}$$
The figure shows three circuits with identical batteries, inductors, and resistors. Rank the circuits according to the current through the battery (a) just after the switch is closed and (b) a long time later, greatest first. (If you have trouble here, work through the next sample problem and then try again.)
The figure shows three circuits with identical batteries, inductors, and resistors. Rank the circuits according to the current through the battery (a) just after the switch is closed and (b) a long time later, greatest first. (If you have trouble here, work through the next sample problem and then try again.)

(a) At \( t \approx 0 \) replace \( L \)'s with breaks & walk the remaining loops.

1. No current \( \Rightarrow i = 0 \)
2. \( V - iR = 0 \) \( \Rightarrow i = V / R \)
3. \( V - iR - iR \) \( \Rightarrow i = V / (2R) \)

\[ i_2 > i_3 > i_1 = 0 \]
CHECKPOINT 6

The figure shows three circuits with identical batteries, inductors, and resistors. Rank the circuits according to the current through the battery (a) just after the switch is closed and (b) a long time later, greatest first. (If you have trouble here, work through the next sample problem and then try again.)

(b) At \( t \approx \infty \) replace \( L \)'s with wire & walk remaining loops.

1. **SeriQ** \( \Rightarrow i = \frac{E}{R_{eq}} = \frac{E}{2R} \)

2. **ParV** \( \Rightarrow i = \frac{E}{R_{eq}} = \frac{E}{(R/2)} = 2\frac{E}{R} \)

3. Tricksy my precious! All \( i \) on path of least resistance:

   Middle \( R \) is like a break!

   \[ E - iR = 0 \Rightarrow i = \frac{E}{R} \]

   \[ i_2 > i_3 > i_1 \]
A solenoid has an inductance of 53 mH and a resistance of 0.37 Ω. If the solenoid is connected to a battery, how long will the current take to reach half its final equilibrium value? (This is a real solenoid because we are considering its small, but nonzero, internal resistance.)

**Calculations:** According to that solution, current $i$ increases exponentially from zero to its final equilibrium value of $\mathcal{E}/R$. Let $t_0$ be the time that current $i$ takes to reach half its equilibrium value. Then Eq. 30-41 gives us

$$\frac{1}{2} \frac{\mathcal{E}}{R} = \frac{\mathcal{E}}{R} (1 - e^{-t_0/\tau_L}).$$

We solve for $t_0$ by canceling $\mathcal{E}/R$, isolating the exponential, and taking the natural logarithm of each side. We find

$$t_0 = \tau_L \ln 2 = \frac{L}{R} \ln 2 = \frac{53 \times 10^{-3} \text{ H}}{0.37 \Omega} \ln 2 = 0.10 \text{ s}. \quad \text{(Answer)}$$

**Fig. 30-16** The circuit of Fig. 30-15 with the switch closed on $a$. We apply the loop rule for the circuit clockwise, starting at $x$. 

**Key Idea**

We can mentally separate the solenoid into a resistance and an inductance that are wired in series with a battery, as in Fig. 30-16. Then application of the loop rule leads to Eq. 30-39, which has the solution of Eq. 30-41 for the current $i$ in the circuit.
A coil has an inductance of 53 mH and a resistance of 0.35 Ω.

(a) If a 12 V emf is applied across the coil, how much energy is stored in the magnetic field after the current has built up to its equilibrium value?

The energy stored in the magnetic field of a coil at any time depends on the current through the coil at that time, according to Eq. 30-49 \( U_B = \frac{1}{2} L i^2 \).

**Calculations:** Thus, to find the energy \( U_{B\infty} \) stored at equilibrium, we must first find the equilibrium current. From Eq. 30-41, the equilibrium current is

\[
i_\infty = \frac{\varepsilon}{R} = \frac{12 \text{ V}}{0.35 \text{ Ω}} = 34.3 \text{ A.} \tag{30-51}
\]

Then substitution yields

\[
U_{B\infty} = \frac{1}{2} L i_\infty^2 = \left(\frac{1}{2}\right)(53 \times 10^{-3} \text{ H})(34.3 \text{ A})^2
= 31 \text{ J.} \tag{Answer}
\]

(b) After how many time constants will half this equilibrium energy be stored in the magnetic field?

**Calculations:** Now we are being asked: At what time \( t \) will the relation

\[
U_B = \frac{1}{2} U_{B\infty}
\]

be satisfied? Using Eq. 30-49 twice allows us to rewrite this energy condition as

\[
\frac{1}{2} L i^2 = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) L i_\infty^2
\]

or

\[
i = \left(\frac{1}{\sqrt{2}}\right) i_\infty. \tag{30-52}
\]

This equation tells us that, as the current increases from its initial value of 0 to its final value of \( i_\infty \), the magnetic field will have half its final stored energy when the current has increased to this value. In general, we know that \( i \) is given by Eq. 30-41, and here \( i_\infty \) (see Eq. 30-51) is \( \varepsilon / R \); so Eq. 30-52 becomes

\[
\frac{\varepsilon}{R} (1 - e^{-\nu \tau_L}) = \frac{\varepsilon}{\sqrt{2}R}.
\]

By canceling \( \varepsilon / R \) and rearranging, we can write this as

\[
e^{-\nu \tau_L} = 1 - \frac{1}{\sqrt{2}} = 0.293,
\]

which yields

\[
\frac{t}{\tau_L} = -\ln 0.293 = 1.23
\]

or

\[
t \approx 1.2\tau_L. \tag{Answer}
\]

Thus, the energy stored in the magnetic field of the coil by the current will reach half its equilibrium value 1.2 time constants after the emf is applied.