Chapter Nine

Radiation

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(1857 - 1894)

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Contents

1 Introduction **1**

2 Radiation by a localized source **3**

2.1 The Near Zone 6

2.2 The Radiation or Far Zone 6

3 Multipole Expansion of the Radiation Field **9**

3.1 Electric Dipole 10

3.1.1 Example: Linear Center-Fed Antenna 13

3.2 Magnetic Dipole 14

3.3 Comparison of Dipoles 16

3.4 Electric Quadrupole 18

3.4.1 Example: Oscillating Charged Spheroid 21

3.5 Large Radiating Systems 23

3.5.1 Example: Linear Array of Dipoles 24

4 Multipole expansion of sources in waveguides **26**
4.1 Electric Dipole 26

5 Scattering of Radiation **31**
5.1 Scattering of Polarized Light from an Electron 31
5.2 Scattering of Unpolarized Light from an Electron 34
5.3 Elastic Scattering From a Molecule 35
5.3.1 Example: Scattering Off a Hard Sphere 38
5.3.2 Example: A Collection of Molecules 40

6 Diffraction **43**
6.1 Scalar Diffraction Theory: Kirchoff Approximation 45
6.2 Babinet's Principle 51
6.3 Fresnel and Fraunhofer Limits 53

7 Example Problems **55**
7.1 Example: Diffraction from a Rectangular Aperture 55
7.2 Example: Diffraction from a Circular Aperture 57
7.3 Diffraction from a Cross 60
7.4 Radiation from a Reciprocating Disk 61

1 Introduction

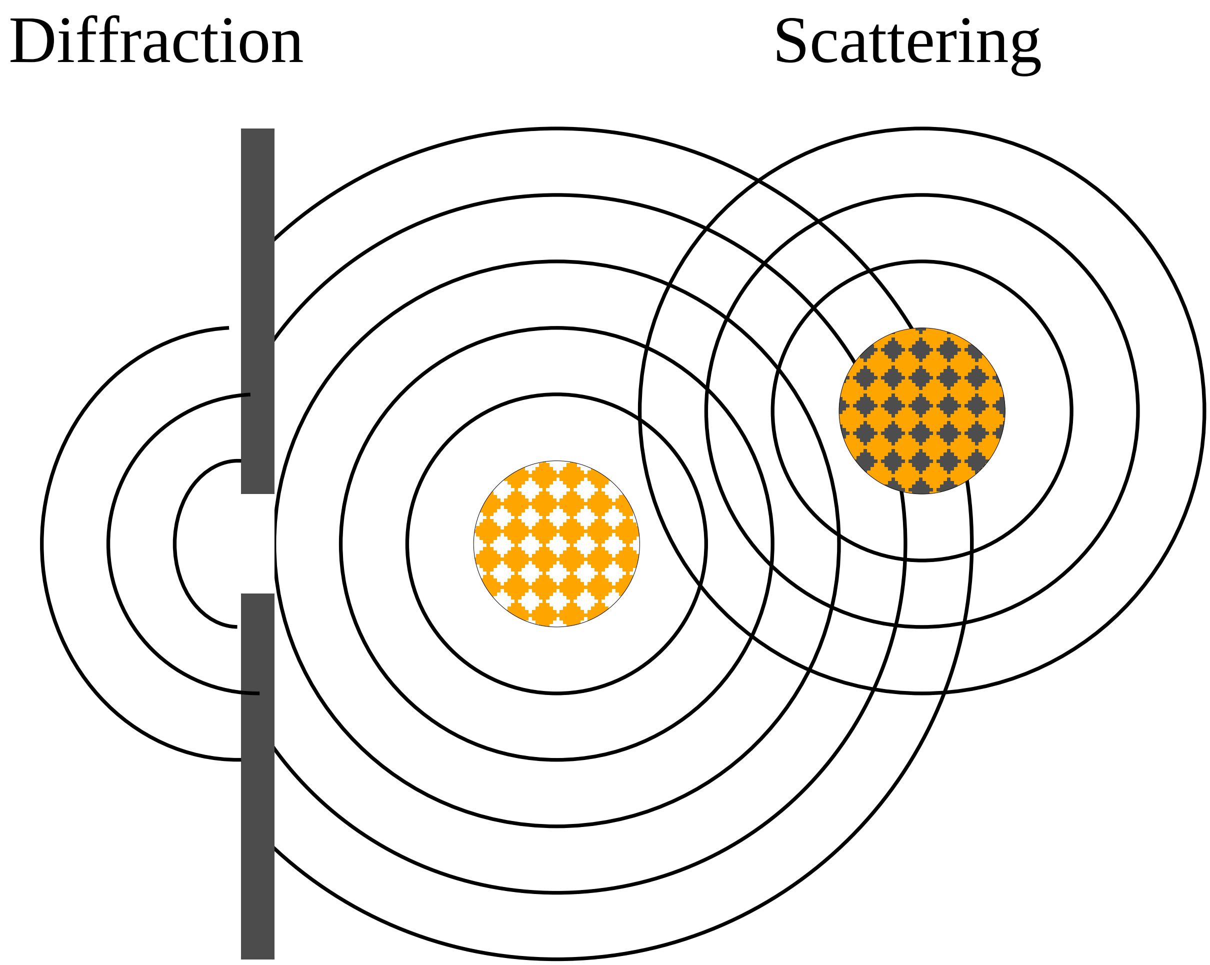
An electromagnetic wave, or electromagnetic **radiation**, has as its sources electric accelerated charges in motion. We have learned a great deal about waves but have not given much thought to the connection between the waves and the sources that produce them. That oversight will be rectified in this chapter.

The **scattering** of electromagnetic waves is produced by bombarding some object (the scatterer) with an electromagnetic wave. Under the influence of the fields in that wave, charges in the scatterer will be set into some sort of coherent motion[1] and these moving charges will produce radiation, called the scattered wave. Hence scattering phenomena are closely related to radiation phenomena.

Diffraction of electromagnetic waves is similar. One starts with a wave incident on an opaque screen with holes, or aperture, in it. Charges in the screen, especially around the apertures, are set in motion and produce radiation which in this case is called the diffracted wave.

Thus radiation, scattering, and diffraction are closely related. We shall start our investigation by considering the radiation produced by some specified localized distribution of charges and currents in harmonic motion. We delay until Chap. 14, the discussion of non-harmonic sources.

[1]The response to a harmonic excitation is of the same frequency, and thus coherent



2 Radiation by a localized source

Suppose that we are given some charge and current densities $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$[2]. These produce potentials which, in the Lorentz gauge (Chap. 6), can be found immediately using the retarded Green's function $G^{(+)}(\mathbf{x}, t; \mathbf{x}', t')$ which we shall write simply as $G(\mathbf{x}, t; \mathbf{x}', t')$:

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{c} \int d^3x' dt' G(\mathbf{x}, t; \mathbf{x}', t') \mathbf{J}(\mathbf{x}', t') \tag{1}$$

[2]In this chapter, we assume $\epsilon = \mu = 1$

$$\Phi(\mathbf{x},t) = \int d^3x'dt'G(\mathbf{x},t;\mathbf{x}',t')\rho(\mathbf{x}',t'). \tag{2}$$

The Green's function itself is given by

$$G(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}. \tag{3}$$

Because all of the equations we shall use to compute fields are linear in the fields themselves, we may conveniently treat just one Fourier component (frequency) of the field at a time. To this end we write

$$\mathbf{J}(\mathbf{x},t) = \frac{1}{2}\int_{-\infty}^{\infty} d\omega \mathbf{J}(\mathbf{x},\omega)e^{-i\omega t} \tag{4}$$

where

$$\mathbf{J}(\mathbf{x},-\omega) = \mathbf{J}^*(\mathbf{x},\omega) \tag{5}$$

is required in order that $\mathbf{J}(\mathbf{x},t)$ be real; Eq. (5) is known as a "reality condition." We may equally well, and more conveniently, replace Eqs. (4) and (5) by

$$\mathbf{J}(\mathbf{x},t) = \Re\int_0^{\infty} d\omega \mathbf{J}(\mathbf{x},\omega)e^{-i\omega t}. \tag{6}$$

We will do this and will in general not bother to write $\Re$ in front of every complex expression whose real part must be taken. We will just have to remember that the real part is the physically meaningful quantity. Similarly, we introduce the Fourier transform of the charge density,

$$\rho(\mathbf{x},t) = \int_0^{\infty} d\omega \rho(\mathbf{x},\omega)e^{-i\omega t}. \tag{7}$$

In the following we shall suppose that the sources have just a single frequency component,

$$\mathbf{J}(\mathbf{x},\omega') = \mathbf{J}(\mathbf{x})\delta(\omega-\omega'), \qquad \omega' > 0 \tag{8}$$

$$\rho(\mathbf{x},\omega') = \rho(\mathbf{x})\delta(\omega-\omega'), \qquad \omega' > 0. \tag{9}$$

Thus, assuming $\omega > 0$,

$$\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x})e^{-i\omega t} \quad \text{and} \tag{10}$$

$$\rho(\mathbf{x},t) = \rho(\mathbf{x})e^{-i\omega t}. \tag{11}$$

From the continuity condition on the sources,

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{x},t) = 0, \tag{12}$$

we find that $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are related by

$$\rho(\mathbf{x}) = -i\frac{\nabla \cdot \mathbf{J}(\mathbf{x})}{\omega}. \tag{13}$$

Using Eq. (10) in Eq. (1) and employing Eq. (3) for the Green's function, we find, upon completing the integration over the time, that

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int d^3x'\, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} e^{ik|\mathbf{x}-\mathbf{x}'|} e^{-i\omega t} \tag{14}$$

where, as usual, $k \equiv \omega/c$. Define $\mathbf{A}(\mathbf{x})$ by

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}; \tag{15}$$

comparison with Eq. (14) gives

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x'\, \mathbf{J}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}. \tag{16}$$

From here the recipe is to find $\mathbf{B}(\mathbf{x},t)$ from the curl of $\mathbf{A}(\mathbf{x},t)$; then the electric field is found[3] from $\nabla \times \mathbf{B}(\mathbf{x},t) = c^{-1}\partial \mathbf{E}(\mathbf{x},t)/\partial t$, which holds in regions where the current density vanishes. These fields have the forms

$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}(\mathbf{x})e^{-i\omega t} \qquad \mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})e^{-i\omega t} \tag{17}$$

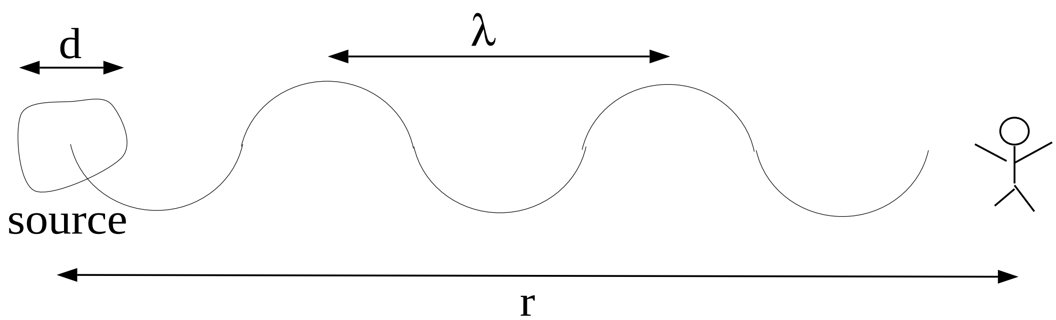
where

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) \qquad \mathbf{E}(\mathbf{x}) = \frac{i}{k}\nabla \times \mathbf{B}(\mathbf{x}). \tag{18}$$

We have reduced everything to a set of straightforward calculations - integrals and derivatives. Doing them exactly can be tedious, so we should spend some time

[3]Notice that we never have to evaluate the scalar potential.

thinking about whether there are any **simplifying approximations** that may have general validity or at least validity in some cases of interest. There are approximations based on expansions in powers of some small parameter. We can see what may be possible by realizing that there are *three relevant lengths* in any radiating system. Provided the origin of coordinates is taken to be somewhere within the source in the integrals above, these are the size of the source, $r' = |\mathbf{x}'|$; the distance of the observer from the source, represented by $r = |\mathbf{x}|$; and the wavelength of the radiation, $\lambda = 2\pi/k$. The magnitude r' is never larger than d, the size of the source. Focusing on just the relative size of λ and r, we identify the three traditional regimes below.



d << r << λ	Near or static zone
d << r ~ λ	Intermediate or induction zone
d << λ << r	Far or radiation zone

Here we have specified also that d be much smaller than the other two lengths. That simplifies the discussion of the three regimes and so is a convenience but it is not always met in practice nor is it always necessary. In particular, the fields far away from the source (in the radiation zone) have characteristic forms independent of the relative size of λ and d provided r is large enough. Also, man-made sources such as antennas (and antenna arrays) are often intentionally constructed to have $\lambda \sim d$ and even $\lambda << d$ in which case the inequalities above are not always satisfied. On the other hand, natural radiating systems, such as atoms and nuclei, typically do satisfy the condition $d << \lambda$ and $d << r$ for any r at which it is practical to detect the radiation.

2.1 The Near Zone

Consider first the **near zone**. Here $d << \lambda$ and $r << \lambda$, so there is a simple expansion of the exponential factor,

$$e^{ik|\mathbf{x}-\mathbf{x}'|} = 1 + ik|\mathbf{x}-\mathbf{x}'| + ... \tag{19}$$

which leads to

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}[1 + ik|\mathbf{x}-\mathbf{x}'| + ...]e^{-i\omega t}. \tag{20}$$

The leading term in this expansion is

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}e^{-i\omega t}. \tag{21}$$

Aside from the harmonic time dependence, this is just the vector potential of a static current distribution $\mathbf{J}(\mathbf{x})$, and that is the origin of the name "static" zone; the magnetic induction here has a spatial dependence which is the same as what one would find for a static current distribution with the spatial dependence of the actual oscillating current distribution. We find this result for the simple reason that in the near zone the exponential factor can be approximated as unity.

2.2 The Radiation or Far Zone

In the **radiation or far zone** ($r \gg \lambda \gg d$), the story is completely different because in this regime the behavior of the exponential dominates the integral. We can most easily see what will happen if we first expand the displacement $|\mathbf{x}-\mathbf{x}'|$ in powers of r'/r (d/r):

$$\begin{aligned} |\mathbf{x}-\mathbf{x}'| &= [(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')]^{1/2} = (r^2 - 2\mathbf{x}\cdot\mathbf{x}' + r'^2)^{1/2} \\ &= r\left[1 - \frac{2\mathbf{x}\cdot\mathbf{x}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2} = r\left[1 - 2\frac{\mathbf{n}\cdot\mathbf{x}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{1/2} \end{aligned} \tag{22}$$

where $\mathbf{n} = \mathbf{x}/r$ is a unit vector in the direction of $\mathbf{x}$. Carrying the expansion to second order in r'/r, we have

$$|\mathbf{x} - \mathbf{x}'| = r\left[1 - \frac{\mathbf{n}\cdot\mathbf{x}'}{r} + \frac{1}{2}\left(\frac{r'}{r}\right)^2 - \frac{1}{2}\left(\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\right)^2\right]. \tag{23}$$

Similarly,

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r}\left\{1 + \frac{\mathbf{n}\cdot\mathbf{x}'}{r} + \frac{1}{2}\left(\frac{r'}{r}\right)^2\left[3\left(\frac{\mathbf{n}\cdot\mathbf{x}'}{r'}\right)^2 - 1\right]\right\}. \tag{24}$$

Using the first of these expansions we can write

$$\begin{aligned} e^{ik|\mathbf{x}-\mathbf{x}'|} &= e^{ikr}e^{-ik(\mathbf{n}\cdot\mathbf{x}')}e^{i\frac{kr}{2}\left[\left(\frac{r'}{r}\right)^2 - \left(\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\right)^2\right]} \\ &= e^{ikr}e^{-ik(\mathbf{n}\cdot\mathbf{x}')}e^{i\frac{kr'^2}{2r}\left[1 - \frac{(\mathbf{n}\cdot\mathbf{x}')^2}{r'^2}\right]}. \end{aligned} \tag{25}$$

Given $r >> r'$ and $kr'^2/r << 1$, it is clear that this exponential function can be approximated by just the first two factors; the third represents a change of phase by an amount small compared to a radian. Further, in the far zone it is sufficient to approximate

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r}. \tag{26}$$

Putting these pieces together we have

$$\mathbf{A}(\mathbf{x}, t) = \frac{e^{i(kr-\omega t)}}{cr}\int d^3x'\,\mathbf{J}(\mathbf{x}')e^{-ik(\mathbf{n}\cdot\mathbf{x}')}\ \ (r \gg d \text{ and } r \gg d^2/\lambda)\,. \tag{27}$$

This expression is always valid for r "large enough" which means $r >> r'$ and $r >> kr'^2$. The relative size of λ and d is unimportant.

The behavior of $\mathbf{A}(\mathbf{x}, t)$ on r and t is explicitly given by the factor in front of the integral; the integral depends on the direction of $\mathbf{x}$ but not on its magnitude. Hence in the far zone the vector potential always takes the form

$$\mathbf{A}(\mathbf{x}, t) = \frac{e^{i(kr-\omega t)}}{r}\mathbf{f}(\theta, \phi) \tag{28}$$

where

$$\mathbf{f}(\theta, \phi) \equiv \frac{1}{c}\int d^3x'\,\mathbf{J}(\mathbf{x}')e^{-ik(\mathbf{n}\cdot\mathbf{x}')}; \tag{29}$$

θ and ϕ specify, in polar coordinates, the direction of $\mathbf{x}$ (or $\mathbf{n}$).

Knowing the form of the potential so precisely makes it easy to see what must be the form of the fields $\mathbf{E}$ and $\mathbf{B}$ in the far zone. First,

$$\mathbf{B}(\mathbf{x},t) = \nabla \times \mathbf{A}(\mathbf{x},t) = \nabla \times \left(\frac{e^{i(kr-\omega t)}}{r}\mathbf{f}(\theta,\phi)\right) \approx ik\frac{e^{i(kr-\omega t)}}{r}[\mathbf{n}\times\mathbf{f}(\theta,\phi)] \qquad (30)$$

where we have discarded terms of relative order λ/r or d/r. Further, from Eqs. (18) and (30), we can find $\mathbf{E}$($\mathbf{x}$,t); to the same order as $\mathbf{B}$, it is

$$\mathbf{E}(\mathbf{x},t) = -ik\frac{e^{i(kr-\omega t)}}{r}[\mathbf{n}\times(\mathbf{n}\times\mathbf{f}(\theta,\phi))] = \mathbf{B}(\mathbf{x},t)\times\mathbf{n}. \qquad (31)$$

There are two essential features of these equations.

- First, both $\mathbf{E}$ and $\mathbf{B}$ in the radiation zone is that the field strengths are proportional to r^{-1}; this is very different from the case for static fields which fall off at least as fast as r^{-2} (consider the static zone).

- Second, the radiation fields are transverse, meaning they are perpendicular to $\mathbf{x}$ or $\mathbf{n}$; they are also perpendicular to each other.

