SUMMARY PHYSICS 707

Electrostatics

The basic differential equations of electrostatics are

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \text{ and } \nabla \times \mathbf{E}(\mathbf{x}) = 0 \tag{1}$$

where $\mathbf{E}(\mathbf{x})$ is the electric field and $\rho(\mathbf{x})$ is the electric charge density. The field is defined by the statement that a charge q at point $\mathbf{x}$ experiences a force $\mathbf{F} = q\mathbf{E}(\mathbf{x})$ where $\mathbf{E}(\mathbf{x})$ is the field produced by all charge other than q itself. These equations have integral equivalents,

$$\oint_S d^2x\, \mathbf{E}(\mathbf{x}) \cdot \mathbf{n} = 4\pi \int_V d^3x\, \rho(\mathbf{x}) = 4\pi Q \tag{2}$$

where Q is the charge enclosed by the surface S surrounding the domain V and $\mathbf{n}$ is a unit outward (from V) normal vector at a point on S; and

$$\oint_C d\mathbf{l} \cdot \mathbf{E}(\mathbf{x}) = 0. \tag{3}$$

Finally, if one applies these equations on the surface of a conductor (inside of which $\mathbf{E} = 0$), then one finds the surface charge density and (negative) pressure are

$$E_n(x) = 4\pi\sigma(x) \quad p = 2\pi\sigma^2 \tag{4}$$

There is an integral solution for $\mathbf{E}(\mathbf{x})$ if one knows ρ everywhere,

$$\mathbf{E}(\mathbf{x}) = \int d^3x'\, \frac{\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{5}$$

Introduce a scalar potential $\Phi(\mathbf{x})$ such that $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$ (This can be done because $\nabla \times \mathbf{E}(\mathbf{x}) = 0$ everywhere). Then

$$\nabla^2\Phi(\mathbf{x}) = -4\pi\rho(\mathbf{x}) \tag{6}$$

which has the integral solution

$$\Phi(\mathbf{x}) = \int d^3x'\, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{7}$$

Note in particular that the solution for a unit point charge is $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ and that this function is such that

$$\nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi\delta(\mathbf{x} - \mathbf{x}'). \tag{8}$$

The meaning of $\Phi(\mathbf{x})$ is that $q\Phi(\mathbf{x})$ is the energy of interaction of q, located at $\mathbf{x}$, with the charges that produce the potential. The energy of a localized charge distribution can be written as

$$W = \frac{1}{8\pi}\int d^3x\, \mathbf{E}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) = \frac{1}{2}\int d^3x\, \rho(\mathbf{x})\Phi(\mathbf{x}). \tag{9}$$

Define the electrostatic energy density as

$$w(\mathbf{x}) = \frac{1}{8\pi}\mathbf{E}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \geq 0. \tag{10}$$

Solution of Boundary Value Problems

We learned how to solve boundary value problems by a variety of methods (For the actual boundary conditions, see the section on macrostatic electrostatics).

1. Image method
2. Green's functions
3. Conformal maps (two-dimensional systems)
4. Orthogonal functions and expansions
   (a) Cartesian coordinates (Fourier series)
   (b) Spherical coordinates (Spherical harmonics, Legendre polynomials)
   (c) Cylindrical coordinates (Bessel functions, Fourier-Bessel series)

In particular, in spherical coordinates a general solution is given by

$$\Phi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_l r^l + B_l/r^{l+1}\right)Y_{l,m}(\theta,\phi); \tag{11}$$

this expansion is crafted in such a way as to provide a complete orthonormal set of functions, the spherical harmonics, on a spherical surface. In cylindrical coordinates, one may write

$$\Phi(\mathbf{x}) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\sinh(x_{mn}z/a) + B_{mn}\cosh(x_{mn}z/a)\right)J_m(x_{mn}\rho/a)e^{im\phi} \tag{12}$$

which is constucted so as to provide a complete orthogonal set of functions, the Bessel functions and exponentials with imaginary argument, on n a disc of radius a; x_{mn} is the n^{th} zero of the Bessel function of order m. Alternatively, one may write

$$\Phi(\mathbf{x}) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} (A_{mn} I_m(n\pi\rho/c) + B_{mn} K_m(n\pi\rho/c)) \sin(n\pi z/c) e^{im\phi} \tag{13}$$

which provides a complete orthogonal set of functions in the form of Fourier series on a surface at constant ρ with $0 \leq z \leq c$.

