Physics 2113
Lecture 07
Electric Fields III

22-6 A POINT CHARGE IN AN ELECTRIC FIELD
A Point Charge in an Electric Field

22-7 A DIPOLE IN AN ELECTRIC FIELD
A Dipole in an Electric Field
First: Given Electric Charges, We Calculate the Electric Field Using $E=\frac{kq}{r^2}$.

Example: the Electric Field Produced By a Single Charge, or by a Dipole:

Second: Given an Electric Field, We Calculate the Forces on Other Charges Using $F=qE$

Examples: Forces on a Single Charge When Immersed in the Field of a Dipole, Torque on a Dipole When Immersed in an Uniform Electric Field.
Continuous Charge Distribution

• Thus Far, We Have Only Dealt With Discrete, Point Charges.

• Imagine Instead That a Charge $q$ Is Smeared Out Over A:
  - LINE
  - AREA
  - VOLUME

• How to Compute the Electric Field $E$? Calculus!!!
• Useful idea: charge density

• Line of charge: charge per unit length = $\lambda$

• Sheet of charge: charge per unit area = $\sigma$

• Volume of charge: charge per unit volume = $\rho$

$\lambda = \frac{q}{L}$

$\sigma = \frac{q}{A}$

$\rho = \frac{q}{V}$
Computing Electric Field of Continuous Charge Distribution

- Approach: Divide the Continuous Charge Distribution into infinitesimally small differential elements.
- Treat each element as a point charge and compute its electric field.
- Sum (integrate) over all elements.
- Always look for symmetry to simplify calculation.

\[ dq = \lambda \, dL \]
\[ dq = \sigma \, dS \]
\[ dq = \rho \, dV \]
Differential Form of Coulomb’s Law

\[ \left| \vec{E}_{12} \right| = \frac{k |q_2|}{r_{12}^2} \]

E-Field at Point

\[ \left| d\vec{E}_{12} \right| = \frac{k |dq_2|}{r_{12}^2} \]

Differential dE-Field at Point
ICPP: Arc of Charge

- Figure shows a uniformly charged rod of charge \(-Q\) bent into a circular arc of radius \(R\), centered at \((0,0)\).
- What is the direction of the electric field at the origin?
  (a) Field is 0.
  (b) Along +y
  (c) Along -y

Choose symmetric elements
- \(x\) components cancel
**Arc of Charge: Quantitative**

- Figure shows a uniformly charged rod of charge \(-Q\) bent into a circular arc of radius \(R\), centered at (0,0).

- ICPP: Which way does net E-field point?

- Compute the direction & magnitude of \(E\) at the origin.

\[
dE_x = [dE] \cos \theta = \left[ \frac{k dq}{R^2} \right] \cos \theta
\]

\[
E_x = \int_0^{\pi/2} \frac{k(\lambda R d\theta) \cos \theta}{R^2} = \frac{k \lambda}{R} \int_0^{\pi/2} \cos \theta d\theta = \frac{k \lambda}{R} \sin \theta \bigg|_0^{\pi/2}
\]

\[
E_x = \frac{k \lambda}{R}
\]

\[
E_y = \frac{k \lambda}{R}
\]

\[
E_{\text{net}} = \sqrt{E_x^2 + E_y^2} = \sqrt{\left(\frac{k \lambda}{R}\right)^2 + \left(\frac{k \lambda}{R}\right)^2} = \frac{k \lambda}{R} \sqrt{2}
\]

\[
\lambda = \frac{Q}{L} = \frac{Q}{2\pi R / 4} = \frac{2Q}{\pi R}
\]

\[
E_{\text{net}} = \sqrt{E_x^2 + E_y^2} = \frac{k \lambda}{R} \sqrt{2}
\]
The figure here shows three nonconducting rods, one circular and two straight. Each has a uniform charge of magnitude $Q$ along its top half and another along its bottom half. For each rod, what is the direction of the net electric field at point $P$?

(a) toward positive $y$;
(b) toward positive $x$;
(c) toward negative $y$
Charged Ring

Canceling Components - Point P is on the axis: In the Figure (right), consider the charge element on the opposite side of the ring. It too contributes the field magnitude $dE$ but the field vector leans at angle $\theta$ in the opposite direction from the vector from our first charge element, as indicated in the side view of Figure (bottom). Thus the two perpendicular components cancel. All around the ring, this cancelation occurs for every charge element and its symmetric partner on the opposite side of the ring. So we can neglect all the perpendicular components.

The components perpendicular to the z axis cancel; the parallel components add.