The Poynting vector in the far zone also has a simple basic form:

$$\begin{aligned}
<\mathbf{S}> &= \frac{c}{8\pi}(\mathbf{E}\times\mathbf{B}^*) = \frac{c}{8\pi}\left(\frac{-ik}{r}\right)^2[\mathbf{n}\times(\mathbf{n}\times\mathbf{f}(\theta,\phi))]\times[\mathbf{n}\times\mathbf{f}^*(\theta,\phi)] \\
&= -\frac{ck^2}{8\pi r^2}\{[\mathbf{n}(\mathbf{n}\cdot\mathbf{f}(\theta,\phi))-\mathbf{f}(\theta,\phi)]\times[\mathbf{n}\times\mathbf{f}^*(\theta,\phi)]\} \\
&= -\frac{ck^2}{8\pi r^2}\Big\{(\mathbf{n}\cdot\mathbf{f}(\theta,\phi))[\mathbf{n}(\mathbf{n}\cdot\mathbf{f}^*(\theta,\phi))-\mathbf{f}^*(\theta,\phi)] - \mathbf{n}|\mathbf{f}(\theta,\phi)|^2 + \mathbf{f}^*(\theta,\phi)(\mathbf{n}\cdot\mathbf{f}(\theta,\phi))\Big\} \\
&= \frac{ck^2}{8\pi r^2}\mathbf{n}[|\mathbf{f}(\theta,\phi)|^2 - |\mathbf{n}\cdot\mathbf{f}(\theta,\phi)|^2]. \qquad (32)
\end{aligned}$$

This is presumably the time-averaged energy current density. Because it points radially outward[4], it also gives directly the angular distribution of radiated power:

$$\frac{d\mathcal{P}}{d\Omega} = \frac{ck^2}{8\pi}[|\mathbf{f}(\theta,\phi)|^2 - |\mathbf{n}\cdot\mathbf{f}(\theta,\phi)|^2]. \qquad (33)$$

[4]In the near or intermediate zones, there are non-zero components in other directions as well.

If we integrate over the solid angle, we find the total power radiated:

$$\mathcal{P} = \oint_S d^2x \, <\mathbf{S}> \cdot \mathbf{n} = \frac{ck^2}{8\pi} \int d\Omega \, [|\mathbf{f}(\theta,\phi)|^2 - |\mathbf{n} \cdot \mathbf{f}(\theta,\phi)|^2]. \tag{34}$$

Notice that the radiated power is, appropriately, independent of r.

3 Multipole Expansion of the Radiation Field

Thus far, we have only assumed that $r \gg \lambda, \; d$. Now we will consider the d/λ. Consider two limits:

- If $d/\lambda \ll 1$, then all elements of the source will essentially be in phase, and thus an observer at r cannot learn about the structure of the source from the emitted radiation. In this limit, we need only consider the first finite moment in d/λ (if the series is convergent).

- If $d/\lambda \gtrsim 1$, then the elements of the source will not radiate in phase, and an observer at r may learn about the details of the structure of the source by analyzing the interference of the radiation pattern (i.e. Bragg diffraction). In this case, to be discussed in sec. V, we need to retain more terms in the series.

Let us now go back and attempt to evaluate $\mathbf{f}(\theta,\phi)$. If $kd << 1$, or $2\pi d/\lambda << 1$, then it is not unreasonable to proceed with the evaluation by expanding the exponential function $e^{-ik(\mathbf{n}\cdot\mathbf{x}')}$,

$$\mathbf{f}(\theta,\phi) = \frac{1}{c} \int d^3x' \, \mathbf{J}(\mathbf{x}') \left[1 - ik(\mathbf{n}\cdot\mathbf{x}') - \frac{1}{2}k^2(\mathbf{n}\cdot\mathbf{x}')^2 + ... \right]. \tag{35}$$

3.1 Electric Dipole

The first term in the expansion is just the volume integral of $\mathbf{J}(\mathbf{x}')$; one can write it as

$$\frac{1}{c} \int d^3x' \, \mathbf{J}(\mathbf{x}') = -\frac{1}{c} \int d^3x' \, [\nabla' \cdot \mathbf{J}(\mathbf{x}')]\mathbf{x}' \tag{36}$$

which one can show by doing integration by parts starting from the right-hand side of this equation. Now employ Eq. (13) to have

$$\frac{1}{c}\int d^3x'\,\mathbf{J}(\mathbf{x}') = -ik\int d^3x'\,\mathbf{x}'\rho(\mathbf{x}'). \tag{37}$$

The right-hand side can be recognized as the electric dipole moment of the amplitude $\rho(\mathbf{x})$ of the harmonically varying charge distribution. Let us define the electric dipole moment $\mathbf{p}$ in the usual way,

$$\mathbf{p} \equiv \int d^3x'\,\mathbf{x}'\rho(\mathbf{x}'). \tag{38}$$

The electric dipole contribution $\mathbf{f}_{ed}(\theta,\phi)$ to $\mathbf{f}(\theta,\phi)$ is thus

$$\mathbf{f}_{ed}(\theta,\phi) = -ik\mathbf{p}; \tag{39}$$

it is in fact independent of θ and ϕ. The corresponding contribution to the vector potential, $\mathbf{A}_{ed}(\mathbf{x},t)$, is

$$\mathbf{A}_{ed}(\mathbf{x},t) = -ik\mathbf{p}\frac{e^{i(kr-\omega t)}}{r}. \tag{40}$$

We have used only $d << \lambda$ and $d << r$; no assumption about the relative size of λ and r has been made. It is somewhat tedious, but nevertheless instructive, to evaluate the fields without making any assumptions so that we can see their form in the near, intermediate, and far zones. First, the magnetic induction is

$$\begin{aligned}\mathbf{B}_{ed}(\mathbf{x}) &= \nabla\times\mathbf{A}_{ed}(\mathbf{x}) = -ik\nabla\left(\frac{e^{ikr}}{r}\right)\times\mathbf{p} \\ &= -ik\left(ik-\frac{1}{r}\right)\frac{e^{ikr}}{r}(\mathbf{n}\times\mathbf{p}) = k^2\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}(\mathbf{n}\times\mathbf{p}).\end{aligned} \tag{41}$$

Further, the electric field is

$$\begin{aligned}\mathbf{E}_{ed}(\mathbf{x}) &= \frac{i}{k}(\nabla\times\mathbf{B}_{ed}(\mathbf{x})) \\ &= ik\nabla\left[\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right]\times(\mathbf{n}\times\mathbf{p}) + ik\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\nabla\times(\mathbf{n}\times\mathbf{p}) \\ &= ik\left\{\left(\frac{ik}{r}-\frac{1}{r^2}\right)\left(1-\frac{1}{ikr}\right)+\frac{1}{ikr^3}\right\}e^{ikr}\mathbf{n}\times(\mathbf{n}\times\mathbf{p})\end{aligned}$$

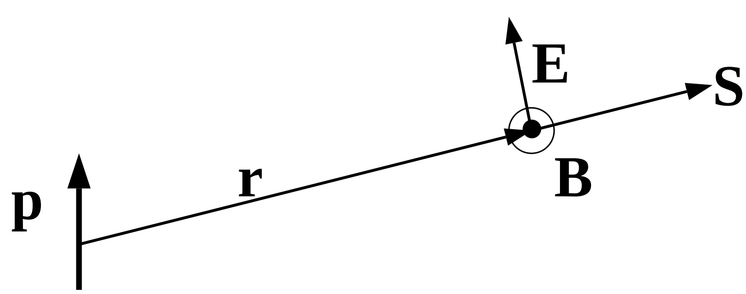
$$
\begin{aligned}
& +ik\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left[(\mathbf{p}\cdot\nabla)\left(\frac{\mathbf{x}}{r}\right)-\mathbf{p}\nabla\cdot\left(\frac{\mathbf{x}}{r}\right)\right] \\
= \; & \frac{k^2}{r}\left[-1+\frac{2}{ikr}+\frac{2}{k^2r^2}\right]e^{ikr}\mathbf{n}\times(\mathbf{n}\times\mathbf{p}) \\
& +\frac{ik}{r}e^{ikr}\left(1-\frac{1}{ikr}\right)\left(\frac{\mathbf{p}}{r}-\frac{(\mathbf{p}\cdot\mathbf{n})\mathbf{n}}{r}-\frac{3\mathbf{p}}{r}+\frac{\mathbf{p}}{r}\right) \\
= \; & \left\{-\frac{k^2}{r}\mathbf{n}\times(\mathbf{n}\times\mathbf{p})+\frac{1}{r^3}(1-ikr)[3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}]\right\}e^{ikr}. \qquad (42)
\end{aligned}
$$

The electric field is divided up in this fashion to bring out, first, the form in the radiation zone which is the first term and, second, the form in the near zone $r << \lambda$ which is the second term. The spatial dependence of the latter is the same as the field of a static dipole, $[3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}]/r^3$, but do not forget that it oscillates with angular frequency ω. In the intermediate zone where all contributions are comparable, the field is complex indeed.[5]

In the radiation zone, where the fields become quite simple, they are

$$
\mathbf{B}_{ed}(\mathbf{x})=\frac{k^2}{r}e^{ikr}(\mathbf{n}\times\mathbf{p}) \qquad \text{and} \qquad \mathbf{E}_{ed}=-\frac{k^2}{r}e^{ikr}\mathbf{n}\times(\mathbf{n}\times\mathbf{p}). \qquad (43)
$$

The same conclusion may be reached much more simply from Eqs. (30), (31), and (39). As remarked earlier, the fields in the far zone are transverse to $\mathbf{n}$. If we let $\mathbf{p}$ define the polar axis, then $\mathbf{B}_{ed}$ is azimuthal, i.e., in the direction of $\boldsymbol{\phi}$. This is a special feature of electric dipole radiation. Further, $\mathbf{E}_{ed}(\mathbf{x})$ is in the direction of $\boldsymbol{\theta}$.



[5]And this is only the electric dipole part of the field which is, along with the magnetic dipole part of the field, by far the simplest.

FromEqs. (32) and (39) we find that the time-averaged Poynting vector in the radiation zone is

$$< \mathbf{S} >= \frac{ck^4}{8\pi r^2}[|\mathbf{p}|^2 - |\mathbf{n}\cdot\mathbf{p}|^2]\mathbf{n} = \frac{ck^4}{8\pi r^2}|\mathbf{p}|^2 \sin^2\theta\,\mathbf{n}, \tag{44}$$

where θ is the usual polar angle, i.e., the angle between the dipole moment and the direction at which the radiation is observed. The radiated power per unit solid angle is

$$\frac{d\mathcal{P}}{d\Omega} = r^2 < \mathbf{S} > \cdot\mathbf{n} = \frac{ck^4}{8\pi}|\mathbf{p}|^2 \sin^2\theta, \tag{45}$$

and the total power radiated is

$$\mathcal{P} = \int d\Omega\,\frac{d\mathcal{P}}{d\Omega} = \frac{ck^4|\mathbf{p}|^2}{8\pi}\int d\phi d\theta\,\sin\theta\sin^2\theta = \frac{ck^4|\mathbf{p}|^2}{3}. \tag{46}$$

3.1.1 Example: Linear Center-Fed Antenna

Consider the short, linear, "center-fed" antenna shown below.

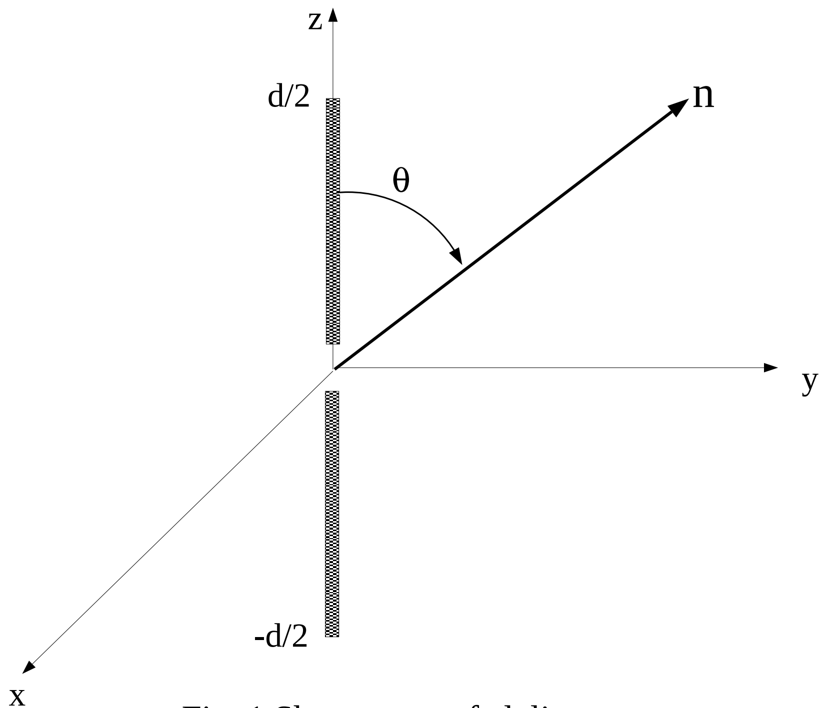


Fig. 1 Short, center-fed, linear antenna.

For such an antenna, the current density can be crudely approximated by

$$\mathbf{J}(\mathbf{x},t) = \boldsymbol{\epsilon}_{\mathbf{3}} I_0 \delta(x)\delta(y)\left(1 - 2\frac{|z|}{d}\right) e^{-i\omega t} \tag{47}$$

for $|z| < d/2$; for $|z| > d/2$, it is zero. Given this current density, we can evaluate the divergence and so find the charge density,

$$\nabla \cdot \mathbf{J}(\mathbf{x}) = -\frac{2I_0}{d}\delta(x)\delta(y)\frac{z}{|z|} = -i\omega\rho(\mathbf{x}) \tag{48}$$

or

$$\rho(\mathbf{x}) = \frac{2iI_0}{\omega d}\delta(x)\delta(y)\frac{z}{|z|}, \qquad\qquad |z| < d/2. \tag{49}$$

Hence,

$$\mathbf{p} = \int d^3x\, \mathbf{x}\rho(\mathbf{x}) = \frac{iI_0 d}{2ck}\boldsymbol{\epsilon}_{\mathbf{3}}. \tag{50}$$

Now we have only to plug Eq. (50) into Eqs. (45) and (46) to find

$$\frac{d\mathcal{P}}{d\Omega} = \frac{I_0^2}{32\pi c}(kd)^2 \sin^2\theta \qquad \text{and} \qquad \mathcal{P} = \frac{I_0^2 k^2 d^2}{12c}. \tag{51}$$

The calculation in this example may be expected to provide a good approximation to the total radiated power provided $kd << 1$ so that the electric dipole term dominates, unless it vanishes. There are many instances where this happens.

3.2 Magnetic Dipole

When this happens it is necessary to look at higher-order terms in the expansion of the phase factor $e^{ik|\mathbf{x}-\mathbf{x}'|}$. Let's look now at the next one. Start from the exact expression for $\mathbf{A}(\mathbf{x})$,

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \frac{1}{c}\int d^3x'\, \mathbf{J}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\ &= \mathbf{A}_{ed}(\mathbf{x}) + \frac{e^{ikr}}{cr}\int d^3x'\, \mathbf{J}(\mathbf{x}')\left(\frac{\mathbf{n}\cdot\mathbf{x}'}{r} - ik(\mathbf{n}\cdot\mathbf{x}')\right)[1+\mathcal{O}(d/r, d/\lambda)]. \end{aligned} \tag{52}$$

where the second term in the () comes from the exponential, and the first comes from the corresponding denominator. The integral we wish to examine is

$$\begin{aligned} &\frac{1}{c}\int d^3x'\, \mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') \\ &= \frac{1}{2c}\int d^3x'\ \{[\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') + \mathbf{x}'(\mathbf{n}\cdot\mathbf{J}(\mathbf{x}'))] + [\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') - \mathbf{x}'(\mathbf{n}\cdot\mathbf{J}(\mathbf{x}'))]\} \\ &= \frac{1}{2c}\int d^3x'\, [\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') + \mathbf{x}'(\mathbf{n}\cdot\mathbf{J}(\mathbf{x}'))] + \frac{1}{2c}\int d^3x'\, \mathbf{n}\times(\mathbf{J}(\mathbf{x}')\times\mathbf{x}') \end{aligned} \tag{53}$$

What is the point of breaking the integral into two pieces, symmetric and antisymmetric under interchange of $\mathbf{x}'$ and $\mathbf{J}(\mathbf{x}')$? There are several related points. One is that in the near zone the second term on the right-hand side produces a magnetic induction which has the form of the induction of a static magnetic dipole while the first term produces an electric field which has the form of the field of a static electric quadrupole. Hence the radiation from the former is called magnetic dipole radiation while that from the latter is known as electric quadrupole radiation. Also, the magnetic dipole part produces a purely transverse electric field in all zones while the electric quadrupole part gives a purely transverse magnetic induction. Recall that for electric dipole radiation, $\mathbf{B}(\mathbf{x}, t)$ is purely transverse in all zones as well.

Let us look at the magnetic dipole fields first. From Eqs. (52) and (53) we have

$$\begin{aligned} \mathbf{A}_{md}(\mathbf{x}) &= \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\mathbf{n} \times \int d^3x'\, \frac{1}{2c}(\mathbf{J}(\mathbf{x}') \times \mathbf{x}') \\ &\equiv -ik\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{n} \times \mathbf{m}) \end{aligned} \tag{54}$$

where

$$\mathbf{m} \equiv \frac{1}{2c}\int d^3x'\, [\mathbf{x}' \times \mathbf{J}(\mathbf{x}')] \tag{55}$$

is the magnetic dipole moment, familiar from our study of magnetostatics.

Rather than plow ahead with with the evaluation of the curl to find $\mathbf{B}$, etc., let us recall the electric dipole results

$$\mathbf{A}_{ed}(\mathbf{x}) = -ik\frac{e^{ikr}}{r}\mathbf{p} \tag{56}$$

$$\mathbf{B}_{ed}(\mathbf{x}) = k^2\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{n} \times \mathbf{p}). \tag{57}$$

We see that $\mathbf{B}_{ed}$ is the same in functional form as $\mathbf{A}_{md}$; consequently, $\mathbf{E}_{ed}$, which is the curl of $\mathbf{B}_{ed}$, must be the same in form as the curl of $\mathbf{A}_{md}$, or $\mathbf{B}_{md}$. Hence we can immediately write, using Eq. (43),

$$\mathbf{B}_{md}(\mathbf{x}) = k^2\frac{e^{ikr}}{r}\mathbf{n} \times (\mathbf{n} \times \mathbf{m}) + \frac{e^{ikr}}{r^3}(1 - ikr)[3\mathbf{n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m}]. \tag{58}$$

Notice that in the near zone $\mathbf{B}_{md}(\mathbf{x})$ is the same as that of a static dipole.

As for the corresponding electric field, we could work through the tedious derivatives of the magnetic induction, but it happens that this is one time when it is much easier to evaluate the the field from the potentials. We know that

$$\mathbf{E}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x}, t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{x}, t)}{\partial t}. \tag{59}$$

Also, in the Lorentz gauge

$$\nabla \cdot \mathbf{A}(\mathbf{x},t) + \frac{1}{c}\frac{\partial \Phi(\mathbf{x},t)}{\partial t} = 0 \qquad \text{or} \qquad \nabla \cdot \mathbf{A}(\mathbf{x}) - ik\Phi(\mathbf{x}) = 0. \tag{60}$$

Using this relation for $\Phi(\mathbf{x})$, we find that

$$\mathbf{E}(\mathbf{x}) = -\frac{i}{k}\nabla(\nabla \cdot \mathbf{A}(\mathbf{x})) + ik\mathbf{A}(\mathbf{x}) \tag{61}$$

for any electric field which is harmonic in time. From our expression for $\mathbf{A}_{md}(\mathbf{x})$, one can easily see that $\nabla \cdot \mathbf{A}_{md} = 0$[6] and so $\mathbf{E}_{md}$ is simply proportional to the vector potential,

$$\mathbf{E}_{md}(\mathbf{x}) = -k^2\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)(\mathbf{n} \times \mathbf{m}). \tag{62}$$

3.3 Comparison of Dipoles

To summarize:

- The electric dipole and magnetic dipole fields are the same with $\mathbf{E}$ and $\mathbf{B}$ interchanged, $\mathbf{E}_{md} \Leftrightarrow -\mathbf{B}_{ed}$ and $\mathbf{B}_{md} \Leftrightarrow \mathbf{E}_{ed}$ when $\mathbf{p} \Leftrightarrow \mathbf{m}$.

- In the near zone $\mathbf{E}_{ed}$ and $\mathbf{B}_{md}$ have the form of static dipole fields, while in all zones, $\mathbf{B}_{ed}$ and $\mathbf{E}_{md}$ are purely azimuthal in direction.