A useful formula (the addition theorem):

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{l,m}(\theta', \phi') Y_{l,m}(\theta, \phi) \tag{14}$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$.

Electric multipoles; Macroscopic Electrostatics

The potential of a localized charge distribution can be expanded as

$$\Phi(\mathbf{x}) = \frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^3} + \frac{1}{2} \sum_{i,j=1}^{3} \frac{x_i Q_{ij} x_j}{r^5} + ... \tag{15}$$

where

$$Q = \int d^3x' \, \rho(\mathbf{x}'), \tag{16}$$

$$\mathbf{p} = \int d^3x' \, \mathbf{x}' \rho(\mathbf{x}'), \tag{17}$$

and

$$Q_{ij} = \int d^3x' \, (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\mathbf{x}'). \tag{18}$$

The energy of a localized charge distribution "centered" at the origin and subjected to an external applied field $\mathbf{E}(\mathbf{x})$ is

$$W_e = Q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{6} \sum_{i,j=1}^{3} Q_{ij} \left. \frac{\partial E_j}{\partial x_i} \right|_0 + ... \tag{19}$$

from which one can find the dipole-dipole interaction

$$W_{d-d} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \mathbf{n})(\mathbf{p}_2 \cdot \mathbf{n})}{r^3}. \tag{20}$$

The electric field of a point dipole at the origin is

$$\mathbf{E}_d(\mathbf{x}) = \frac{3(\mathbf{p}\cdot\mathbf{n})\mathbf{n} - \mathbf{p}}{r^3} - \frac{4\pi}{3}\mathbf{p}\delta(\mathbf{x}). \tag{21}$$

In macroscopic electrostatics the sources are the macroscopic charge density $\rho(\mathbf{x})$ and the polarization or electric dipole moment per unit volume, $\mathbf{P}(\mathbf{x})$. The macroscopic electric field is written as $\mathbf{E}(\mathbf{x})$ and one defines the electric displacement $\mathbf{D}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) + 4\pi\mathbf{P}(\mathbf{x})$. The differential equations for the fields are

$$\nabla\cdot\mathbf{D}(\mathbf{x}) = 4\pi\rho \text{ and } \nabla\times\mathbf{E}(\mathbf{x}) = 0; \tag{22}$$

the latter implies that there is a scalar potential for $\mathbf{E}(\mathbf{x})$. There is a real charge density associated with the polarization, $\rho_p = -\nabla\cdot\mathbf{P}$; at a boundary between two materials, this has a surface-charge component, $\sigma_p = -(\mathbf{P}_2 - \mathbf{P}_1)\cdot\mathbf{n}$ where $\mathbf{n}$ is the unit normal to the boundary pointing into material 2. Boundary conditions on the fields are

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n} = 4\pi\sigma \text{ and } \mathbf{n}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0. \tag{23}$$

Here, σ is the macroscopic surface-charge density; it does not include the polarization surface-charge density. In order to solve for the fields, one must have a constitutive relation involving (at least) two of $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$. The simplest such relation, valid for uniform, linear, isotropic materials, is $\mathbf{P} = \chi_e\mathbf{E}$ in which case one can write $\mathbf{D} = \epsilon\mathbf{E}$ with the dielectric constant given by $\epsilon = 1 + 4\pi\chi_e$.

The energy change accompanying an infinitesimal change $\delta\mathbf{D}(\mathbf{x})$ in the displacement can be written quite generally as

$$\delta W = \frac{1}{4\pi}\int d^3x\,\mathbf{E}(\mathbf{x})\cdot\delta\mathbf{D}(\mathbf{x}); \tag{24}$$

if the materials are linear, then one can further write

$$W = \frac{1}{8\pi}\int d^3x\,\mathbf{E}(\mathbf{x})\cdot\mathbf{D}(\mathbf{x}) = \frac{1}{2}\int d^3x\,\Phi(\mathbf{x})\rho(\mathbf{x}). \tag{25}$$

Finally, the force on a piece of dielectric can be determined from energy-conservation-based arguments to be

$$F_\eta = -\left(\frac{\partial W}{\partial\eta}\right)_Q = \left(\frac{\partial W}{\partial\eta}\right)_V. \tag{26}$$

We also discussed the local field, the connection between the dielectric constant and the molecular polarizability, and the physical origins of the polarizability.

