The basic differential equations of electrostatics are
\[ \nabla \cdot \mathbf{E}(\mathbf{x}) = 4\pi \rho(\mathbf{x}) \text{ and } \nabla \times \mathbf{E}(\mathbf{x}) = 0 \]  \hspace{1cm} (1)
where \( \mathbf{E}(\mathbf{x}) \) is the electric field and \( \rho(\mathbf{x}) \) is the electric charge density. The field is defined by the statement that a charge \( q \) at point \( \mathbf{x} \) experiences a force \( \mathbf{F} = q\mathbf{E}(\mathbf{x}) \) where \( \mathbf{E}(\mathbf{x}) \) is the field produced by all charge other than \( q \) itself. These equations have integral equivalents,
\[ \int_{S} d^{2}x \mathbf{E}(\mathbf{x}) \cdot \mathbf{n} = 4\pi \int_{V} d^{3}x \rho(\mathbf{x}) = 4\pi Q \]  \hspace{1cm} (2)
where \( Q \) is the charge enclosed by the surface \( S \) surrounding the domain \( V \) and \( \mathbf{n} \) is a unit outward (from \( V \)) normal vector at a point on \( S \); and
\[ \int_{C} d\mathbf{l} \cdot \mathbf{E}(\mathbf{x}) = 0. \]  \hspace{1cm} (3)
Finally, if one applies these equations on the surface of a conductor (inside of which \( \mathbf{E} = 0 \)), then one finds the surface charge density and (negative) pressure are
\[ E_{n}(\mathbf{x}) = 4\pi \sigma(\mathbf{x}) \quad p = 2\pi \sigma^{2} \]  \hspace{1cm} (4)
There is an integral solution for \( \mathbf{E}(\mathbf{x}) \) if one knows \( \rho \) everywhere,
\[ \mathbf{E}(\mathbf{x}) = \int d^{3}x' \frac{\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{3}}. \]  \hspace{1cm} (5)
Introduce a scalar potential \( \Phi(\mathbf{x}) \) such that \( \mathbf{E}(\mathbf{x}) = -\nabla \Phi(\mathbf{x}) \) (This can be done because \( \nabla \times \mathbf{E}(\mathbf{x}) = 0 \) everywhere). Then
\[ \nabla^{2} \Phi(\mathbf{x}) = -4\pi \rho(\mathbf{x}) \]  \hspace{1cm} (6)
which has the integral solution
\[ \Phi(\mathbf{x}) = \int d^{3}x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \]  \hspace{1cm} (7)
Note in particular that the solution for a unit point charge is \( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \) and that this function is such that
\[ \nabla^{2} \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi \delta(\mathbf{x} - \mathbf{x}'). \]  \hspace{1cm} (8)
The meaning of $\Phi(x)$ is that $q\Phi(x)$ is the energy of interaction of $q$, located at $x$, with the charges that produce the potential. The energy of a localized charge distribution can be written as

$$W = \frac{1}{8\pi} \int d^3x \mathbf{E}(x) \cdot \mathbf{E}(x) = \frac{1}{2} \int d^3x \rho(x)\Phi(x).$$

(9)

Define the electrostatic energy density as

$$w(x) = \frac{1}{8\pi} \mathbf{E}(x) \cdot \mathbf{E}(x) \geq 0.$$  

(10)

**Solution of Boundary Value Problems**

We learned how to solve boundary value problems by a variety of methods (For the actual boundary conditions, see the section on macrostatic electrostatics).

1. Image method

2. Green’s functions

3. Conformal maps (two-dimensional systems)

4. Orthogonal functions and expansions

   (a) Cartesian coordinates (Fourier series)

   (b) Spherical coordinates (Spherical harmonics, Legendre polynomials)

   (c) Cylindrical coordinates (Bessel functions, Fourier-Bessel series)

In particular, in spherical coordinates a general solution is given by

$$\Phi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( A_{l}\phi^{l} + B_{l}\phi^{l+1} \right) Y_{l,m}(\theta, \phi);$$

(11)

this expansion is crafted in such a way as to provide a complete orthonormal set of functions, the spherical harmonics, on a spherical surface. In cylindrical coordinates, one may write

$$\Phi(x) = \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} (A_{mn} \sinh(x_{mn}z/a) + B_{mn} \cosh(x_{mn}z/a)) J_{m}(x_{mn}\rho/a) e^{im\phi}$$

(12)
which is constructed so as to provide a complete orthogonal set of functions, the Bessel functions and exponentials with imaginary argument, on a disc of radius \(a\); \(x_{mn}\) is the \(n\)th zero of the Bessel function of order \(m\). Alternatively, one may write

\[
\Phi(x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} (A_{mn}J_m(n\pi\rho/c) + B_{mn}K_m(n\pi\rho/c)) \sin(n\pi z/c)e^{im\phi} \tag{13}
\]

which provides a complete orthogonal set of functions in the form of Fourier series on a surface at constant \(\rho\) with \(0 \leq z \leq c\).