A ring of uniform positive charge. A differential element of charge occupies a length $ds$ (greatly exaggerated for clarity). This element sets up an electric field $dE$ at point $P$. 
Charged Ring

Adding Components. From the figure (bottom), we see that the parallel components each have magnitude $dE \cos \theta$. We can replace $\cos \theta$ by using the right triangle in the Figure (right) to write

$$\cos \theta = \frac{z}{r} = \frac{z}{(z^2 + R^2)^{1/2}}$$

$$dE \cos \theta = \frac{1}{4\pi\varepsilon_0} \frac{z\lambda}{(z^2 + R^2)^{3/2}} ds.$$
Charged Ring

Integrating. Because we must sum a huge number of these components, each small, we set up an integral that moves along the ring, from element to element, from a starting point (call it $s=0$) through the full circumference ($s=2\pi R$). Only the quantity $s$ varies as we go through the elements. We find

$$E = \int dE \cos \theta = \frac{z \lambda}{4 \pi \varepsilon_0 (z^2 + R^2)^{3/2}} \int_0^{2\pi R} ds$$

Finally,

$$E = \frac{q z}{4 \pi \varepsilon_0 (z^2 + R^2)^{3/2}} \quad \text{(charged ring)}$$

The components perpendicular to the $z$ axis cancel; the parallel components add.

A ring of uniform positive charge. A differential element of charge occupies a length $ds$ (greatly exaggerated for clarity). This element sets up an electric field $dE$ at point $P$. 
ICPP: Field on Axis of Charged Disk

- A uniformly charged circular disk (with positive charge)
- What is the direction of E at point P on the axis?

(a) Field is 0
(b) Along +z
(c) Somewhere in the x-y plane

(a) Field is 0
(b) Along +z
(c) Somewhere in the x-y plane
Charged Disk is Integral of Charged Rings

A disk of radius $R$ and uniform positive charge. The ring shown has radius $r$ and radial width $dr$. It sets up a differential electric field $dE$ at point $P$ on its central axis.

\[
 \sigma = \frac{Q}{\pi R^2} \quad dq = \sigma dA = \sigma 2\pi r dr
\]

\[
 dE = \frac{dq \ z}{4\pi \varepsilon_0 (z^2 + r^2)^{3/2}}.
\]

\[
 E = \int dE = \frac{\sigma z}{4\varepsilon_0} \int_0^R \left( \frac{z}{z^2 + r^2} \right)^{3/2} (2r) \ dr,
\]

\[
 E = \frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \quad \text{(charged disk)}
\]

Taking $R \gg z$ gives E-field above an infinite charged plane:

\[
 E_{\text{plane}} = \frac{\sigma}{2\varepsilon_0}
\]
Charged Disk is Integral of Charged Rings

\[ E = \frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \quad \text{(charged disk)} \]

Taking \( z \gg R \) gives E-field a distance \( z \) from a point charge \( Q \):

\[
E_{\text{point}} = \frac{\sigma}{2\varepsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + R^2}} \right] = \frac{\sigma}{2\varepsilon_0} \left[ 1 - \frac{z}{z\sqrt{1 + \left( R / z \right)^2}} \right]
\]

\[
= \frac{\sigma}{2\varepsilon_0} \left[ 1 - \left( 1 + \left( R / z \right)^2 \right)^{-1/2} \right] \approx \frac{\sigma}{2\varepsilon_0} \left[ 1 - \left( 1 - \frac{1}{2} \left( R / z \right)^2 \right) \right]
\]

\[
= \frac{\sigma}{4\varepsilon_0} \frac{R^2}{z^2} = \frac{Q}{4\pi\varepsilon_0 R^2} \frac{R^2}{z^2}
\]

\[ = \frac{kQ}{z^2} \quad \text{(Coulomb's Law for Point Charge)} \]

\[ \sigma = \frac{Q}{\pi R^2} \quad dq = \sigma dA = \sigma 2\pi r dr \]

A disk of radius \( R \) and uniform positive charge. The ring shown has radius \( r \) and radial width \( dr \). It sets up a differential electric field \( dE \) at point \( P \) on its central axis.
Definition of Electric Field:

\[ \vec{E} = \frac{\vec{F}}{q} \]

Force on Charge Due to Electric Field:

\[ \vec{F} = q\vec{E} \]
Force on a Charge in Electric Field

Positive Charge
Force in Same Direction as $\mathbf{E}$-Field (Follows)

Negative Charge
Force in Opposite Direction as $\mathbf{E}$-Field (Opposes)
CHECKPOINT 3

(a) In the figure, what is the direction of the electrostatic force on the electron due to the external electric field shown?

(b) In which direction will the electron accelerate if it is moving parallel to the $y$ axis before it encounters the external field? (c) If, instead, the electron is initially moving rightward, will its speed increase, decrease, or remain constant?