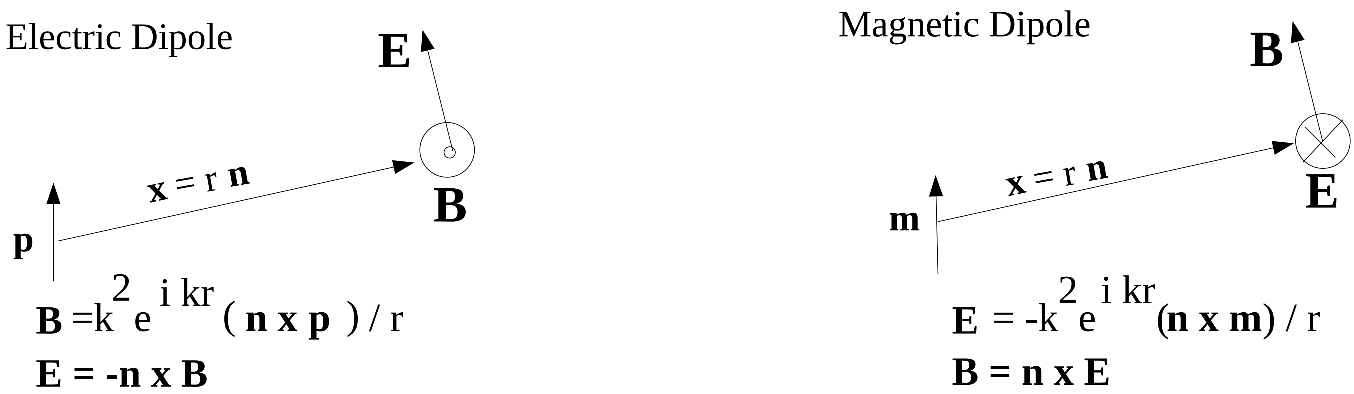
- The Poynting vector in the far zone has the same form for both electric dipole and magnetic dipole fields; in the latter case it is

$$< \mathbf{S} >= r^2\mathbf{n}\frac{ck^4}{8\pi}|\mathbf{m}|^2 \sin^2\theta, \tag{63}$$

  leading to results for the power distribution and total power which are the same as for electric dipole radiation, Eqs. (45) and (46), with $\mathbf{m}$ in place of $\mathbf{p}$. Notice particularly that if one measures $d\mathcal{P}/d\Omega$, and finds the $\sin^2\theta$ pattern, then one

[6] An equivalent statement is that a magnetic dipole is always charge neutral, so that $\Phi = 0$.

knows that the radiation is indeed[7] dipole radiation; however, it is impossible to distinguish electric dipole radiation from magnetic dipole radiation without examining its polarization.



$$\mathbf{p} \longrightarrow \mathbf{m}$$
$$\mathbf{B} \longrightarrow -\mathbf{E}$$
$$\mathbf{E} \longrightarrow \mathbf{B}$$

These formulas *suggest* that for a given set of moving charges, one should get as much power out of an oscillating magnetic dipole as an oscillating electric dipole. This is not so, since the magnetic dipole moment is

$$\mathbf{m} \equiv \frac{1}{2c} \int d^3x' \, [\mathbf{x}' \times \mathbf{J}(\mathbf{x}')] \tag{64}$$

thus

$$|\mathbf{m}| \sim \frac{v}{c} Q d \tag{65}$$

where v is the speed of the moving charge, Q is the magnitude of this charge, and d is the size of the current loop. The size of the corresponding electric dipole moment

[7] Higher-order multipoles produce radiation with distinctive angular distributions which are never proportional to $\sin^2\theta$.

is roughly Qd. Thus,

$$|\mathbf{m}| \sim \frac{v}{c}|\mathbf{p}| \tag{66}$$

We see that

$$\mathcal{P}_{md} \approx \left(\frac{v}{c}\right)^2 \mathcal{P}_{ed}\,. \tag{67}$$

Thus, in a system with both an electric and magnetic dipole moment, the latter is usually a relativistic correction to the former.

3.4 Electric Quadrupole

Let's go now to the other contribution to the vector potential in Eq. (53). This is called the electric quadrupole term; knowing that, we naturally expect to find that it produces an electric field in the near zone which has the characteristic form of a static electric quadrupole field. The vector potential is

$$\begin{aligned} A_{eq}(\mathbf{x}) &= \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\frac{1}{2c}\int d^3x'\,[\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') + \mathbf{x}'(\mathbf{n}\cdot\mathbf{J}(\mathbf{x}'))] \\ &= -\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\frac{1}{2c}\int d^3x'\,(\mathbf{n}\cdot\mathbf{x}')(\nabla'\cdot\mathbf{J}(\mathbf{x}'))\mathbf{x}'. \end{aligned} \tag{68}$$

To demonstrate this algebraic step, consider the i^{th} component of the final expression:

$$\begin{aligned} \frac{1}{2c}\int d^3x'\,(\mathbf{n}\cdot\mathbf{x}')\left(\sum_j \frac{\partial J_j}{\partial x'_j}\right)x'_i &= -\frac{1}{2c}\sum_j\int d^3x'\,\frac{\partial}{\partial x'_j}[x'_i(\mathbf{n}\cdot\mathbf{x}')]J_j \\ &= -\frac{1}{2c}\int d^3x'\,\sum_j[\delta_{ij}(\mathbf{n}\cdot\mathbf{x}') + x'_i n_j]J_j \\ &= -\frac{1}{2c}\int d^3x'\,[J_i(\mathbf{n}\cdot\mathbf{x}') + x'_i(\mathbf{n}\cdot\mathbf{J})] \end{aligned} \tag{69}$$

which matches the i^{th} component of the original expression in Eq. (68). Making use of Eq. (13) for $\nabla\cdot\mathbf{J}(\mathbf{x})$ and also using $\omega = ck$, we find, from Eq. (68), that

$$\mathbf{A}_{eq}(\mathbf{x}) = -\frac{k^2}{2}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\int d^3x'\,\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\rho(\mathbf{x}'). \tag{70}$$

There are nine components to the integral since the factor of $\mathbf{n}$ in the integrand can be used to project out three numbers by, for example, letting $\mathbf{n}$ be each of the Cartesian basis vectors. That is, the basic integral over the source, $\rho(\mathbf{x})$, which appears here is a dyadic, $\int d^3x\, \mathbf{x}\mathbf{x}\rho(\mathbf{x})$. It is symmetric and so has at most six independent components. Notice also that $\mathbf{A}_{eq}$ depends on the direction of $\mathbf{n}$, which was not the case for either $\mathbf{A}_{ed}$ or $\mathbf{A}_{md}$; the evaluation of the fields is further complicated as a consequence.

The electric quadrupole vector potential can be written in terms of the components Q_{ij} of the electric quadrupole moment tensor which we defined long ago. Recall that

$$Q_{ij} \equiv \int d^3x\,(3x_ix_j - r^2\delta_{ij})\rho(\mathbf{x}). \tag{71}$$

Take combinations of these to construct the components Q_i of a vector $\mathbf{Q}$:

$$Q_i(\mathbf{n}) \equiv \sum_{j=1}^{3} Q_{ij}n_j. \tag{72}$$

From these definitions one can show quite easily that

$$\frac{1}{3}\mathbf{n}\times\mathbf{Q}(\mathbf{n}) \equiv \mathbf{n}\times\left(\int d^3x'\,\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\rho(\mathbf{x}')\right). \tag{73}$$

To see this, note that the i^{th} component of $\mathbf{Q}/3$ is

$$\frac{1}{3}Q_i = \int d^3x'\,[x_i'(\sum_j n_jx_j')\rho(\mathbf{x}') - r'^2n_i\rho(\mathbf{x}')/3]. \tag{74}$$

The first term on the right-hand side of this relation produces the i^{th} component of the integral in Eq. (73); the second term is some i-independent quantity multiplied by n_i; the three terms in $\mathbf{Q}$ of this form give something which is directly proportional to $\mathbf{n}$ and so they do not contribute to $\mathbf{n}\times\mathbf{Q}$.

Thus have we established the validity of Eq. (73). But what good is it? It tells us that we can write $\mathbf{n}\times\mathbf{A}_{eq}$ in terms of $\mathbf{n}\times\mathbf{Q}$, but does not allow us to write $\mathbf{A}_{eq}$ itself in terms of $\mathbf{Q}$. However, if we restrict our attention to the radiation zone, then

$\mathbf{n} \times \mathbf{A}$ is all we will need, because in this zone, $\mathbf{B} = ik(\mathbf{n} \times \mathbf{A})$ and $\mathbf{E} = -\mathbf{n} \times \mathbf{B}$. Thus, in the far zone,

$$\mathbf{B}_{eq}(x) = -\frac{ik^3}{6}\frac{e^{ikr}}{r}[\mathbf{n} \times \mathbf{Q}(\mathbf{n})] \tag{75}$$

and

$$\mathbf{E}_{eq}(\mathbf{x}) = \frac{ik^3}{6}\frac{e^{ikr}}{r}\mathbf{n} \times [\mathbf{n} \times \mathbf{Q}(\mathbf{n})]. \tag{76}$$

The time-averaged power radiated is

$$\begin{aligned}\frac{d\mathcal{P}}{d\Omega} = r^2 < \mathbf{S} \cdot \mathbf{n} > &= \frac{c}{8\pi}r^2[\mathbf{E}(\mathbf{x}) \times \mathbf{B}^*(\mathbf{x})] \cdot \mathbf{n} = \frac{c}{8\pi}r^2[\mathbf{B}^*(\mathbf{x}) \times \mathbf{n}] \cdot \mathbf{E}(\mathbf{x}) \\ &= \frac{c}{8\pi}\frac{k^6}{36}|\mathbf{n} \times [\mathbf{n} \times \mathbf{Q}(\mathbf{n})]|^2 = \frac{ck^6}{288\pi}|\mathbf{n} \times \mathbf{Q}(\mathbf{n})|^2. \end{aligned} \tag{77}$$

The right-hand side does not have any single form as a function of θ and ϕ that we can extract because $\mathbf{Q}(\mathbf{n})$ depends in an unknown way (in general) on these angles. We can proceed to a general result only up to a point:

$$\begin{aligned}|\mathbf{n} \times \mathbf{Q}|^2 = &\ (\mathbf{n} \times \mathbf{Q}) \cdot (\mathbf{n} \times \mathbf{Q}^*) = [(\mathbf{n} \times \mathbf{Q}) \times \mathbf{n}] \cdot \mathbf{Q}^* \\ = &\ -\mathbf{Q}^* \cdot [\mathbf{n}(\mathbf{Q} \cdot \mathbf{n}) - \mathbf{Q}] = |\mathbf{Q}|^2 - |\mathbf{n} \cdot \mathbf{Q}|^2; \end{aligned} \tag{78}$$

this is a brilliant derivation of the statement that $\sin^2\theta = 1 - \cos^2\theta$. Further,

$$\mathbf{Q} \cdot \mathbf{Q}^* - (\mathbf{n} \cdot \mathbf{Q})(\mathbf{n} \cdot \mathbf{Q}^*) = \sum_{ijk} Q_{ij}n_jQ^*_{ik}n_k - \sum_{ijkl} n_iQ_{ij}n_jn_kQ^*_{kl}n_l. \tag{79}$$

The n_i's are direction cosines and so obey the identities

$$\int d\Omega\, n_in_j = \frac{4\pi}{3}\delta_{ij} \tag{80}$$

and

$$\int d\Omega\, n_in_jn_kn_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{81}$$

These enable us at least to get a simple result for the total radiated power:

$$\begin{aligned}\mathcal{P} &= \int d\Omega\, \frac{d\mathcal{P}}{d\Omega} \\ &= \frac{ck^6}{288\pi}\left\{\frac{4\pi}{3}\sum_{ij}|Q_{ij}|^2 - \frac{4\pi}{15}\sum_{ik}[Q_{ii}Q^*_{kk} + |Q_{ik}|^2 + Q_{ij}Q^*_{ji}]\right\} \\ &= \frac{ck^6}{360}\sum_{ij}|Q_{ij}|^2 \end{aligned} \tag{82}$$

where we have used the facts that $Q_{ij} = Q_{ji}$ and $\sum_i Q_{ii} = 0$.

By choosing appropriate axes (the principal axes of the quadrupole moment matrix or tensor) one can always put the matrix of quadrupole moments into diagonal form. Further, only two of the diagonal elements can be chosen independently because the trace of the matrix must vanish. Hence any quadrupole moment matrix is a linear combination of two basic ones.

3.4.1 Example: Oscillating Charged Spheroid

A commonly occurring example is an oscillating spheroidal charge distribution leading to

$$Q_{33} = Q_0 \quad \text{and} \quad Q_{22} = Q_{11} = -Q_0/2. \tag{83}$$

Then

$$Q_i = \sum_j Q_{ij} n_j = Q_{ii} n_i \tag{84}$$

or

$$\mathbf{Q} = Q_0 \left[\cos\theta \boldsymbol{\epsilon_3} - \frac{1}{2}\sin\theta(\cos\phi\, \boldsymbol{\epsilon_1} + \sin\phi\, \boldsymbol{\epsilon_2})\right]. \tag{85}$$

From this we have

$$\begin{aligned} |\mathbf{Q}|^2 &= Q_0^2 \left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right), \\ \mathbf{n}\cdot\mathbf{Q} &= Q_0 \left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right), \\ |\mathbf{n}\cdot\mathbf{Q}|^2 &= Q_0^2 \left(\cos^4\theta - \cos^2\theta\sin^2\theta + \frac{1}{4}\sin^4\theta\right), \end{aligned} \tag{86}$$

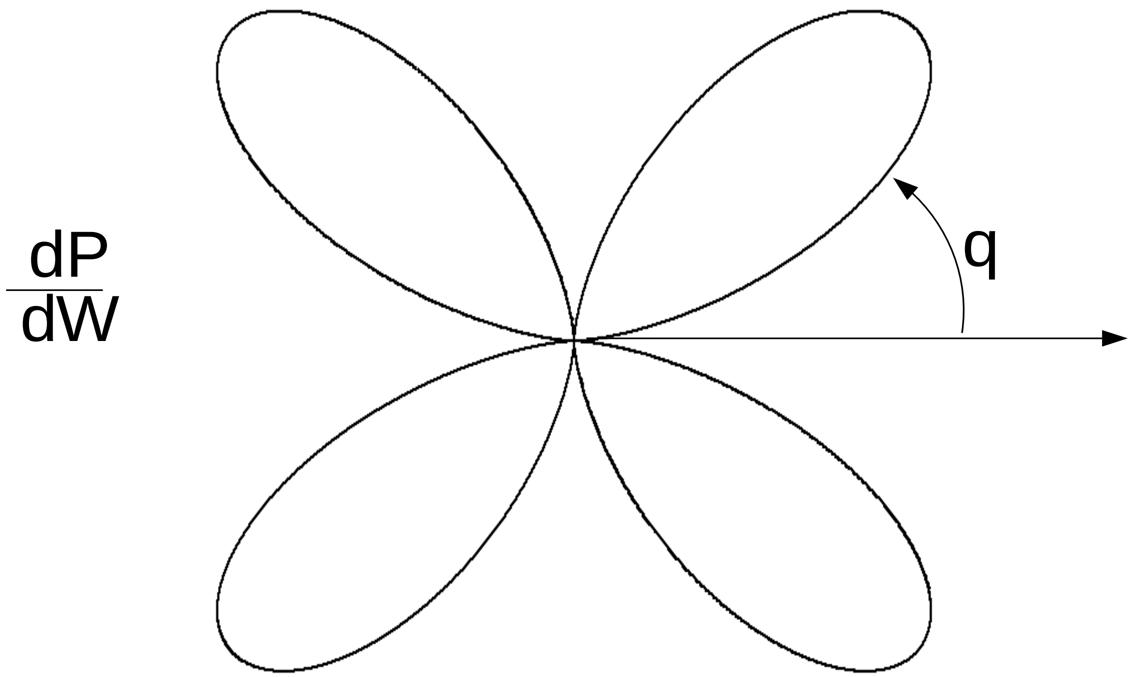
and so

$$\begin{aligned} |\mathbf{Q}|^2 - |\mathbf{n}\cdot\mathbf{Q}|^2 &= Q_0^2 \left[\cos^2\theta + \frac{1}{4}\sin^2\theta - \cos^4\theta + \sin^2\theta\cos^2\theta - \frac{1}{4}\sin^4\theta\right] \\ &= Q_0^2 \left[\cos^2\theta\sin^2\theta + \frac{1}{4}\cos^2\theta\sin^2\theta + \cos^2\theta\sin^2\theta\right] \\ &= \frac{9}{4}Q_0^2\cos^2\theta\sin^2\theta. \end{aligned} \tag{87}$$

Hence,

$$\frac{d\mathcal{P}}{d\Omega} = \frac{ck^6}{128\pi} Q_0^2 \sin^2\theta \cos^2\theta. \tag{88}$$

This is a typical, but not uniquely so, electric quadrupole radiation distribution.



Radiation Pattern of an oscillating charged spheriod

Higher-order multipole radiation (including magnetic quadrupole radiation) is found by expanding the factor $e^{-ik(\mathbf{n}\cdot\mathbf{x}')}$ in Eq. (27) to higher order in powers of d/λ. One can do this in a complete and systematic fashion after first developing some appropriate mathematical machinery by generalizing the spherical harmonics to vector fields,[8] the purpose being to construct an orthonormal set of basis functions for the electromagnetic fields.

[8]Thereby producing the so-called *vector spherical harmonics*. This is the subject matter of Chapter 16.

3.5 Large Radiating Systems

Before abandoning the topic of simple radiating systems, let us look at one example of an antenna which is not small compared the wavelength of the emitted radiation. Our example is an array of antennas, each of which is itself small compared to λ and each of which will be treated as a point dipole. We take the current density of this array to be

$$\mathbf{J}(\mathbf{x}) = I_0 a \sum_j \delta(\mathbf{x}-\mathbf{x}_j) e^{i\phi_j} \boldsymbol{\epsilon}_3. \tag{89}$$

One can easily see that this is an array of point antennas located at positions $\mathbf{x}_j$; they all have the same current but are not necessarily in phase, the phase of the j^{th} antenna being given by ϕ_j (the additional phase factor $e^{-i\omega t}$, common to all antennas, has been omitted, as usual).

No matter how large the array, we can specify that $\mathbf{x}$ is large enough that the observation point is in the far zone in which case the vector potential can be taken as

$$\mathbf{A}(\mathbf{x}) = \frac{e^{ikr}}{r}\mathbf{f}(\theta,\phi) \tag{90}$$

where

$$\begin{aligned}\mathbf{f}(\theta,\phi) &= \boldsymbol{\epsilon}_3 \frac{I_0 a}{c}\int d^3x' \sum_j \delta(\mathbf{x}'-\mathbf{x}_j) e^{-ik(\mathbf{n}\cdot\mathbf{x}')} e^{i\phi_j} \\ &= \boldsymbol{\epsilon}_3 \frac{I_0 a}{c}\sum_j e^{i[\phi_j - k(\mathbf{n}\cdot\mathbf{x}_j)]}.\end{aligned} \tag{91}$$

Referring back to Eq. (33) we find that the distribution of radiated power is

$$\frac{d\mathcal{P}}{d\Omega} = \frac{k^2 I_0^2 a^2}{8\pi c}\sin^2\theta |w|^2 \tag{92}$$

where

$$w \equiv \sum_j e^{i(\phi_j - k\mathbf{n}\cdot\mathbf{x}_j)}. \tag{93}$$

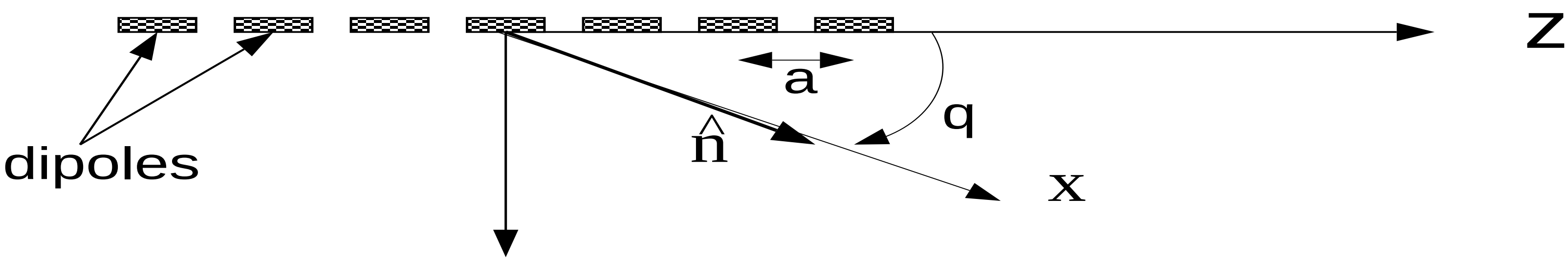
The factor of $\sin^2\theta$ arises because each element of this array is treated in the dipole approximation. The factor $|w|^2$ contains all of the information about the relative phases of the amplitudes from the different elements, i.e., about the interference of the waves from the different elements.