A useful formula (the addition theorem):

\[
P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{l,m}^{*}(\theta', \phi')Y_{l,m}(\theta, \phi) \tag{14}
\]

where \(\gamma\) is the angle between \(x\) and \(x'\).

**Electric multipoles; Macroscopic Electrostatics**

The potential of a localized charge distribution can be expanded as

\[
\Phi(x) = \frac{Q}{r} + \frac{p \cdot x}{r^3} + \frac{1}{2} \sum_{i,j=1}^{3} \frac{x_iQ_{ij}x_j}{r^5} + \ldots \tag{15}
\]

where

\[
Q = \int d^3x' \rho(x'),
\]

\[
p = \int d^3x' x' \rho(x'),
\]

and

\[
Q_{ij} = \int d^3x' (3x_i'x_j' - r^2\delta_{ij})\rho(x'). \tag{18}
\]

The energy of a localized charge distribution “centered” at the origin and subjected to an external applied field \(\mathbf{E}(x)\) is

\[
W_e = Q\Phi(0) - p \cdot \mathbf{E}(0) - \frac{1}{6} \sum_{i,j=1}^{3} Q_{ij} \frac{\partial E_j}{\partial x_i} \bigg|_0 + \ldots \tag{19}
\]

from which one can find the dipole-dipole interaction

\[
W_{d-d} = \frac{p_1 \cdot p_2 - 3(p_1 \cdot n)(p_2 \cdot n)}{r^3}. \tag{20}
\]
The electric field of a point dipole at the origin is
\[ \mathbf{E}_d(x) = \frac{3(\mathbf{p} \cdot \mathbf{n}) \mathbf{n} - \mathbf{p}}{r^3} - \frac{4\pi}{3} \mathbf{p}\delta(x). \] (21)

In macroscopic electrostatics the sources are the macroscopic charge density \( \rho(x) \) and the polarization or electric dipole moment per unit volume, \( \mathbf{P}(x) \). The macroscopic electric field is written as \( \mathbf{E}(x) \) and one defines the electric displacement \( \mathbf{D}(x) = \mathbf{E}(x) + 4\pi\mathbf{P}(x) \). The differential equations for the fields are
\[ \nabla \cdot \mathbf{D}(x) = 4\pi\rho \quad \text{and} \quad \nabla \times \mathbf{E}(x) = 0; \quad (22) \]
the latter implies that there is a scalar potential for \( \mathbf{E}(x) \). There is a real charge density associated with the polarization, \( \rho_p = -\nabla \cdot \mathbf{P} \); at a boundary between two materials, this has a surface-charge component, \( \sigma_p = -(\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{n} \) where \( \mathbf{n} \) is the unit normal to the boundary pointing into material 2. Boundary conditions on the fields are
\[ (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = 4\pi\sigma \quad \text{and} \quad \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0. \quad (23) \]
Here, \( \sigma \) is the macroscopic surface-charge density; it does not include the polarization surface-charge density. In order to solve for the fields, one must have a constitutive relation involving (at least) two of \( \mathbf{D}, \mathbf{E}, \) and \( \mathbf{P} \). The simplest such relation, valid for uniform, linear, isotropic materials, is \( \mathbf{P} = \chi \epsilon \mathbf{E} \) in which case one can write \( \mathbf{D} = \epsilon \mathbf{E} \) with the dielectric constant given by \( \epsilon = 1 + 4\pi\chi \epsilon \).

The energy change accompanying an infinitesimal change \( \delta \mathbf{D}(x) \) in the displacement can be written quite generally as
\[ \delta W = \frac{1}{4\pi} \int d^3 x \, \mathbf{E}(x) \cdot \delta \mathbf{D}(x); \quad (24) \]
if the materials are linear, then one can further write
\[ W = \frac{1}{8\pi} \int d^3 x \, \mathbf{E}(x) \cdot \mathbf{D}(x) = \frac{1}{2} \int d^3 x \, \Phi(x) \rho(x). \quad (25) \]
Finally, the force on a piece of dielectric can be determined from energy-conservation-based arguments to be
\[ F_\eta = -\left( \frac{\partial W}{\partial \eta} \right)_Q = \left( \frac{\partial W}{\partial \eta} \right)_V. \quad (26) \]
We also discussed the local field, the connection between the dielectric constant and the molecular polarizability, and the physical origins of the polarizability.
Magnetostatics

The differential equations of magnetostatics are

\[ \nabla \cdot \mathbf{B}(\mathbf{x}) = 0 \quad \text{and} \quad \nabla \times \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c} \mathbf{J}(\mathbf{x}); \]  

the magnetic induction \( \mathbf{B}(\mathbf{x}) \) is defined by the torque produced on a test dipole of moment \( \mathbf{m} \) at point \( \mathbf{x} \), \( \mathbf{N} = \mathbf{m} \times \mathbf{B}(\mathbf{x}) \). In magnetostatics, \( \nabla \cdot \mathbf{J}(\mathbf{x}) = 0 \).