(a) left

(b) left

(c) decrease
Electric Dipole in a Uniform Field

• Net force on dipole = 0; center of mass stays where it is.

• Net TORQUE $\tau$: INTO page. Dipole rotates to line up in direction of $E$.
  
  $$|\tau| = 2(qE)(d/2)(\sin \theta)$$
  
  $$= (qd)(E)\sin \theta$$
  
  $$= |p| E \sin \theta$$
  
  $$= |p \times E|$$

• The dipole tends to “align” itself with the field lines.

• ICPP: What happens if the field is NOT UNIFORM??

Distance Between Charges = $d$
Electric Dipole in a Uniform Field

- Net force on dipole = 0; center of mass stays where it is.
- Potential Energy $U$ is smallest when $\vec{p}$ is aligned with $\vec{E}$ and largest when $\vec{p}$ anti-aligned with $\vec{E}$.
- The dipole tends to “align” itself with the field lines.

Distance Between Charges = $d$

$$U = -\vec{p} \cdot \vec{E} = -pE \cos \theta$$

For $\theta = 0^\circ$, $U = -pE \cos 0^\circ = -pE$

For $\theta = 180^\circ$, $U = -pE \cos 180^\circ = +pE$
CHECKPOINT 4

The figure shows four orientations of an electric dipole in an external electric field. Rank the orientations according to (a) the magnitude of the torque on the dipole and (b) the potential energy of the dipole, greatest first.

1 and 3 are “uphill”.
2 and 4 are “downhill”.
U1 = U3 > U2 = U4

\[
U_1 = -pE \cos(135^\circ) = +0.71pE \\
U_2 = -pE \cos(+45^\circ) = -0.71pE \\
U_3 = -pE \cos(-135^\circ) = +0.71pE \\
U_4 = -pE \cos(-45^\circ) = -0.71pE
\]

|\tau_1| = pE \sin(45^\circ + 45^\circ + 45^\circ) = pE \sin(135^\circ) = 0.71pE
|\tau_2| = pE \sin(45^\circ) = 0.71pE
|\tau_3| = pE \sin(-135^\circ) = 0.71pE
|\tau_4| = pE \sin(-45^\circ) = 0.71pE

(a) all tie;
(b) 1 and 3 tie, then 2 and 4 tie
sample problem

Torque and energy of an electric dipole in an electric field

A neutral water molecule (H₂O) in its vapor state has an electric dipole moment of magnitude \(6.2 \times 10^{-30} \text{ C} \cdot \text{m}\).

(a) How far apart are the molecule’s centers of positive and negative charge?

**Key Idea**

A molecule’s dipole moment depends on the magnitude \(q\) of the molecule’s positive or negative charge and the charge separation \(d\).

**Calculations:** There are 10 electrons and 10 protons in a neutral water molecule; so the magnitude of its dipole moment is

\[ p = qd = (10e)(d), \]

in which \(d\) is the separation we are seeking and \(e\) is the elementary charge. Thus,

\[ d = \frac{p}{10e} = \frac{6.2 \times 10^{-30} \text{ C} \cdot \text{m}}{(10)(1.60 \times 10^{-19} \text{ C})} \]

\[ = 3.9 \times 10^{-12} \text{ m} = 3.9 \text{ pm}. \quad \text{(Answer)} \]

This distance is not only small, but it is also actually smaller than the radius of a hydrogen atom.

(b) If the molecule is placed in an electric field of \(1.5 \times 10^4 \text{ N/C}\), what maximum torque can the field exert on it? (Such a field can easily be set up in the laboratory.)

**Key Idea**

The work done by an external agent (by means of a torque applied to the molecule) is equal to the change in the molecule’s potential energy due to the change in orientation.

**Calculations:** From Eq. 22-40, we find

\[ W_a = U_{180^\circ} - U_0 \]

\[ = (-pE \cos 180^\circ) - (-pE \cos 0) \]

\[ = 2pE = (2)(6.2 \times 10^{-30} \text{ C} \cdot \text{m})(1.5 \times 10^4 \text{ N/C}) \]

\[ = 1.9 \times 10^{-25} \text{ J}. \quad \text{(Answer)} \]
Summary

• The electric field produced by a system of charges at any point in space is the force per unit charge they produce at that point.
• We can draw field lines to visualize the electric field produced by electric charges.
• Electric field of a point charge: $E = \frac{kq}{r^2}$
• Electric field of a dipole: $E \sim \frac{kp}{r^3}$
• An electric dipole in an electric field rotates to align itself with the field.
• Use CALCULUS to find E-field from a continuous charge distribution.
“Bob! You fool. ... Don't plug that thing in!”