3.5.1 Example: Linear Array of Dipoles

As a special and explicit example, suppose that

$$\phi_j = j\phi_0 \qquad \text{and} \qquad \mathbf{x}_j = aj\boldsymbol{\epsilon_3}, \tag{94}$$

meaning that the elements are equally spaced in a line along the z-axis and that they have relative phases that increase linearly along the array.



Then

$$w = \sum_j e^{i(\phi_0 - ka\cos\theta)j} \equiv \sum_j x^j \tag{95}$$

where

$$x = e^{i(\phi_0 - ka\cos\theta)}. \tag{96}$$

The sum is easy to evaluate if we know where j begins and ends:

$$\sum_{j=j_1}^{j_2} x^j = x^{j_1}\frac{1 - x^{j_2 - j_1 + 1}}{1 - x}, \tag{97}$$

so

$$|w|^2 = \left|\frac{1 - e^{i\alpha(\theta)(j_2 - j_1 + 1)}}{1 - e^{i\alpha(\theta)}}\right|^2 \tag{98}$$

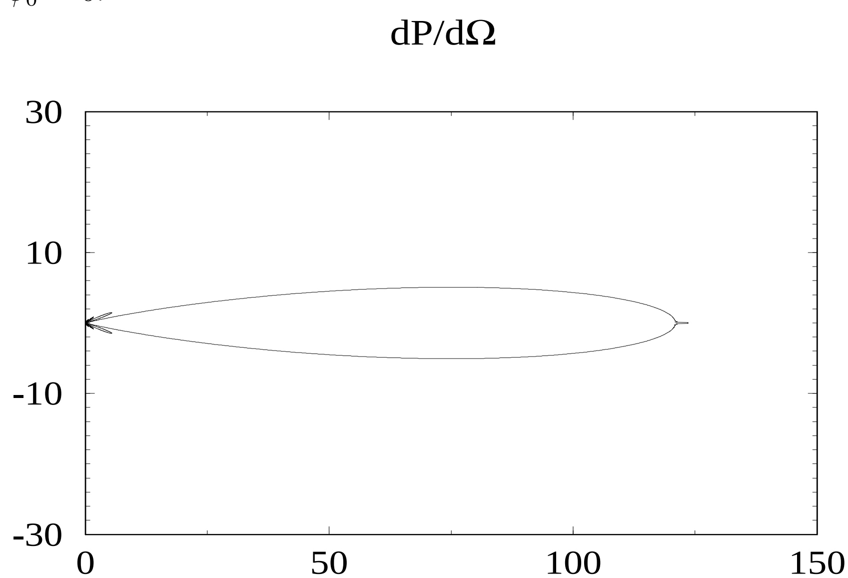
where $\alpha(\theta) \equiv \phi_0 - ka\cos\theta$. Let $j_1 = -n$ and $j_2 = n$, corresponding to an array of $2n+1$ elements centered at the origin. Then

$$|w|^2 = \left|\frac{1 - e^{i\alpha(2n+1)}}{1 - e^{i\alpha}}\right|^2 = \frac{1 - \cos[(2n+1)\alpha]}{1 - \cos\alpha}. \tag{99}$$

This is a function which is in general of order unity and which oscillates as a function of θ. It has, however, a large peak of size $(2n+1)^2$ when $\alpha(\theta)$ is an integral multiple of 2π. The peak occurs at that angle θ_0 where[9] $\alpha(\theta_0) = 0$ or $\cos\theta_0 = \phi_0/ka$. If we choose $\phi_0 = 0$, the peak is at $\theta_0 = \pi/2$; further, if $ka < 2\pi$ or $a < \lambda$, there is no other such peak. The width of the peak can be determined from the fact that the factor w goes to zero when $(2n+1)\alpha(\theta) = 2\pi$. Assuming that $n >> 1$, one finds that the corresponding angle θ differs from θ_0 by η where

$$\eta = \frac{2\pi}{(2n+1)ka[1 - (\phi_0/ka)^2]} \sim (2n+1)^{-1}. \tag{100}$$

Hence the antenna becomes increasingly directional with increasing n. The accompanying figure shows the power distribution in units of $k^2 I_0^2 a^2/8\pi c$ for 5 elements with $ka = \pi$ and $\phi_0 = 0$.



[9]More generally, $\alpha(\theta_0) = 2m\pi$ where m is an integer; because we control ϕ_0, we can make it small enough that the peak corresponds to the particular case $m = 0$.

4 Multipole expansion of sources in waveguides

We saw in Chapter 8 that a field in a waveguide could be expanded in the normal modes of the waveguide with an integral over the sources, consisting of a current distribution and apertures, determining the coefficients in the expansion. If the sources are small in size compared to distances over which fields in the normal modes vary, then one can do the integrals in an approximate fashion by making a multipole expansion of the sources.

4.1 Electric Dipole

Consider first the part of the amplitude which is produced by some explicit current distribution. It is

$$A_\lambda^{(\pm)} = -\frac{2\pi Z_\lambda}{c}\int d^3x'\,\mathbf{J}(\mathbf{x}')\cdot\mathbf{E}_\lambda^{(\mp)}. \tag{101}$$

The field in the mode is given by

$$\mathbf{E}_\lambda^{(\pm)}(\mathbf{x}') = [\mathbf{E}_\lambda(x',y') \pm \boldsymbol{\epsilon}_\mathbf{3} E_{z\lambda}(x',y')]e^{\pm ik_\lambda z'}. \tag{102}$$

The origin of the coordinate $\mathbf{x}'$ is at some appropriately chosen point which is probably not near the center of the source distribution. Let us use a different coordinate $\mathbf{x}$ having as origin a point near the center of the source. Also, let's let the electric field in the mode be called $\mathbf{E}(\mathbf{x})$ as a matter of convenience. Then we have to do an integral of the form

$$\int d^3x\,\mathbf{J}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) = \int d^3x\,\mathbf{J}(\mathbf{x})\cdot\left[\mathbf{E}(0) + \sum_{i.j}\left.\frac{\partial E_i}{\partial x_j}\right|_0 \boldsymbol{\epsilon_i} x_j + ...\right] \tag{103}$$

where we are assuming that the field varies little over the size of the source. In this integral the first term can be converted to an integral over the charge density,

$$\int d^3x\,\mathbf{J}(\mathbf{x}) = -i\omega\int d^3x\,\mathbf{x}\rho(\mathbf{x}) \tag{104}$$

provided the integration by parts can be done without picking up a contribution from the surface of the integration volume. That is not automatic here because the surface includes some points on the wall of the waveguide including those points where the current is fed into the guide, so one has to exercise some care in applying this formula. Assuming that it is okay, we see that the leading term in the expansion of the integrand produces to a term in the coefficient $A_\lambda^{(\pm)}$ which is proportional to $\mathbf{p}\cdot\mathbf{E}(0)$.

The next term in the expansion can, as we have seen, be divided into symmetric and antisymmetric parts:

$$\begin{aligned}\sum_{i,j}\frac{\partial E_i}{\partial x_j}\bigg|_0\int d^3x\,J_i(\mathbf{x})x_j &= \frac{1}{4}\sum_{i,j}\left(\frac{\partial E_i}{\partial x_j}-\frac{\partial E_j}{\partial x_i}\right)\bigg|_0\int d^3x\,[J_ix_j-J_jx_i] \\ &+\frac{1}{2}\frac{\partial E_i}{\partial x_j}\bigg|_0\int d^3x\,[J_ix_j+J_jx_i]. \qquad (105)\end{aligned}$$

The first term on the right-had side is set up in such a way as to display explicitly a component of $\nabla\times\mathbf{E}$, which is $i(\omega/c)\mathbf{B}$, and the same component of the magnetic dipole moment. Hence this term is proportional to $\mathbf{m}\cdot\mathbf{B}(0)$. The remaining term can be handled in the way that we treated the electric quadrupole part of the vector potential earlier; it becomes

$$-\frac{i\omega}{2}\sum_{i,j}\frac{\partial E_i}{\partial x_j}\bigg|_0\int d^3x\,x_ix_j\rho(\mathbf{x}) = -\frac{i\omega}{6}\sum_{i,j}Q_{ij}\,\frac{\partial E_i}{\partial x_j}\bigg|_0 \qquad (106)$$

provided one can throw away the contributions that come from the surface when the integration by parts is done. The final step is achieved by making use of the fact that $\nabla\cdot\mathbf{E}=0$.

In the next order, not shown in Eq. (103), the antisymmetric part provides the magnetic quadrupole contribution. Without delving into the algebra of the derivation, we state that the result is of the same form as Eq. (106) but with the *magnetic quadrupole moment tensor* Q_{ij}^M in place of the electric quadrupole moment tensor, $\mathbf{B}$ in place of $\mathbf{E}$, and an overall relative $(-)$. The components of the magnetic quadrupole moment tensor are defined in the same way as those of the electric quadrupole moment tensor except that in place of $\rho(\mathbf{x})$ there is

$$\rho^M(\mathbf{x}) = -\frac{1}{2c}\nabla\cdot[\mathbf{x}\times\mathbf{J}(\mathbf{x})]. \tag{107}$$

The final result for the amplitude $A_\lambda^{(\pm)}$ with all indices in place is

$$A_\lambda^{(\pm)} = i\frac{2\pi\omega}{c}\left\{\mathbf{p}\cdot\mathbf{E}_\lambda^{(\mp)}(0) - \mathbf{m}\cdot\mathbf{B}_\lambda^{(\mp)} + \frac{1}{6}\sum_{i,j}\left[Q_{ij}\left.\frac{\partial E_{i\lambda}^{(\mp)}}{\partial x_j}\right|_0 - Q_{ij}^M\left.\frac{\partial B_{i\lambda}^{(\mp)}}{\partial x_j}\right|_0\right] + ...\right\}. \tag{108}$$

We can see immediately some interesting features of this result. For example, if one wants to produce TM modes, which have z components of $\mathbf{E}$ but not of $\mathbf{B}$, then this is most efficiently done by designing the source to have a large p_z. At the same time, hardly any TE mode will be generated if there is only a z component of $\mathbf{p}$ because the TE mode has no E_z to couple to $\mathbf{p}$. Hence this expression gives one a good idea how to design a source to produce, or not produce, modes of a given kind.

Now let's look at the same expansion if the source of radiation is an aperture rather than an explicit current distribution. We derived in chapter 8 that in this case

$$\begin{aligned} A_\lambda^{(\pm)} &= -\frac{Z_\lambda}{2}\int_{S_a} d^2x'\,\mathbf{n}\cdot\left[\mathbf{E}(\mathbf{x}')\times\mathbf{H}_\lambda^{(\mp)}(\mathbf{x}')\right] \\ &= -\frac{Z_\lambda}{2}\int_{S_a} d^2x'\,[\mathbf{n}\times\mathbf{E}(\mathbf{x}')]\cdot\mathbf{H}_\lambda^{(\mp)}(\mathbf{x}') \end{aligned} \tag{109}$$

where the integral is over the aperture S_a, $\mathbf{E}(\mathbf{x}')$ is the electric field actually present at point $\mathbf{x}'$, and $\mathbf{n}$ is the inward directed normal at the aperture. Notice that only the tangential component of the electric field contributes to this integral. Assuming that the aperture is small compared to distances over which the fields in the normal modes vary, we can expand the latter and find, after changing the origin to a point in the aperture,

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_{S_a} d^2x\,(\mathbf{n}\times\mathbf{E}_{tan})\cdot\left[\mathbf{H}_\lambda^{(\mp)}(0) + \sum_{i,j}\boldsymbol{\epsilon_i}\left.\frac{\partial H_{i\lambda}^{(\pm)}}{\partial x_j}\right|_0 x_j + ...\right]$$

$$= -\frac{Z_\lambda}{2}\left\{\frac{\mathbf{B}_\lambda^{(\mp)}(0)}{\mu}\int_{S_a} d^2x\,(\mathbf{n}\times\mathbf{E}_{tan}) + \sum_{i.j}\int_{S_a} d^2x\,(\mathbf{n}\times\mathbf{E}_{tan})_i\left.\frac{\partial H_{i\lambda}^{(\mp)}}{\partial x_j}\right|_0 x_j + ...\right\}. \tag{110}$$

The first term here has already an appropriate form. As for the second one, note that we can break up the integrand into even and odd pieces,

$$(\mathbf{n}\times\mathbf{E}_{tan})_i \frac{\partial H_{i\lambda}}{\partial x_j} x_j \quad = \quad \frac{1}{2}[(\mathbf{n}\times\mathbf{E}_{tan})_i x_j - (\mathbf{n}\times\mathbf{E}_{tan})_j x_i] \left.\frac{\partial H_{i\lambda}^{(\mp)}}{\partial x_j}\right|_0$$
$$+\frac{1}{2}[(\mathbf{n}\times\mathbf{E}_{tan})_i x_j + (\mathbf{n}\times\mathbf{E}_{tan})_j x_i] \left.\frac{\partial H_{i\lambda}^{(\mp)}}{\partial x_j}\right|_0 . \quad (111)$$

Take just the antisymmetric part of this expression and complete the integral over $\mathbf{x}$. The contribution to the amplitude, aside from a factor of $-Z_\lambda/2$, is

$$\frac{1}{4}\sum_{i,j}\int_{S_a} d^2x[(\mathbf{n}\times\mathbf{E}_{tan})_i x_j - (\mathbf{n}\times\mathbf{E}_{tan})_j x_i] \left.\left(\frac{\partial H_{i\lambda}^{(\mp)}}{\partial x_j} - \frac{\partial H_{j\lambda}^{(\mp)}}{\partial x_i}\right)\right|_0$$
$$= \frac{1}{4}\sum_{i,j,k}\int_{S_a} d^2x\,[(\mathbf{n}\times\mathbf{E}_{tan})_i x_j - (\mathbf{n}\times\mathbf{E}_{tan})_j x_i]\epsilon_{kij}\left(-\frac{i\omega\epsilon}{c}E_{k\lambda}^{(\mp)}(0)\right)$$
$$= -\frac{i\omega\epsilon}{4c}\int_{S_a} d^2x\,\mathbf{E}_\lambda^{(\mp)}(0)\cdot[2\mathbf{x}\times(\mathbf{n}\times\mathbf{E}_{tan})]$$
$$= -\frac{i\omega\epsilon}{2c}\int_{S_a} d^2x\mathbf{E}_\lambda^{(\mp)}(0)\cdot[\mathbf{x}\times(\mathbf{n}\times\mathbf{E}_{tan})] = -\frac{i\omega\epsilon}{2c}\int_{S_a} d^2x\,\mathbf{E}_\lambda^{(\mp)}(0)\cdot\mathbf{n}(\mathbf{x}\cdot\mathbf{E}_{tan}). \quad (112)$$

In this string of algebra we have made use of the fact that $\nabla\times\mathbf{H} = -i(\omega/c)\epsilon\mathbf{E}$ and that $\mathbf{E}_{tan}$ is orthogonal to $\mathbf{E}_\lambda^{(\mp)}$ in the aperture because of the boundary conditions on the latter field.

Putting this piece into the expression for $A_\lambda^{(\pm)}$ along with the leading one, we find

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\left\{\frac{1}{\mu}\mathbf{B}_\lambda^{(\mp)}(0)\cdot\int_{S_a} d^2x\,\mathbf{n}\times\mathbf{E}_{tan}(\mathbf{x}) - \frac{i\omega\epsilon}{2c}\mathbf{E}_\lambda^{(\mp)}(0)\cdot\mathbf{n}\int_{S_a} d^2x\mathbf{x}\cdot\mathbf{E}_{tan}(\mathbf{x})\right\} \quad (113)$$

Defining

$$\mathbf{p}_{eff} \equiv \frac{\epsilon}{4\pi}\mathbf{n}\int_{S_a} d^2x\,\mathbf{x}\cdot\mathbf{E}_{tan}(\mathbf{x}) \quad (114)$$

and

$$\mathbf{m}_{eff} \equiv \frac{c}{2\pi i\mu\omega}\int_{S_a} d^2x\,\mathbf{n}\times\mathbf{E}_{tan}(\mathbf{x} \quad (115)$$

we can write

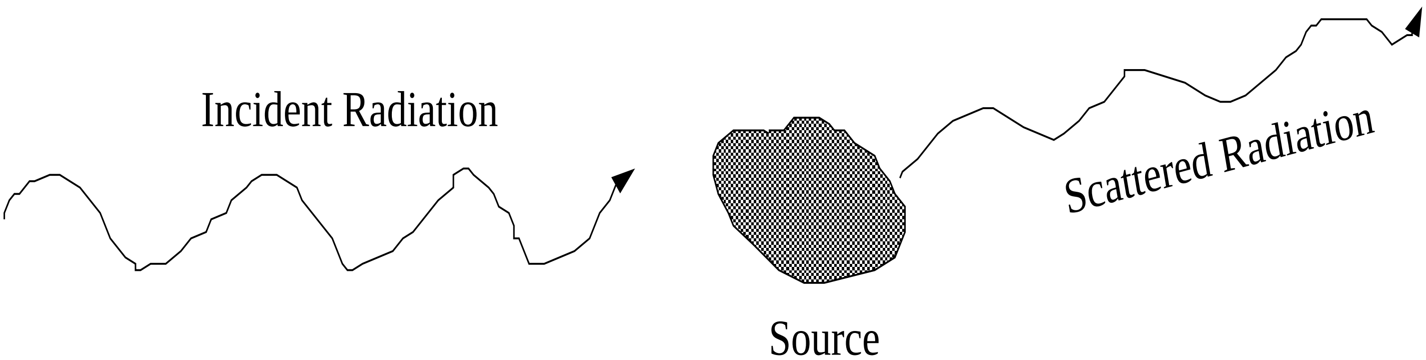
$$A_\lambda^{(\pm)} = \frac{i\pi\omega Z_\lambda}{c}[\mathbf{p}_{eff}\cdot\mathbf{E}_\lambda^{(\mp)}(0) - \mathbf{m}_{eff}\cdot\mathbf{B}_\lambda^{(\mp)}(0) + ...] \quad (116)$$

Thus we find that in the small wavelength limit, apertures are equivalent to dipole sources. The effective dipole moments are found by solving for, or using some simple approximation for, the fields in the aperture. See Jackson for a description of the particular case of circular apertures.

5 Scattering of Radiation

So far, we have discussed the radiation produced by a harmonically moving source without discussing the origin of the source's motion. In this section we will address the fields which set the source in motion.

Consider the case where the motion is excited by some incident radiation. The incident radiation is absorbed by the source, which then begins to oscillate coherently, and thus generates new radiation. This process is generally called scattering.



5.1 Scattering of Polarized Light from an Electron

For simplicity, let's consider a plane electromagnetic wave incident upon a single electron of charge $-e$. We will assume that the incident field has the form

$$\mathbf{E}_{in}(\mathbf{x},t) = \boldsymbol{\epsilon_1} E_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{117}$$

Since the equations which govern the motion of the electron are linear, we expect the electron to respond by oscillating along $\boldsymbol{\epsilon_1}$ at the same frequency (assuming that the magnetic force on the electron is small as long as the electron's motion is nonrelativistic). This will produce a time varying electric dipole moment.