Magnetostatics

The differential equations of magnetostatics are

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0 \text{ and } \nabla \times \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}); \tag{27}$$

the magnetic induction $\mathbf{B}(\mathbf{x})$ is defined by the torque produced on a test dipole of moment $\mathbf{m}$ at point $\mathbf{x}$, $\mathbf{N} = \mathbf{m} \times \mathbf{B}(\mathbf{x})$. In magnetostatics, $\nabla \cdot \mathbf{J}(\mathbf{x}) = 0$.

The integral forms of the preceding equations are

$$\oint_S d^2x\, \mathbf{B}(\mathbf{x}) \cdot \mathbf{n} = 0 \text{ and } \oint_C d\mathbf{l} \cdot \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c} \int_S d^2x\, \mathbf{J}(\mathbf{x}) \cdot \mathbf{n} = \frac{4\pi}{c} I_S. \tag{28}$$

. Given $\mathbf{J}(\mathbf{x})$, the integral solution for $\mathbf{B}(\mathbf{x})$ is

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c} \int d^3x'\, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{29}$$

The force on a localized current distribution $\mathbf{J}(\mathbf{x})$ produced by an applied field $\mathbf{B}(\mathbf{x})$ can be obtained by integrating over the force density which is

$$\mathbf{f}(\mathbf{x}) = \frac{1}{c}\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}); \tag{30}$$

the torque on the distributin is

$$\mathbf{N} = \frac{1}{c} \int d^3x\, [\mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}))]. \tag{31}$$

Because the divergence of $\mathbf{B}$ is always zero, there is a vector potential $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$. An integral solution for $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c} \int d^3x'\, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}; \tag{32}$$

the differential equation for $\mathbf{A}$ is

$$\nabla^2 \mathbf{A}(\mathbf{x}) - \nabla(\nabla \cdot \mathbf{A}(\mathbf{x})) = -\frac{4\pi}{c}\mathbf{J}(\mathbf{x}). \tag{33}$$

If one chooses $\mathbf{A}$ so that $\nabla \cdot \mathbf{A} = 0$, which is always possible, then the Cartesian components of $\mathbf{A}$ obey Poisson equations. More generally, one can choose the divergence of the vector potential arbitrarily; doing so is called choosing a gauge.

Multipoles and Macroscopic Magnetostatics

A localized current distribution may for many purposes be treated in the dipole approximation. The magnetic dipole moment of the distribution is defined as

$$\mathbf{m} = \frac{1}{2c} \int d^3x \, \mathbf{x} \times \mathbf{J}(\mathbf{x}) \equiv \int d^3x \, \mathcal{M}(\mathbf{x}) \tag{34}$$

where we introduce also the magnetic moment density. In the dipole approximation, the vector potential is

$$\mathbf{A}_d(\mathbf{x}) = \frac{\mathbf{m} \times \mathbf{x}}{r^3} \tag{35}$$

and the field is

$$\mathbf{B}_d(\mathbf{x}) = \frac{3(\mathbf{m}\cdot)\mathbf{n} - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{x}); \tag{36}$$

the singular piece is strictly valid only for a point dipole. The force on the distribution given an applied field $\mathbf{B}(\mathbf{x})$ is, in this approximation, $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}(\mathbf{x}))$; the torque is $\mathbf{N} = \mathbf{m} \times \mathbf{B}$.

The sources in macroscopic magnetostatics are the macroscopic or free current density $\mathbf{J}(\mathbf{x})$ and the magnetization or magnetic dipole moment density $\mathbf{M}(\mathbf{x})$ which comes from charges bound on molecules. The differential field equations are

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0 \text{ and } \nabla \times \mathbf{H}(\mathbf{x}) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}) \tag{37}$$

where the magnetic field is $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$. For linear, isotropic, uniform materials, $\mathbf{M} = \chi_m\mathbf{H}$ which gives $\mathbf{B} = \mu\mathbf{H}$; μ is called the magnetic permeability. One introduces a magnetization current density $J_M(\mathbf{x}) = c\nabla \times \mathbf{M}(\mathbf{x})$; at a boundary between two materials, there is a surface-current density component in J_M; it is $K_M = \mathbf{n} \times (\mathbf{M}_2 - \mathbf{M}_1)$ where $\mathbf{n}$ is the unit normal into material 2 at the boundary.

The continuity conditions on the macroscopic fields are

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0 \text{ and } \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \frac{4\pi}{c}\mathbf{K} \tag{38}$$

where $\mathbf{K}$ is the surface-current density at the boundary (it does not include the magnetization surface-current density) and $\mathbf{n}$ is as above.