The integral forms of the preceding equations are

\[ \oint_S d^2 \mathbf{x} \mathbf{B}(\mathbf{x}) \cdot \mathbf{n} = 0 \quad \text{and} \quad \oint_C d\mathbf{l} \cdot \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c} \int_S d^2 \mathbf{x} \mathbf{J}(\mathbf{x}) \cdot \mathbf{n} = \frac{4\pi}{c} I_S. \]  

Given \( \mathbf{J}(\mathbf{x}) \), the integral solution for \( \mathbf{B}(\mathbf{x}) \) is

\[ \mathbf{B}(\mathbf{x}) = \frac{1}{c} \int d^3 \mathbf{x}' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \]  

The force on a localized current distribution \( \mathbf{J}(\mathbf{x}) \) produced by an applied field \( \mathbf{B}(\mathbf{x}) \) can be obtained by integrating over the force density which is

\[ \mathbf{f}(\mathbf{x}) = \frac{1}{c} \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}); \]  

the torque on the distribution is

\[ \mathbf{N} = \frac{1}{c} \int d^3 \mathbf{x} [\mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}))]. \]  

Because the divergence of \( \mathbf{B} \) is always zero, there is a vector potential \( \mathbf{A} \) such that \( \mathbf{B} = \nabla \times \mathbf{A} \). An integral solution for \( \mathbf{A} \) is

\[ \mathbf{A}(\mathbf{x}) = \frac{1}{c} \int d^3 \mathbf{x}' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}; \]  

the differential equation for \( \mathbf{A} \) is

\[ \nabla^2 \mathbf{A}(\mathbf{x}) - \nabla (\nabla \cdot \mathbf{A}(\mathbf{x})) = \frac{4\pi}{c} \mathbf{J}(\mathbf{x}). \]  

If one chooses \( \mathbf{A} \) so that \( \nabla \cdot \mathbf{A} = 0 \), which is always possible, then the Cartesian components of \( \mathbf{A} \) obey Poisson equations. More generally, one can choose the divergence of the vector potential arbitrarily; doing so is called choosing a gauge.
Multipoles and Macroscopic Magnetostatics

A localized current distribution may for many purposes be treated in the dipole approximation. The magnetic dipole moment of the distribution is defined as

$$ m = \frac{1}{2c} \int d^3 x \times J(x) \equiv \int d^3 x \mathcal{M}(x) $$

(34)

where we introduce also the magnetic moment density. In the dipole approximation, the vector potential is

$$ A_d(x) = \frac{m \times x}{r^3} $$

(35)

and the field is

$$ B_d(x) = \frac{3(m \cdot n - m)}{r^3} + \frac{8\pi}{3} m \delta(x); $$

(36)

the singular piece is strictly valid only for a point dipole. The force on the distribution given an applied field $\mathbf{B}(x)$ is, in this approximation, $\mathbf{F} = \nabla \cdot (\mathbf{m} \cdot \mathbf{B}(x))$; the torque is $\mathbf{N} = \mathbf{m} \times \mathbf{B}$.

The sources in macroscopic magnetostatics are the macroscopic or free current density $\mathbf{J}(x)$ and the magnetization or magnetic dipole moment density $\mathbf{M}(x)$ which comes from charges bound on molecules. The differential field equations are

$$ \nabla \cdot \mathbf{B}(x) = 0 \quad \text{and} \quad \nabla \times \mathbf{H}(x) = \frac{4\pi}{c} \mathbf{J}(x) $$

(37)

where the magnetic field is $\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$. For linear, isotropic, uniform materials, $\mathbf{M} = \chi_m \mathbf{H}$ which gives $\mathbf{B} = \mu \mathbf{H}$; $\mu$ is called the magnetic permeability. One introduces a magnetization current density $J_M(x) = c \nabla \times \mathbf{M}(x)$; at a boundary between two materials, there is a surface-current density component in $J_M$; it is $K_M = \mathbf{n} \times (\mathbf{M}_2 - \mathbf{M}_1)$ where $\mathbf{n}$ is the unit normal into material 2 at the boundary.

The continuity conditions on the macroscopic fields are

$$ (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \frac{4\pi}{c} \mathbf{K} $$

(38)

where $\mathbf{K}$ is the surface-current density at the boundary (it does not include the magnetization surface-current density) and $\mathbf{n}$ is as above.