We write the electron position as

$$\mathbf{x}(t) = \mathbf{x}_0 + \delta\mathbf{x}(t) \tag{118}$$

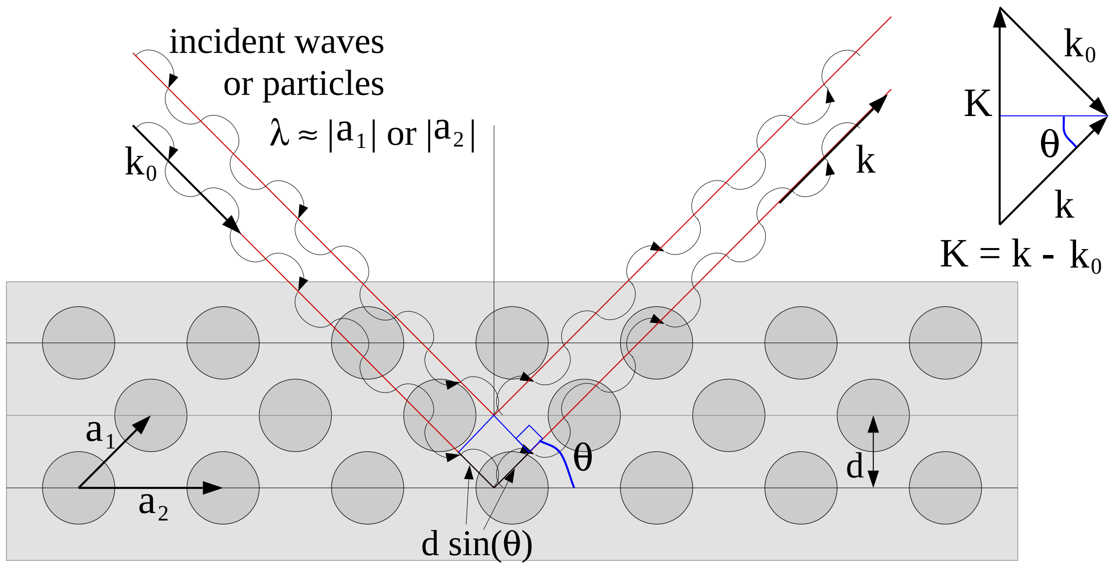
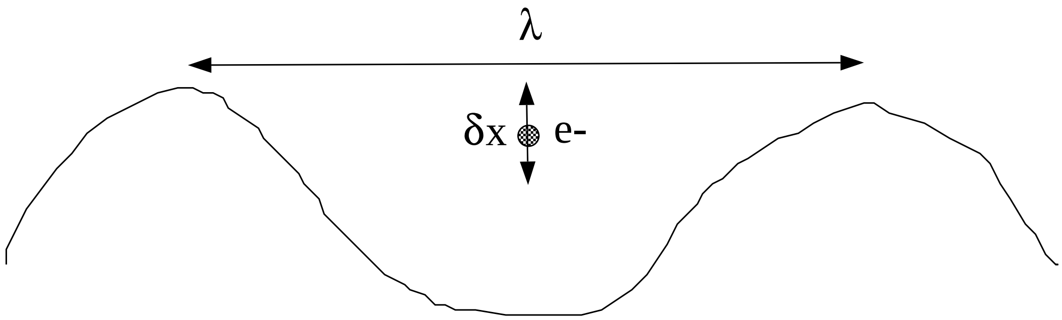


Figure 1: *Scattering of waves or particles with wavelength of roughly the same size as the lattice repeat distance allows us to learn about the lattice structure. We will assume that each electron acts as a dipole scatterer.*

If E_0 is small enough, then $\delta\mathbf{x}$ will be small compared to the wavelength λ of the incident radiation.



This approximation is also consistent with the non-relativistic assumption. When this is case, we can write

$$e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \approx e^{i(\mathbf{k}\cdot\mathbf{x}_0-\omega t)} \tag{119}$$

so that the field of the incident radiation does not change over the distance traveled by the oscillating electron. If we assume a harmonic form for the electronic motion

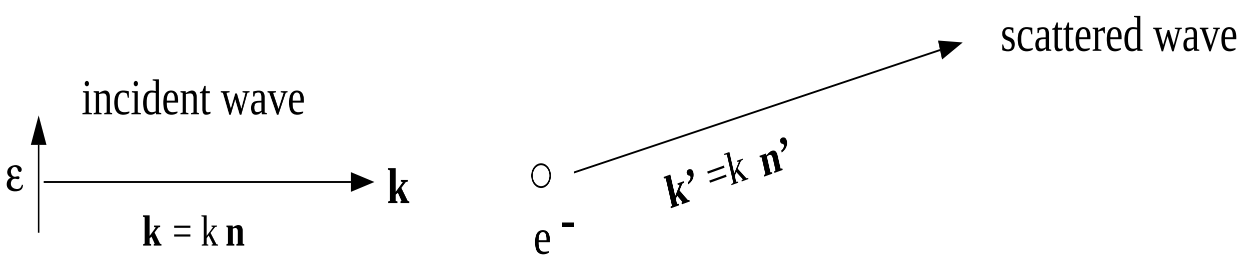
$\delta\mathbf{x}(t) = \delta\mathbf{x}e^{-i\omega t}$, insert this into Newton's law[10] and solve, we get

$$\delta\mathbf{x} = \boldsymbol{\epsilon_1}\left(\frac{eE_0e^{i\mathbf{k}\cdot\mathbf{x}_0}}{m\omega^2}\right) \tag{120}$$

The dipole moment of the oscillating electron will then be $\mathbf{p}(t) = \mathbf{p}e^{-i\omega t}$, with

$$\mathbf{p} = -e\delta\mathbf{x} = \boldsymbol{\epsilon_1}\left(\frac{-e^2E_0e^{i\mathbf{k}\cdot\mathbf{x}_0}}{m\omega^2}\right) \tag{121}$$

Now we will solve for the scattered radiation. Let the geometry of the scattering be as shown below.



From our electric dipole formulas, the angular distribution of the scattered power Eq. (45) is

$$\frac{d\mathcal{P}_{scat}}{d\Omega} = r^2 < \mathbf{S} > \cdot\mathbf{n}' = \frac{ck^4}{8\pi}[|\mathbf{p}|^2 - |\mathbf{n}'\cdot\mathbf{p}|^2] \tag{122}$$

or

$$\frac{d\mathcal{P}_{scat}}{d\Omega} = \frac{e^4E_0^2}{8\pi m^2c^3}[1 - (\mathbf{n}'\cdot\boldsymbol{\epsilon_1})^2] \tag{123}$$

Note that gives zero radiated power for $\mathbf{n}'$ along $\boldsymbol{\epsilon_1}$. I.e. there is no power radiated in the direction along which the dipole oscillates.

[10] $\mathbf{F} = m\mathbf{a} \rightarrow -e\mathbf{E} = -e\boldsymbol{\epsilon_1}E_0e^{i\mathbf{k}\cdot\mathbf{x}_0} = m\delta\ddot{\mathbf{x}}(t) = -m\omega^2\delta\mathbf{x}$

We may determine the differential cross section for scattering by dividing the radiated power per solid angle obtained above by the incident power per unit area (the incident Poynting vector).

$$\mathbf{S}_{in} = \frac{c}{8\pi}\left(\mathbf{E} \times \mathbf{B}^*\right) = \frac{c}{8\pi} E_0^2 \tag{124}$$

thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{|\mathbf{S}_{in}|}\frac{d\mathcal{P}_{scat}}{d\Omega} = \frac{e^4}{m^2 c^4}\left(1 - (\mathbf{n'} \cdot \boldsymbol{\epsilon_1})^2\right) \tag{125}$$

or

$$\frac{d\sigma}{d\Omega} = r_0^2\left(1 - (\mathbf{n'} \cdot \boldsymbol{\epsilon_1})^2\right) \tag{126}$$

where $r_0 = e^2/mc^2 = 2.8 \times 10^{13}$ cm is the classical scattering radius of an electron. This is the formula for Thomson scattering of incident light polarized along $\boldsymbol{\epsilon_1}$.

5.2 Scattering of Unpolarized Light from an Electron

Now suppose that the incident light is unpolarized. Then the cross section is the average of the cross sections for the two possible polarizations $\boldsymbol{\epsilon_1}$ and $\boldsymbol{\epsilon_2}$ of the incident wave (why?).

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2}\sum_{i=1}^{2} r_0^2\left(1 - (\mathbf{n'} \cdot \boldsymbol{\epsilon_i})^2\right) \tag{127}$$

$$= r_0^2 - \frac{1}{2}\sum_{i=1}^{2} r_0^2\left((\mathbf{n'} \cdot \boldsymbol{\epsilon_i})^2\right). \tag{128}$$

From the fact that ($\boldsymbol{\epsilon_1}$, $\boldsymbol{\epsilon_2}$, and $\mathbf{n}$) form an orthonormal triad of unit vectors, one may show that

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2} r_0^2\left(1 + (\mathbf{n'} \cdot \mathbf{n})^2\right) \tag{129}$$

This may be rewritten as

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2} r_0^2\left(1 + \cos^2(\theta)\right) \tag{130}$$

where θ is the scattering angle of the unpolarized radiation. From this we can see that the scattering in predominantly forward ($\theta = 0$), and backscattering ($\theta = \pi$).

The total cross sections may be obtained by integrating the differential cross sections. Thus

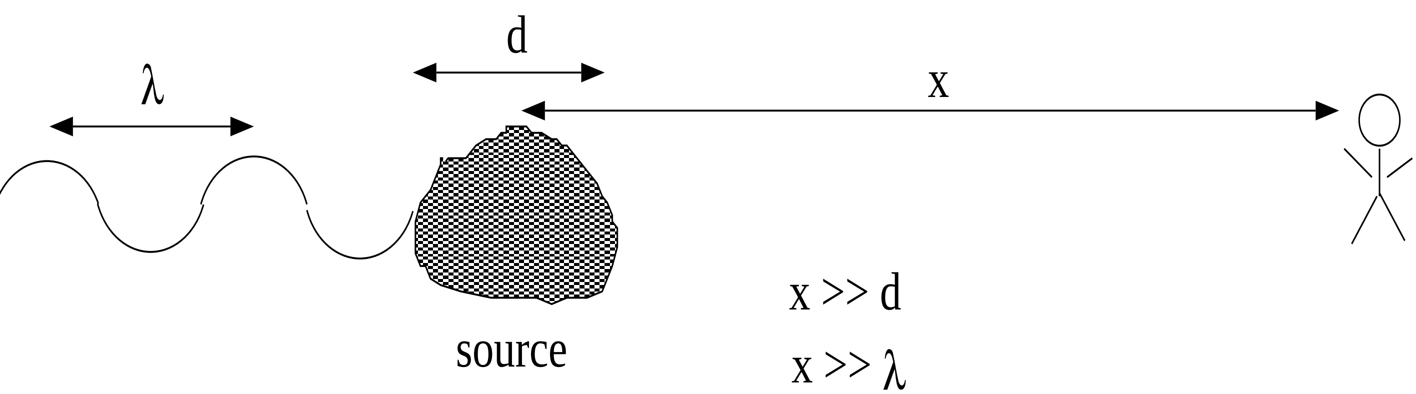
$$\sigma_{pol} = \frac{8\pi}{3} r_0^2 \tag{131}$$

and

$$\sigma_{unpol} = \frac{1}{2}\left(\sigma_{pol}(\boldsymbol{\epsilon_1}) + \sigma_{pol}(\boldsymbol{\epsilon_2})\right) = \sigma_{pol} \tag{132}$$

5.3 Elastic Scattering From a Molecule

Let us now consider the elastic scattering from a molecule[11] in which many electrons are generally present. In calculating the field produced by the molecular electrons when light is incident on them, we will again assume that we are making our observation very far away from the molecule, so that



We will *not* assume however, that $\lambda \gg d$, since we want to use the scattering to learn something about the structure of the molecule. This is only possible if our resolution (limited by λ) is smaller than d. Thus we expect our results to include interference effects from different charges in the molecule

[11]Here we shall use molecule to identify any small collection of charges

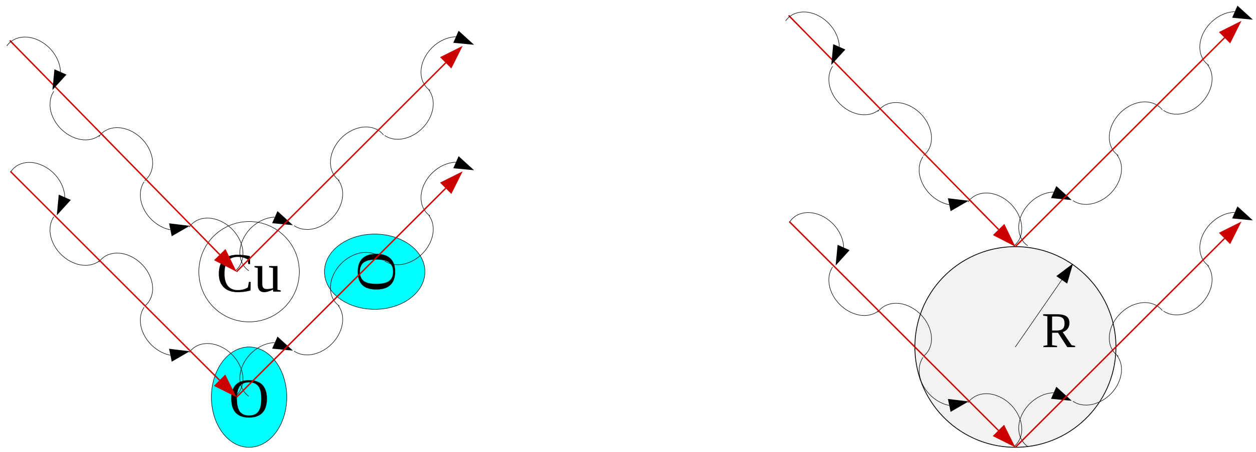


Figure 2: *Rays scattered from different elements of the basis, and from different places on the atom, interfere giving the scattered intensity additional structure described by the form factor S and the atomic form factor f, respectively.*

From page 9, the vector potential produced by the i'th electron in the molecule is

$$\mathbf{A}_i(\mathbf{x}) = \frac{e^{ikr}}{cr} \int d^3x' \, \mathbf{J}_i(\mathbf{x}') e^{-ik(\mathbf{n}\cdot\mathbf{x}')}. \tag{133}$$

This expression includes just the current of the i'th electron, and assumes $x \gg \lambda$ and $x \gg d$, as well as a harmonic time dependence. To find the current $\mathbf{J}_i(\mathbf{x}')$ for the i'th electron, we note that the classical current density of an electron is

$$\mathbf{J}_i(\mathbf{x}', t) = -e\mathbf{v}_i(t)\delta(\mathbf{x}' - \mathbf{x}_i) \tag{134}$$

where $(\mathbf{x}_i, \mathbf{v}_i)$ are the location and velocity of the i'th electron. If the electron is exposed to an incident electromagnetic plane wave, we find that

$$\mathbf{x}_i(t) = \mathbf{x}_{i,0} + \delta\mathbf{x}_i(t) \tag{135}$$

$$\delta\mathbf{x}_i(t) = \boldsymbol{\epsilon}_\mathbf{1} \left(\frac{eE_0}{m\omega^2} e^{i\mathbf{k}\cdot\mathbf{x}_i} e^{-i\omega t} \right) \tag{136}$$

From this we can find the electron velocity, and hence the current

$$\mathbf{J}_i(\mathbf{x}') = \boldsymbol{\epsilon}_\mathbf{1} \left(\frac{ie^2E_0}{m\omega} e^{i\mathbf{k}\cdot\mathbf{x}_i} \delta(\mathbf{x}' - \mathbf{x}_i) \right). \tag{137}$$

In this we have taken out the harmonic time term. Thus the vector potential of the scattered field generated by the i'th electron in the molecule is

$$\mathbf{A}_i(\mathbf{x}) = \boldsymbol{\epsilon}_\mathbf{1} \frac{ie^2 E_0}{m\omega} \frac{e^{ikr}}{cr} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}_i} \tag{138}$$

where $\mathbf{k}' = k\mathbf{n}'$ is the scattered wave vector and $\mathbf{k} = k\mathbf{n}$ is the incident wave vector. We define the vector $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ which is ($1/\hbar$ times) the momentum transfer from the photon to the electron, then the total vector potential of the scattered field is

$$\mathbf{A}(\mathbf{x}) = \sum_i \mathbf{A}_i(\mathbf{x}) = \boldsymbol{\epsilon}_\mathbf{1} \frac{ie^2 E_0}{m\omega} \frac{e^{ikr}}{cr} \sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i} \tag{139}$$

This differs from the vector potential due to a single scattering electron only through the factor $\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}$. Since this factor is independent of the observation point $\mathbf{x}$, the resulting $\mathbf{B}$ and $\mathbf{E}$ fields are likewise those of a single scattering multiplied by a factor of $\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}$. Thus we may immediately write down the differential cross sections. For polarized light

$$\left(\frac{d\sigma}{d\Omega}\right)_{pol} = \left[r_0^2\left(1 - (\mathbf{n}'\cdot\boldsymbol{\epsilon}_\mathbf{1})^2\right)\right]\left[\left|\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}\right|^2\right] \tag{140}$$

where the first term in brackets is the single-electron result, and the second term is the structure factor. For unpolarized light

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \left[\frac{1}{2}r_0^2\left(1 + (\mathbf{n}'\cdot\mathbf{n})^2\right)\right]\left[\left|\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}\right|^2\right] \tag{141}$$

We see that from a measurement of the differential cross section, we may learn about the structure of the object from which we are scattering light. Let's examine the structure factor in more detail.

$$\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i} = \int_V d^3x \sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}\delta(\mathbf{x}-\mathbf{x}_i) \tag{142}$$

$$= \int_V d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}} \sum_i \delta(\mathbf{x}-\mathbf{x}_i) \tag{143}$$

However,

$$\sum_i \delta(\mathbf{x} - \mathbf{x}_i) = n_{el}(\mathbf{x}) \tag{144}$$

which is the electron number density at position $\mathbf{x}$, thus

$$\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i} = \int_V d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}} n_{el}(\mathbf{x}) = n_{el}(-\mathbf{q}) \tag{145}$$

where $n_{el}(-\mathbf{q})$ is the Fourier transform of the electron density. Thus, our differential cross sections are, for polarized light:

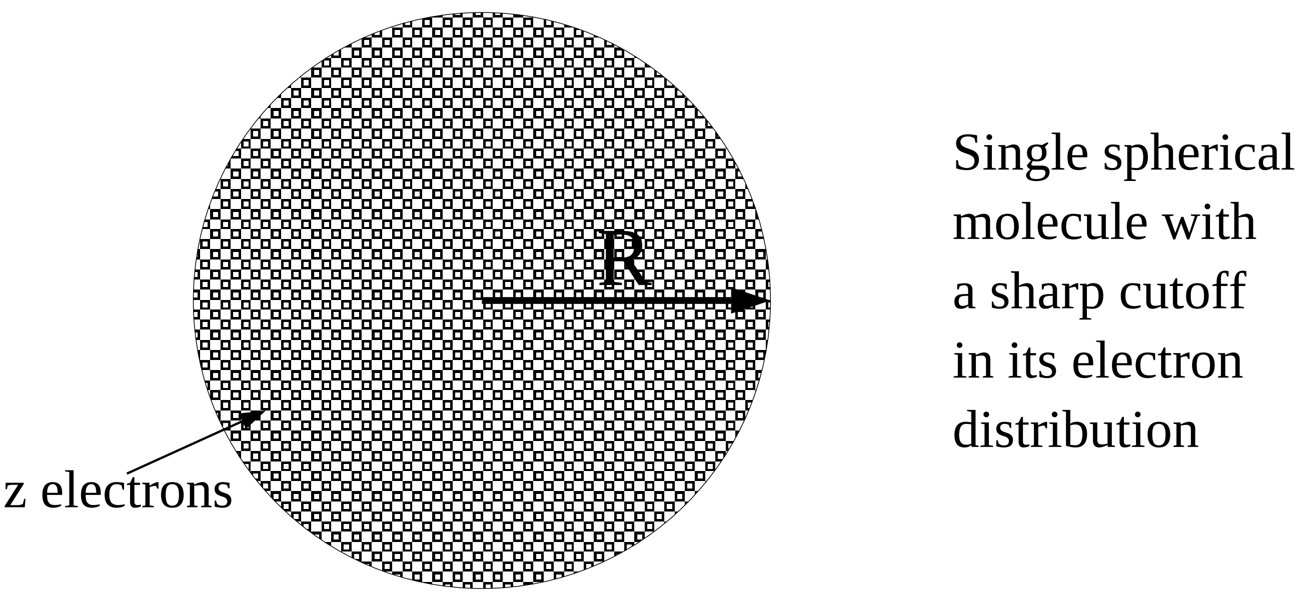
$$\left(\frac{d\sigma}{d\Omega}\right)_{pol} = \left[r_0^2\left(1 - (\mathbf{n}' \cdot \boldsymbol{\epsilon}_1)^2\right)\right]\left[|n_{el}(-\mathbf{q})|^2\right], \tag{146}$$

and for unpolarized light

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \left[\frac{1}{2}r_0^2\left(1 + (\mathbf{n}' \cdot \mathbf{n})^2\right)\right]\left[|n_{el}(-\mathbf{q})|^2\right] \tag{147}$$

Since $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ depends upon the scattering angle and the wave number of the incident photon, a scan over θ and/or $\mathbf{k}$ gives you information about the electron density on the target.

5.3.1 Example: Scattering Off a Hard Sphere



Consider scattering from a single molecule with a sharp cutoff in its electron distribution.

$$n_{el}(\mathbf{x}) = \frac{3z}{4\pi R^3}\theta(R - |\mathbf{x}|) \tag{148}$$

where R is the radius of the sphere and z is the number of electrons, then

$$n_{el}(-\mathbf{q}) = \int_V d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}} n_{el}(\mathbf{x}) = \frac{3z}{q^3R^3}\left[\sin(qR) - qR\cos(qR)\right] , \tag{149}$$

where $j_1(x) = (\sin x/x - \cos x)$ is a spherical Bessel function. Note that this has zeroes at

$$qR = \tan(qR) \tag{150}$$

so that we expect zeroes in the scattering cross section. These are clearly related to the size of the molecule. For forward scattering ($q = 2k\sin(0) = 0$), we have

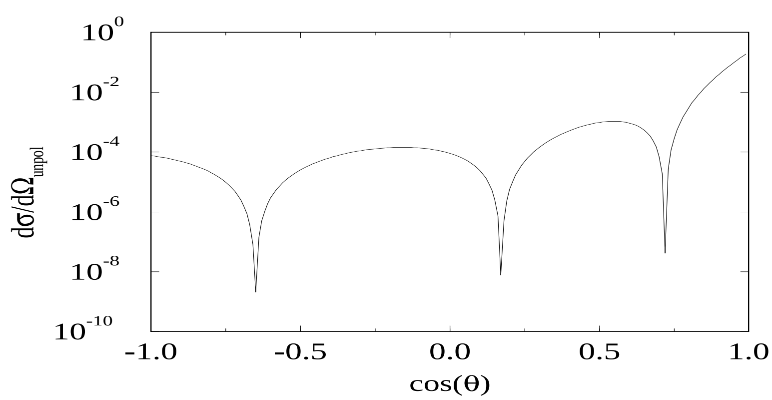
$$n_{el}(-\mathbf{q} = 0) = z \tag{151}$$

This is the maximum value, i.e. all the scattering electrons give constructive interference for forward scattering.

A sketch of the unpolarized cross section

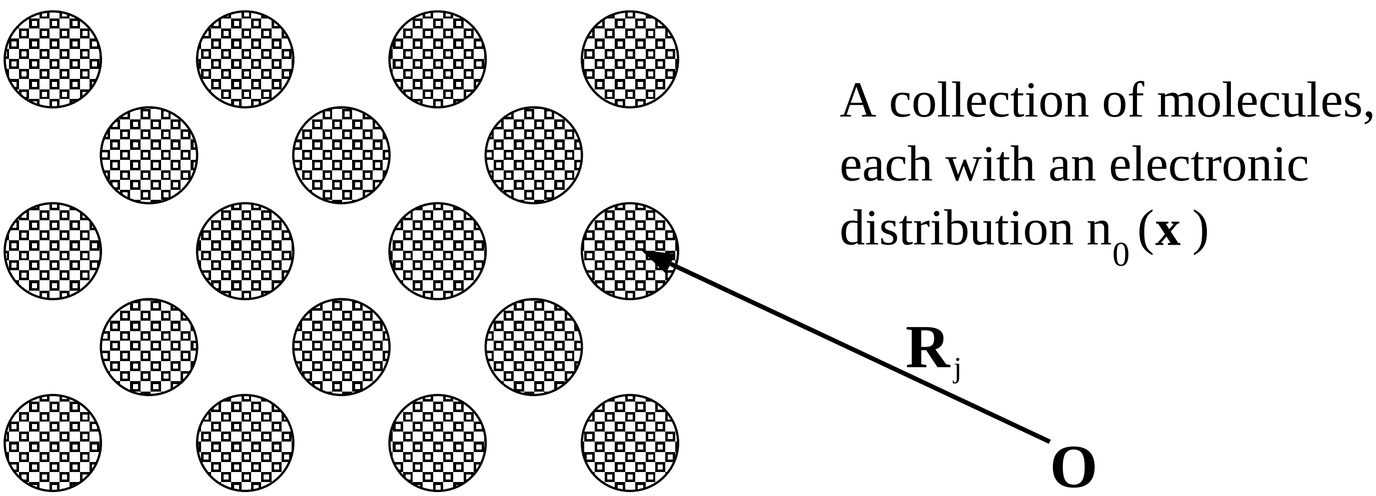
$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2}\mathbf{r}_0^2\left(1 + \cos^2(\theta)\right)|n_{el}(-\mathbf{q})|^2$$

looks like



If we measured this, we could learn the spatial distribution of electrons by inverting the transform. It is important to note that we could not learn the details of the electron wavefunctions since all the phase information is lost. I.e., the cross section just measures the modulus squared of n_{el}, and so phase information is lost.

5.3.2 Example: A Collection of Molecules



In a macroscopic sample, the total electron density is made up of two parts: the electron distribution within the molecules, and the distribution of the molecules within the sample. For simplicity, let's assume that the sample contains just one type of molecule, each of which has an electron density $n_0(\mathbf{x})$. The total electron number density is then

$$n_{el}(\mathbf{x}) = \sum_j n_0(\mathbf{x} - \mathbf{R}_j) \tag{152}$$

where $\mathbf{R}_j$ is the position of the j'th molecule. The Fourier transform of this is

$$n_{el}(-\mathbf{q}) = \int_V d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}} n_{el}(\mathbf{x}) = \sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j} n_0(-\mathbf{q}) \tag{153}$$

which is a weighted sum of the Fourier transforms for each molecule. We see that then the cross section will include the factor

$$|n_0(-\mathbf{q})|^2 \left|\sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j}\right|^2 \tag{154}$$

From above, we know that $n_0(-\mathbf{q} = 0) = z$, the number of electrons in a molecule, so we may write

$$n_0(-\mathbf{q}) = zF(\mathbf{q}) \tag{155}$$

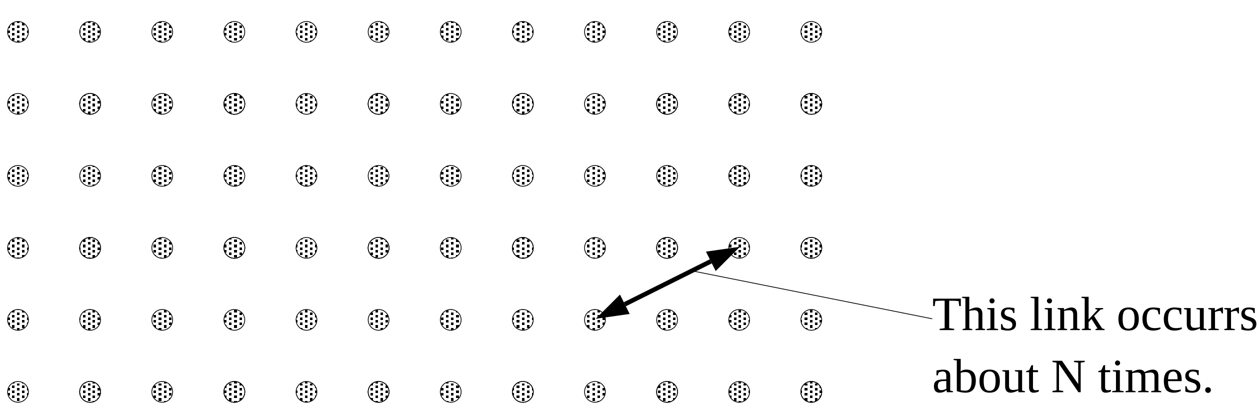
where $F(\mathbf{q})$ is called the form factor of the molecule which is normalized to unity for $\mathbf{q} = 0$. Thus the unpolarized differential cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2} z^2 r_0^2 (1 + \cos^2(\theta)) \, |F(\mathbf{q})|^2 \left|\sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j}\right|^2 \tag{156}$$

In general, we will not know where all the molecules are, nor do we necessarily care. Thus we will look at an average of the term which describes the distribution of the molecules.

$$\left\langle \left|\sum_j e^{i\mathbf{q}\cdot\mathbf{R}_j}\right|^2 \right\rangle = \sum_{jj'} \left\langle e^{i\mathbf{q}\cdot(\mathbf{R}_j - \mathbf{R}_{j'})} \right\rangle \approx N \sum_j \left\langle e^{i\mathbf{q}\cdot(\mathbf{R}_j - \mathbf{R}_0)} \right\rangle \equiv NS(\mathbf{q}) \tag{157}$$

where N is the number of molecules in the sample, $\mathbf{R}_0$ is the position of the origin, and $S(\mathbf{q})$ is the structure factor of the sample. The approximation is justified by the fact that (neglecting finite-size effects) each link between sites occurs about N times.



$$S(\mathbf{q}) = \sum_j \left\langle e^{i\mathbf{q}\cdot(\mathbf{R}_j - \mathbf{R}_0)} \right\rangle = 1 + \sum_{j\neq 0} \left\langle e^{i\mathbf{q}\cdot(\mathbf{R}_j - \mathbf{R}_0)} \right\rangle \tag{158}$$

Note that this goes to 1 as $|\mathbf{q}| \to \infty$. Then, using the same procedure detailed before, this becomes

$$NS(\mathbf{q}) = V \int_V d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}} \langle n(\mathbf{x})n(0)\rangle \tag{159}$$

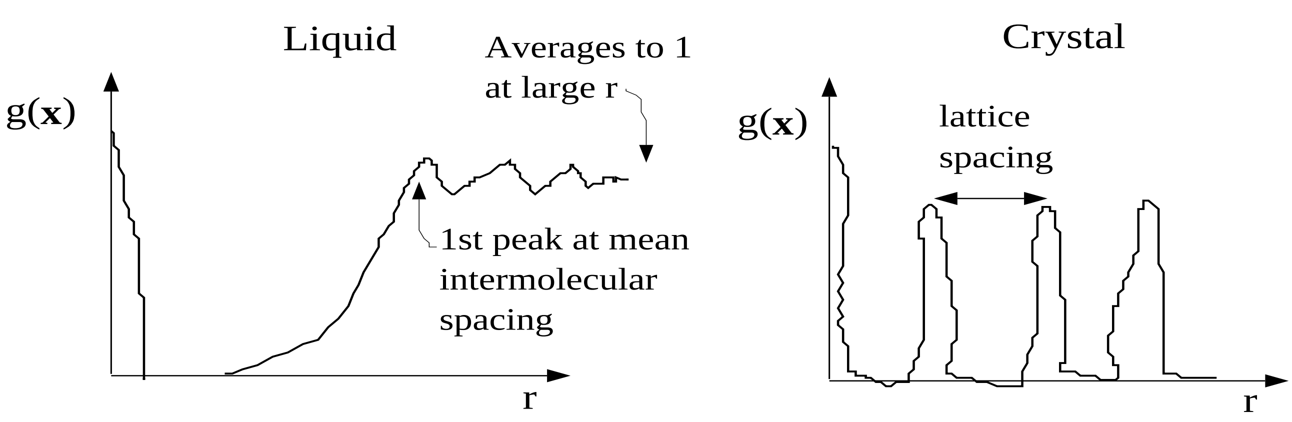
where V is the sample volume, and $n(\mathbf{x})$ is the number density of molecules in the sample.

Here, $\langle n(\mathbf{x})n(0)\rangle$ tell us about correlations between molecular positions.

$$\langle n(\mathbf{x})n(0)\rangle = (n(0))^2\, g(\mathbf{x}) \tag{160}$$

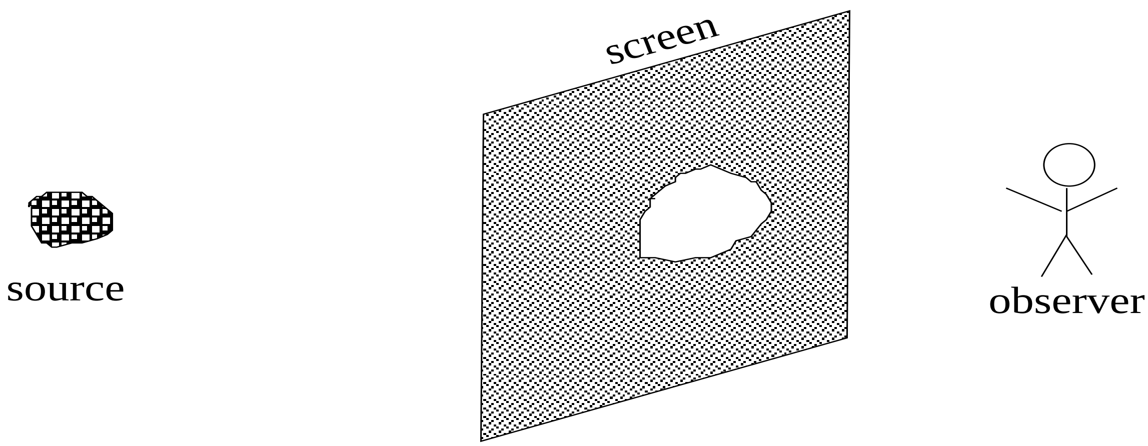
where $g(\mathbf{x})$ is just the probability of finding a molecule at $\mathbf{x}$ if there is one at the origin. This is the normalized two-body correlation function. We see that if the molecular form factor is known, then a measurement of the differential cross section tells us about the Fourier transform of $g(\mathbf{x})$. This is the key to using x-ray (or neutron, etc) scattering to determine the internal structure of materials.

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2} z^2 r_0^2 (1+\cos^2(\theta))\, |F(\mathbf{q})|^2\, V\, (n(0))^2\, g(\mathbf{q})$$



6 Diffraction

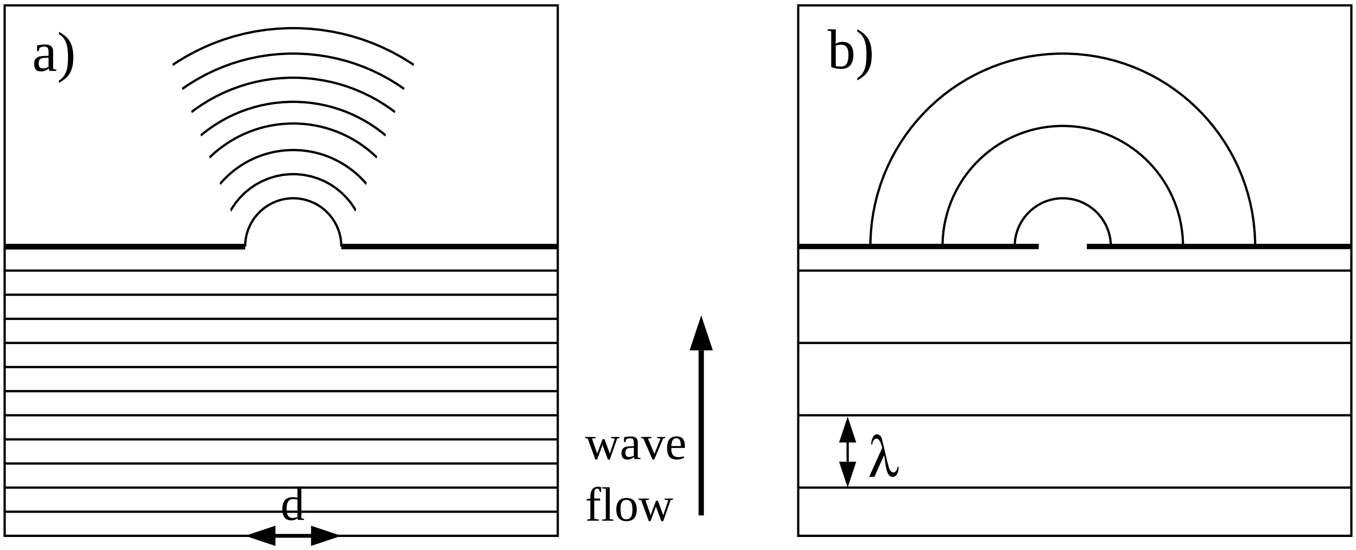
In this section we will discuss diffraction which is related to scattering. The prototypical setup is shown below in which radiation from a source is diffracted through an opaque screen with one or more holes in it.



Clearly, diffraction is the study of the propagation of light or radiation, or rather the deviation of light from rectilinear propagation. As undergraduates we all learned that the propagation of light was governed by **Huygen's Principle** that *every point on a primary wavefront serves a the source of spherical secondary wavelets such that the primary wavefront at some later time is the envelope of these wavelets. Moreover, the wavelets advance with a speed and frequency equal to that of the primary wave at each point in space.*[12]. This serves as the paradigm for our study of geometric optics. Diffraction is the first crisis in this paradigm.

To see this consider wave propagation in a ripple tank through an aperture.

[12]Hecht-Zajac, page 60



In the figure, wavefronts are propagating toward the aperture from the bottom of each box. In case a) the wavelength λ is much smaller than the size of the aperture d. In this case, the diffracted waves interfere destructively except immediately in front of the aperture. In case b), $\lambda \gg d$, and no such interference is is observed.

Huygen's principle cannot explain the difference between cases a) and b), since it is independent of any wavelength considerations, and thus would predict the same wavefront in each case. The difficulty was resolved by Fresnel. The corresponding **Huygens-Fresnel principle** states that *every unobstructed point of a wavefront,at a given instant of in time, serves as a source of spherical secondary wavelets of the same frequency as the source. The amplitude of the diffracted wave is the sum of the wavelets considering their amplitudes and relative phases*[13]. Applying these ideas clarifies case a). Here what is happening is that the wavelets from the right and left sides of the aperture interfere constructively in front of the aperture (since they travel the same distance and hence remain in phase), whereas these wavelets interfere destructively to the sides of the aperture (since they travel two paths with length differences of order $\lambda/2$). In case b), we approach the limit of a single point source

[13]Hecht-Zajac, page 330

of spherical waves. Beside being rather hypothetical, the Huygens-Fresnel principle involves some approximations which we will discuss later; however, Kirchoff showed that the Huygen's-Fresnel principle is a direct consequence of the wave equation.

6.1 Scalar Diffraction Theory: Kirchoff Approximation

This problem could be solved using the techniques we just developed to treat scattering. I.e. by considering the dynamics of the charged particles in the screen, and then calculating the scattered radiation generated by these particles. However, diffraction is conventionally treated as a boundary value problem in which the presence of the screen is taken into account with boundary conditions on the wave.

Several excellent references for this problem are worth noting:

1. B.B. Baker and E.T. Copson, **The Mathematical Theory of Huygens Principle**, (Clarendon Press, Oxford, 1950).

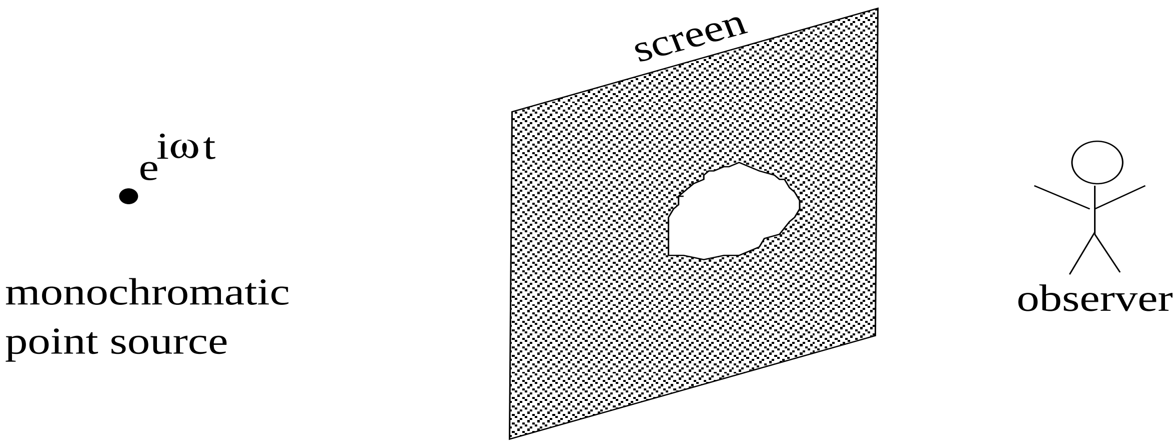
2. Landau and Lifshitz, **The Classical Theory of Fields**.

3. L. Eyges, **The Classical Electromagnetic Field**, (Addison-Wesley, Reading, 1972).

4. Hecht-Zajac, **Optics**, (Addison-Wesley, Reading, 1979).

The Baker reference, especially, has a good discussion of the limits of the Kirchoff approximation, of course, Landau and Lifshitz have an excellent discussion of the physics, but Hecht-Zajac's discussion is perhaps the most elementary, and will often be quoted here.

The typical question we ask is, given a strictly monochromatic point source S, what resulting radiation is observed at the point O on the opposite side of the screen. At first this approach may seem to limit us just to point sources. The case of a real extended source which emits non-monochromatic light does not, however, require special treatment. This is because of the linearity of our equations and the complete independence (incoherence) of the light emitted by different points of the source. The interference terms average to zero. Thus the total diffraction pattern is simply the sum of the intensity distributions obtained from the diffraction of the independent components of the light.



To treat the theory of diffraction, a number of approximations will be necessary.

- First, we assume that we can neglect the vector nature of the electromagnetic fields, and work instead with a scalar complex function ψ (a component of $\mathbf{E}$ or $\mathbf{B}$, or the single polarization observed in the ripple tank discussed above). In principle, ψ is any of the three components of either $\mathbf{E}$ or $\mathbf{B}$. In practice, however, the polarization of the radiation is usually ignored and the intensity of the radiation at a point is usually taken as $|\psi(\mathbf{x}, t)|^2$. This first assumption limits the number of geometries we can treat.

- Second, we will generally assume that $\lambda/d \ll 1$, where d is the linear dimension of the aperture or obstacle.

- The third assumption is that we will only look for the first correction to geometric optics due to diffraction. This is often called the Kirchoff approximation, which will be discussed a bit later. (This set of approximations are sometimes also called the Kirchoff approximation scheme.)

We then impose the boundary condition[14]

$$\psi(\mathbf{x}, t) = 0 \quad \text{everywhere on the screen} \tag{161}$$

We will assume that ψ obeys the wave equation, and the source is harmonic so that it emits radiation of frequency ω, hence

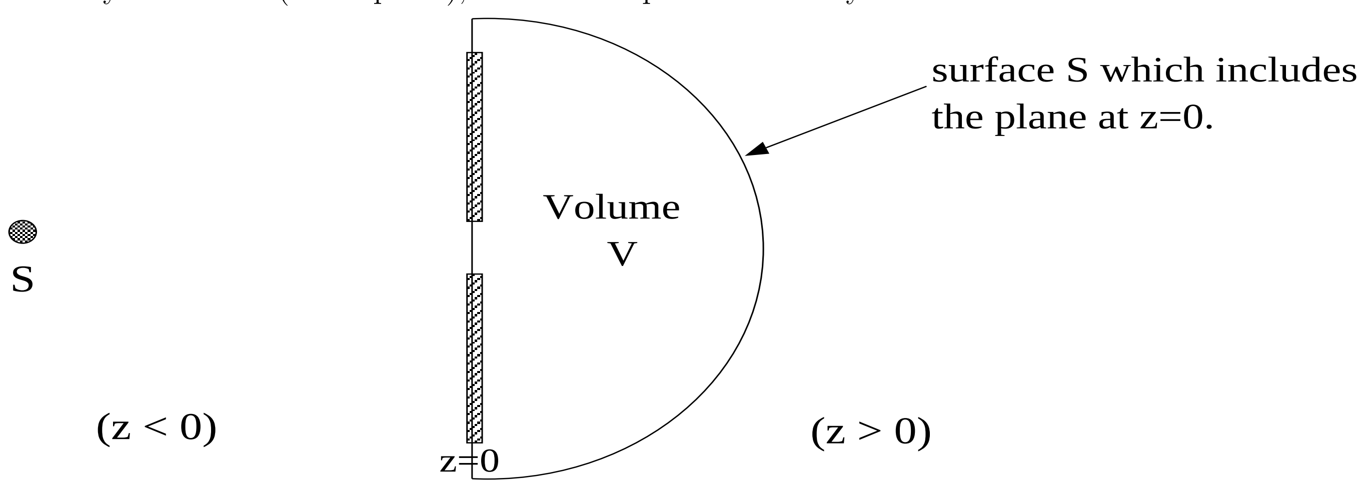
$$\psi(\mathbf{x}, t) = \psi(\mathbf{x}) e^{-i\omega t} \,. \tag{162}$$

Thus the spatial wave function obeys the Helmholtz equation

$$\Box^2 \psi = (\nabla^2 + k^2)\psi(\mathbf{x}) = 0 \quad \text{, with} \;\; k = \omega/c \tag{163}$$

if we assume that the waves propagate in a homogeneous medium and restrict our observations to points away from the source S. As always, the physical quantities will be the *real* amplitude $\Re\left(\psi(\mathbf{x})e^{-i\omega t}\right)$, and the modulus squared which denotes the time-averaged intensity.

Let's consider just the observation points in the volume V below which is bounded by the screen ($z = 0$ plane), and a hemisphere at infinity.



[14]Note that we only impose one boundary condition on the screen. This is to avoid the difficulties when both boundary conditions (Neumann and Dirichlet) are imposed, as discussed in Jackson on page 429

Then, everywhere within V, the Helmholtz equation $\Box^2\psi = 0$ is obeyed since the source lies outside of V. To find ψ within V, we use Green's theorem

$$\int_V d^3x' \left(\psi(\mathbf{x}')\nabla'^2\phi(\mathbf{x}') - \phi(\mathbf{x}')\nabla'^2\psi(\mathbf{x}')\right) = \int_S d^2x' \, \mathbf{n}'\cdot(\psi(\mathbf{x}')\nabla'\phi(\mathbf{x}') - \phi(\mathbf{x}')\nabla'\psi(\mathbf{x}')) \ , \tag{164}$$

or, adding and subtracting $k^2\psi(\mathbf{x}')\phi(\mathbf{x}')$ from the first integrand, we get

$$\int_V d^3x' \left(\psi(\mathbf{x}')(\nabla'^2 + k^2)\phi(\mathbf{x}') - \phi(\mathbf{x}')(\nabla'^2 + k^2)\psi(\mathbf{x}')\right) = \int_S d^2x' \, \mathbf{n}'\cdot(\psi(\mathbf{x}')\nabla'\phi(\mathbf{x}') - \phi(\mathbf{x}')\nabla'\psi(\mathbf{x}')) \tag{165}$$

This works for any two functions ψ and ϕ. We will take $\psi(\mathbf{x}')$ to be the wave amplitude, and take $\phi(\mathbf{x}') = G(\mathbf{x}, \mathbf{x}')$, where the Dirichlet Green's function satisfies

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \text{ in V} \tag{166}$$

i.e., it is the response to a unit point source so that in free space $G(\mathbf{x}, \mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$. However, need to solve for G with boundary conditions

$$G(\mathbf{x}, \mathbf{x}') = 0 \text{ for } \mathbf{x}' \text{ on } S \, . \tag{167}$$

since we will be using Dirichlet boundary conditions on ψ: We will specify the value of $\psi(\mathbf{x})$ (as opposed to its derivative) on the boundary S. Now using the facts that

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \text{ in V} \tag{168}$$

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = 0 \text{ in V} \tag{169}$$

and that

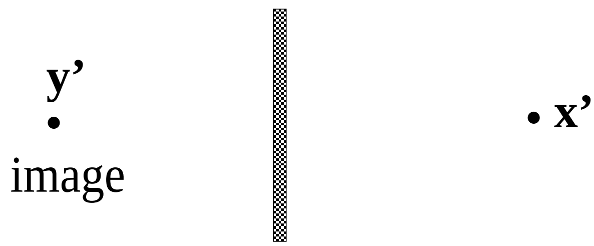
$$G(\mathbf{x}, \mathbf{x}') = 0 \text{ for } \mathbf{x}' \text{ on } S \tag{170}$$

this Green's theorem becomes (the Kirchoff Integral)

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int_S d^2x' \, \mathbf{n}' \cdot (\psi(\mathbf{x}')\nabla' G(\mathbf{x}, \mathbf{x}')) \tag{171}$$

To proceed further we must determine the form of $G(\mathbf{x}, \mathbf{x}')$ and we must now specify $\psi(\mathbf{x}')$ on the surface S. For an infinite planar screen the Green's function is given by the method of images (analogously to the electrostatic case)

$$G(\mathbf{x},\mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} - \frac{e^{ik|\mathbf{x}-\mathbf{y}'|}}{|\mathbf{x}-\mathbf{y}'|} \tag{172}$$



where $\mathbf{x}' = (x', y', z')$ is in V, but $\mathbf{y}' = (x', y', -z')$ (the image point) is not. This satisfies $(\nabla^2 + k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ in V, and vanishes both on the plane $z' = 0$ and on the hemisphere at infinity. Note that it vanishes as $1/r'^2$ as $r' \to \infty$. From this it is clear that the integral

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int_S d^2x'\, \mathbf{n}' \cdot (\psi(\mathbf{x}')\nabla' G(\mathbf{x},\mathbf{x}')) \tag{173}$$

has a vanishing contribution from the hemisphere at infinity, since $\psi(\mathbf{x}')$ vanishes at least as fast as $1/r'$ as $r' \to \infty$ (since it is a solution of the wave equation for a finite source). Furthermore, from the boundary condition

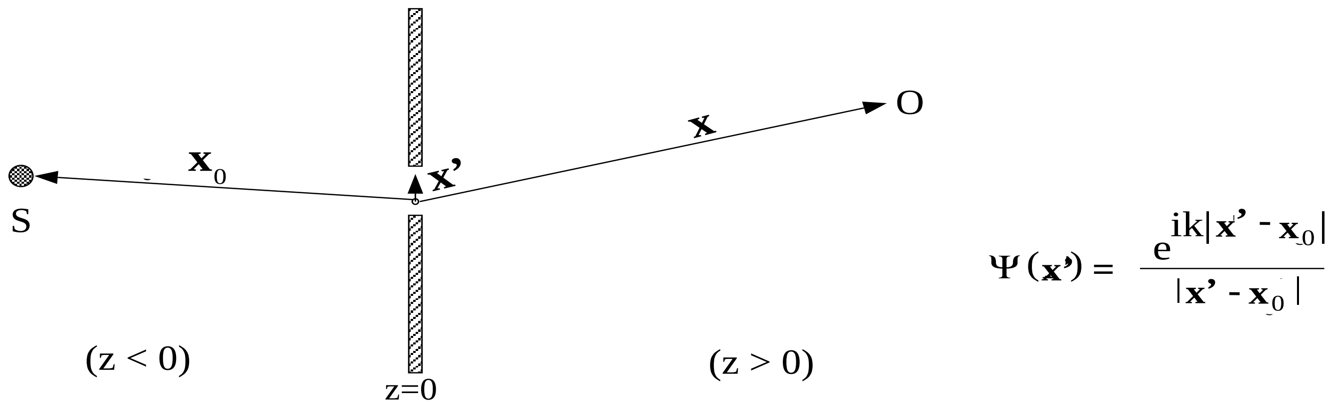
$$\psi(\mathbf{x}') = 0 \ \text{ on the screen} \tag{174}$$

we can see that the integral only gets a nonzero contribution from the opening

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{opening} d^2x'\, \mathbf{n}' \cdot (\psi(\mathbf{x}')\nabla' G(\mathbf{x},\mathbf{x}')) \ . \tag{175}$$

So far we have made an exact evaluation of our *scalar* theory. However, to proceed we must make an approximation. We will assume that the value of $\psi(\mathbf{x}')$ in the opening will be the *same* as if the screen was not there at all. This means that the wavelength we are considering must be small compared to to characteristic size of the problem (i.e. the size of the opening). As a result, our formalism will yield just the the lowest-order correction due to diffraction to the results of geometrical or ray optics. This approximation is called the **Kirchoff approximation**.

Now to implement this approximation we need two things. First, we need the field strength at the opening. For a source of unit strength at position $\mathbf{x}_0$, the field at position $\mathbf{x}'$ in the opening is taken to be



which is just a spherical wave. Second, we need the component of $\nabla' G(\mathbf{x}, \mathbf{x}')$ in the direction of $\mathbf{n}'$ in the opening. Since $\mathbf{n}'$ is the outward normal direction from the volume V, this means we need

$$-\frac{d}{dz'} G(\mathbf{x}, \mathbf{x}') = -\frac{d}{dz'} \left(\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} - \frac{e^{ik|\mathbf{x}-\mathbf{y}'|}}{|\mathbf{x}-\mathbf{y}'|} \right) . \tag{176}$$

Since we are using a short wavelength approximation, it follows that $k|\mathbf{x} - \mathbf{x}'| \gg 1$ and $k|\mathbf{x} - \mathbf{y}'| \gg 1$. Consequently, in taking the derivative of $G(\mathbf{x}, \mathbf{x}')$, the derivatives coming from the denominators are negligible compared to those from the exponentials. Thus

$$-\frac{d}{dz'} G(\mathbf{x}, \mathbf{x}') \approx \frac{ik(z - z')}{|\mathbf{x}-\mathbf{x}'|} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + \frac{ik(z + z')}{|\mathbf{x}-\mathbf{y}'|} \frac{e^{ik|\mathbf{x}-\mathbf{y}'|}}{|\mathbf{x}-\mathbf{y}'|}. \tag{177}$$

We only need to evaluate this in the opening ($z' = 0$)

$$-\frac{d}{dz'} G(\mathbf{x}, \mathbf{x}') \bigg|_{z'=0} \approx \frac{2ikz}{|\mathbf{x}-\mathbf{x}'|} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{178}$$

since $\mathbf{x}' = \mathbf{y}'$ here.

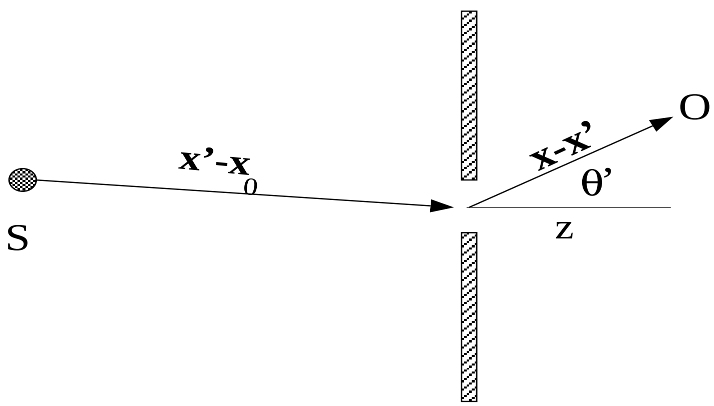
Thus, our Kirchoff approximation yields the following expression for the field observed at $\mathbf{x}$

$$\begin{aligned}\psi(\mathbf{x}) &= -\frac{1}{4\pi}\int_{opening} d^2x'\,\mathbf{n}'\cdot(\psi(\mathbf{x}')\nabla G(\mathbf{x},\mathbf{x}')) \qquad (179)\\ &= -\frac{1}{4\pi}\int_{opening} d^2x'\,\frac{2ikz}{|\mathbf{x}-\mathbf{x}'|}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\frac{e^{ik|\mathbf{x}_0-\mathbf{x}'|}}{|\mathbf{x}_0-\mathbf{x}'|}\,. \qquad (180)\end{aligned}$$

We may also write

$$\frac{z}{|\mathbf{x}-\mathbf{x}'|} = \cos(\theta')\,, \qquad (181)$$

where θ' is the angle from the normal to the opening at integration point $\mathbf{x}'$ to the observation point $\mathbf{x}$.



Note that θ' is different for each integration point $\mathbf{x}'$. Thus our expression becomes

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\int_{opening} d^2x'\,\frac{e^{ik|\mathbf{x}_0-\mathbf{x}'|}}{|\mathbf{x}_0-\mathbf{x}'|}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\cos(\theta') \qquad (182)$$

The first term is the field from the source located at $\mathbf{x}_0$ and received at point $\mathbf{x}'$ in the opening. The second term is the field from the "source" at $\mathbf{x}'$ in the opening received at observation point $\mathbf{x}$. The third term is the inclination factor. Thus, this integral is just an expression of the Huygens-Fresnel principle.

6.2 Babinet's Principle

Let's explore the consequences of this formula. First, consider two different screens which are complimentary.

One has an opening, labeled σ, while the other is just a disk. For the second screen the "opening" is the entire plane $z = 0$ except for the disk. This opening is the compliment of σ, so we will call it $\bar{\sigma}$.

We notice something interesting about the amplitudes received at the point O in the two complimentary cases, we have

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\sigma} d^2x'\, \mathbf{n}' \cdot (\psi(\mathbf{x}')\nabla G(\mathbf{x}, \mathbf{x}')) \tag{183}$$

$$\overline{\psi(\mathbf{x})} = -\frac{1}{4\pi}\int_{\bar{\sigma}} d^2x'\, \mathbf{n}' \cdot (\psi(\mathbf{x}')\nabla G(\mathbf{x}, \mathbf{x}')) \tag{184}$$

The *sum* of these two amplitudes is then

$$\psi(\mathbf{x}) + \overline{\psi(\mathbf{x})} = -\frac{1}{4\pi}\int_{\bar{\sigma}+\sigma} d^2x'\, \mathbf{n}' \cdot (\psi(\mathbf{x}')\nabla G(\mathbf{x}, \mathbf{x}')) \tag{185}$$

However, $\bar{\sigma} + \sigma$ represents the entire plane $z = 0$. Thus $\psi(\mathbf{x}) + \overline{\psi(\mathbf{x})}$ represents the amplitude detected at $\mathbf{x}$ if there was no screen at all. In other words $\psi(\mathbf{x}) + \overline{\psi(\mathbf{x})}$ includes no diffraction.

To interpret this, note that $\psi(\mathbf{x})$ may be written as

$$\psi(\mathbf{x}) = f\frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} + \psi_{diff}(\mathbf{x}) \tag{186}$$

where f is a ray optic amplitude. If $f = 0$, then there is no line-of-sight from the observer to the source, while if $f = 1$, then there is. The second term is the amplitude due to diffraction of the wave. Returning to our amplitudes $\psi(\mathbf{x})$ and $\overline{\psi(\mathbf{x})}$, we see that one will have $f = 0$ and the other $f = 1$, so that the sum will include a term $e^{ik|\mathbf{x}-\mathbf{x}_0|}/|\mathbf{x} - \mathbf{x}_0|$. *In fact, this is all it will include, since $\psi+\overline{\psi}$ includes no diffraction. Thus, we see that the diffraction amplitudes for complimentary screens cancel.* This is **Babinet's Principle**. It says that solving one diffraction problem is tantamount to solving its compliment. However, Babinet's principle does *not* say that the intensities will cancel in the two cases! We will see some consequences of Babinet's principle in what follows

6.3 Fresnel and Fraunhofer Limits

Lets return to our expression for $\psi(\mathbf{x})$ in the Kirchoff approximation. Since we assume that the wavelength is small, it follows that

$$ka \gg 1 \quad kr_0 \gg 1 \quad kr \gg 1 \tag{187}$$

where a is the typical aperture size, r_0 is the distance from the screen to the source, and r is the distance to the observer.

In general these approximations should be applied to the expression for $\psi(\mathbf{x})$ only after the integral over the opening has been carried out. This can be done for some simple geometries (as in the homework). What we will do here is to insert the above limits into the Kirchoff approximate expressions for $\psi(\mathbf{x})$. In what follows, we will restrict our attention to apertures, not discs. By doing this it follows that r' will have an upper limit a, so $kr' \gg 1$. In practice, the restriction to apertures is not a real limitation since Babinet's principle allows us to calculate the the diffraction amplitude for a disk from the diffraction amplitude for the complimentary screen.

Once again, our expression for $\psi(\mathbf{x})$ is

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\int_{opening} d^2x' \, \frac{e^{ik|\mathbf{x}_0-\mathbf{x}'|}}{|\mathbf{x}_0 - \mathbf{x}'|}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}\cos(\theta')\,. \tag{188}$$

Now if a/r_0 and/or a/r are not small, then we are limited in what further approximations are possible since r' is not necessarily small compared to either r or r_0.

This limit is called **Fresnel Diffraction**, in which the source and/or the observation points are close enough to the aperture that we must worry about the diffraction of spherical waves. This presents a difficult problem, which we will not treat (but please see the homework and Landau and Lifshitz, Classical Theory of Fields, Sec. 60).

In the opposite limit, which we will treat, is called **Fraunhofer Diffraction**. In this limit

$$ka > 1 \quad kr_0 \gg 1 \quad kr \gg 1 \quad a \ll r_0 \text{ and } a \ll r \tag{189}$$

Note that we specify $ka > 1$ rather than $ka \gg 1$ since we want to solve for the first finite corrections to the latter inequality, so that

$$\frac{ka^2}{r_0} = ka\left(\frac{a}{r_0}\right) \ll 1 \text{ and } \frac{ka^2}{r'} \ll 1 \tag{190}$$

Thus we are looking at plane waves instead of spherical waves in this limit.

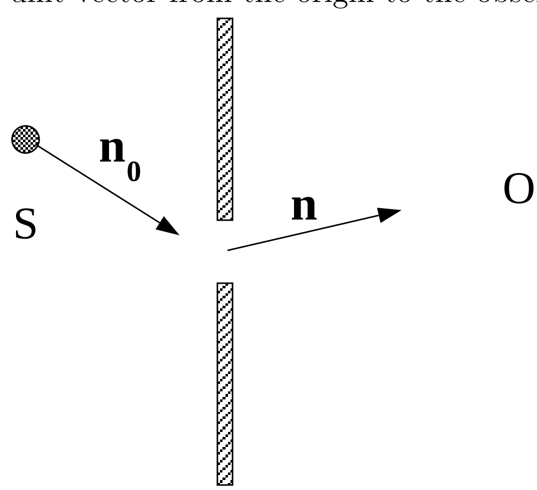
It is possible to simplify our expression for $\psi(\mathbf{x})$ even further in this limit. Then

$$|\mathbf{x}' - \mathbf{x}_0| \approx r_0 - \frac{\mathbf{x}_0 \cdot \mathbf{x}'}{r_0} = r_0 + \mathbf{n}_0 \cdot \mathbf{x}' \tag{191}$$

where $\mathbf{n}_0$ is a unit vector from the source to the origin (taken to be the center of the aperture), so that $\mathbf{x}_0 = -\mathbf{n}_0 r_0$. Similarly,

$$|\mathbf{x} - \mathbf{x}'| \approx r - \mathbf{n} \cdot \mathbf{x}' \tag{192}$$

where $\mathbf{n}$ is a unit vector from the origin to the observer so that $\mathbf{x} = \mathbf{n}r$.



Thus,

$$\frac{e^{ik|\mathbf{x}'-\mathbf{x}_0|}}{|\mathbf{x}'-\mathbf{x}_0|}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{e^{ik(r+r_0)}}{rr_0} e^{ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{x}'} \tag{193}$$

where we have dropped second-order terms coming from the denominators. Furthermore, since r is large compared to the aperture size, it is appropriate to take

$$\cos(\theta') = \frac{z}{|\mathbf{x}-\mathbf{x}'|} \approx 1 \tag{194}$$

inside the integral. Thus in the Fraunhofer limit

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\frac{e^{ik(r+r_0)}}{rr_0}\int_{opening} d^2x'\, e^{ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{x}'} \tag{195}$$

But the incident wave vector was $k\mathbf{n}$, so

$$k(\mathbf{n}_0 - \mathbf{n}) = \mathbf{k}_{in} - \mathbf{k}_{diff} \equiv \mathbf{q} \tag{196}$$

so we finally have

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\frac{e^{ik(r+r_0)}}{rr_0}\int_{opening} d^2x'\, e^{i\mathbf{q}\cdot\mathbf{x}'} \tag{197}$$

7 Example Problems

Let's consider some examples.

7.1 Example: Diffraction from a Rectangular Aperture

Here the opening is given by $|x'| < a$ and $|y'| < b$, so the integral above is

$$\int_{opening} d^2x'\, e^{i\mathbf{q}\cdot\mathbf{x}'} = \int_{-a}^{a} dx'\, e^{iq_x x'} \int_{-b}^{b} dy'\, e^{iq_y y'} = 2\frac{\sin(q_x a)}{q_x}\, 2\frac{\sin(q_y b)}{q_y} \tag{198}$$

The amplitude at $\mathbf{x}$ is

$$\psi(\mathbf{x}) = -\frac{2ikab}{\pi}\frac{e^{ik(r+r_0)}}{rr_0}\left(\frac{\sin(q_x a)}{aq_x}\right)\left(\frac{\sin(q_y b)}{bq_y}\right) \tag{199}$$

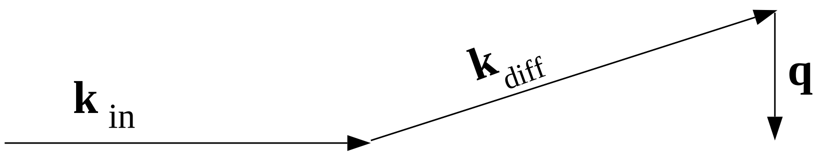
thus, the intensity of the diffracted wave is

$$\mathbf{I}(\mathbf{x}) = |\psi(\mathbf{x})|^2 = I_0 \left(\frac{\sin(q_x a)}{a q_x}\right)^2 \left(\frac{\sin(q_y b)}{b q_y}\right)^2 \tag{200}$$

with

$$I_0 = \frac{4k^2 a^2 b^2}{\pi^2 r_0^2 r^2} \tag{201}$$

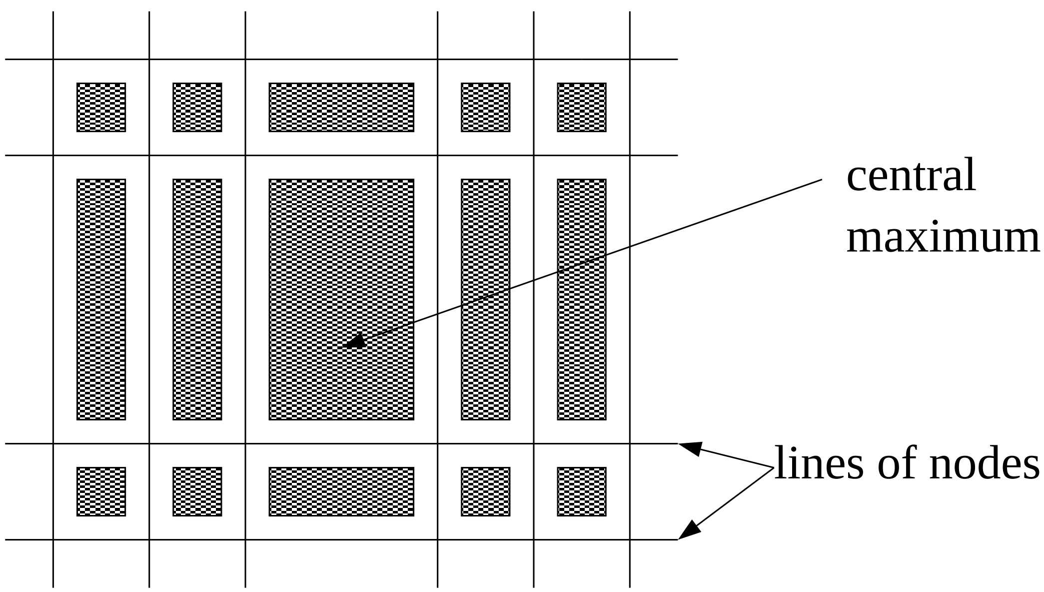
We see that all of the angular dependence of in this is embodied in the dependence on $\mathbf{q}$, the momentum transfer. Since we a looking at small-angle scattering, we are taking $\mathbf{q}$ to lie essentially in the x-y plane.



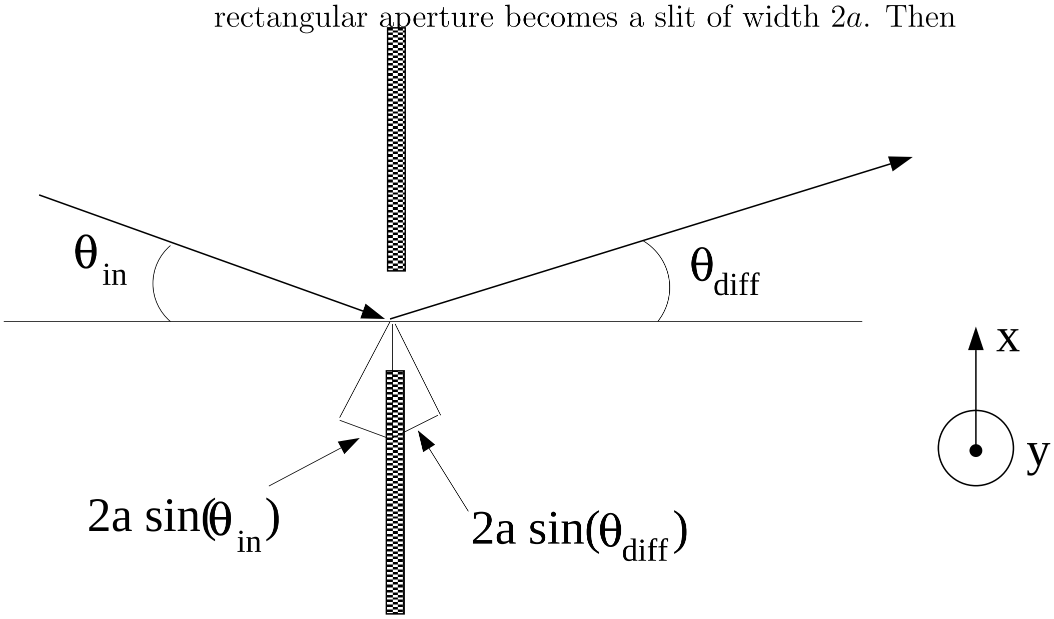
We see that the nodes of $\mathbf{I}(\mathbf{x})$ occur for

$$q_x = \frac{m\pi}{a}, \quad m \neq 0, \quad \text{and/or} \quad q_y = \frac{n\pi}{b}, \quad n \neq 0 \tag{202}$$

The global maximum ($I(\mathbf{x}) = I_0$) occurs for $q_x = q_y = 0$, i.e. for forward scattering. Thus the intensity of the diffracted light looks something like



We can recover the formula for single-slit diffraction by taking $b \gg a$, so that our rectangular aperture becomes a slit of width $2a$. Then



$$q_x = (k_x)_{in} - (k_x)_{diff} = k\sin(\theta_{in}) - k\sin(\theta_{diff}) \tag{203}$$

and the condition that there is a node is

$$q_x = k\left(\sin(\theta_{in}) - \sin(\theta_{diff})\right) = \frac{m\pi}{a} \quad m \neq 0\,. \tag{204}$$

Since $k = 2\pi/\lambda$, this may be written

$$2a\left(\sin(\theta_{in}) - \sin(\theta_{diff})\right) = m\lambda \quad m \neq 0 \tag{205}$$

which is the usual expression for a node in single-slit diffraction.

7.2 Example: Diffraction from a Circular Aperture

Here the opening is given by $r' < a$. Our Fraunhofer expression for the amplitude at $\mathbf{x}$ is then

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\frac{e^{ik(r+r_0)}}{rr_0}\int_0^a dr'\, r' \int_0^{2\pi} d\phi' e^{iqr'\cos(\phi')} \tag{206}$$

where we have taken $\mathbf{q}$ to lie entirely in the plane of the aperture, as is appropriate in small-angle diffraction. Now

$$\int_0^{2\pi} d\phi' e^{iqr'\cos(\phi')} = 2\pi J_0(qr') \tag{207}$$

where J_0 is the zeroth-order Bessel function. Our expression for the diffraction amplitude is then

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\frac{e^{ik(r+r_0)}}{rr_0}\int_0^a dr'\, r'\, 2\pi J_0(qr') = -ika^2\frac{e^{ik(r+r_0)}}{rr_0}\left(\frac{J_1(qa)}{qa}\right) \tag{208}$$

where we have used $J_1(0) = 0$. The diffracted intensity is then

$$I(\mathbf{x}) = I_0\left(\frac{J_1(qa)}{qa}\right)^2, \quad I_0 = \frac{k^2\, a^4}{r^2\, r_0^2} \tag{209}$$

this reaches a maximum at $q = 0$, it then has a minima of zero and maxima which decrease as q increases. For incident light with $\mathbf{k}$ normal to the opening,

$$q = k_{diff}\sin(\theta_{diff}) = k\sin(\theta_{diff}) \tag{210}$$

since the magnitude of the wavevector does not change. Thus,

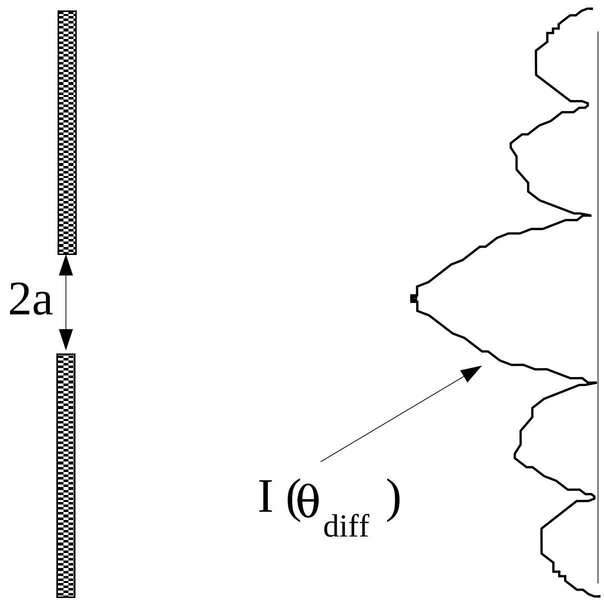
$$I(\theta_{diff}) = I_0\left(\frac{J_1\left(ka\sin(\theta_{diff})\right)}{ka\sin(\theta_{diff})}\right)^2 \tag{211}$$

The first minima of $J_1(x)$ is at $x = 3.83$, hence the first minima of diffraction will occur when

$$ka\sin(\theta_{diff}) = 3.83\,, \quad \text{or} \quad sin(\theta_{diff}) = 1.22\left(\frac{\lambda}{2a}\right)\,. \tag{212}$$

Since, by assumption, λ is small compared to a, $sin(\theta_{diff}) \approx \theta_{diff}$, and so the angle of the first node is roughly

$$\theta_{diff} = 1.22\left(\frac{\lambda}{2a}\right)\,. \tag{213}$$



We notice from these examples of Fraunhofer diffraction that all of the light falling on the aperture is deflected. Granted, the intensity for forward scattering ($\mathbf{q} = 0$) is nonzero, but the total amount of light with $\mathbf{q} = 0$ is

$$|\psi(\mathbf{q})|^2 \, q^2 \, dq \, d\Omega\Big|_{\mathbf{q}=0} \to 0 \tag{214}$$

Thus, it is legitimate to say that all of the light is deflected. Let us apply this observation to a disc which is totally absorbing (i.e. a perfect black-body). If plane waves are incident on this disc, then all the light that falls directly upon it is absorbed. If there was no diffraction, then there would be a geometrically perfect shadow cast by the object, and the total cross section of the disk would be its area A. But of course there is diffraction, so in addition to absorbing light, some will also be deflected by the disk. By Babinet's principle, as much light will be diffracted by the disc as by an aperture of the same shape as the disc but in a screen. But we have said that in Fraunhofer diffraction all of the light falling on an aperture is deflected. Thus for our disc, the cross section for deflection of light will be the disc area A as well. Thus the total cross section, including both absorption (inelastic cross section) and diffraction (elastic cross section) is

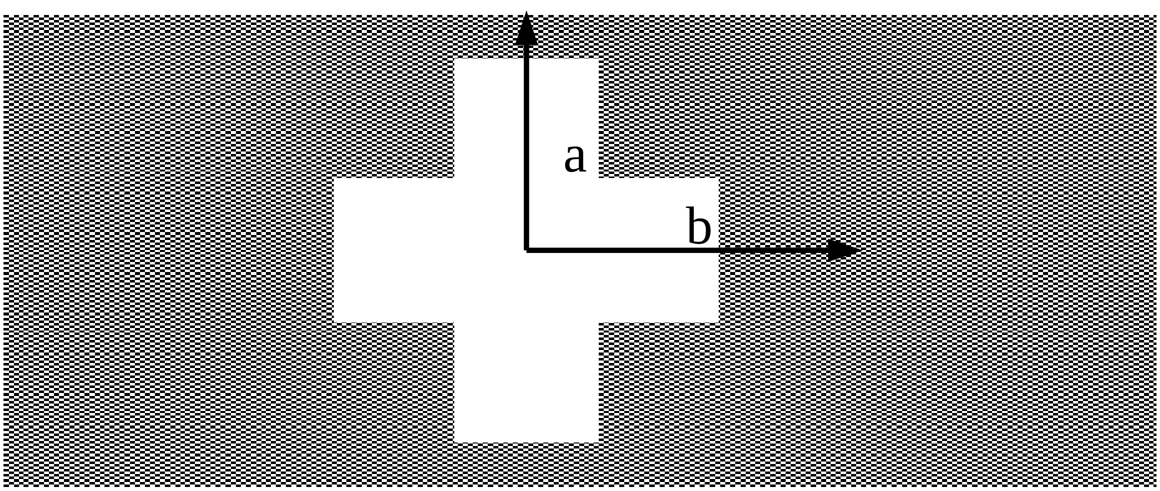
$$\sigma_{total} = \sigma_{inelastic} + \sigma_{elastic} = A + A = 2A \tag{215}$$

This is twice the area of the disc. Recall that in this argument we assume that $ka \gg 1$, where a is the characteristic size of the disc. In this limit in quantum mechanics one finds that the cross section for scattering from a sphere of radius a is $2\pi a^2$. The factor of two has the same origins here as it does in quantum mechanics.

7.3 Diffraction from a Cross

1. Given a normally incident wave of frequency ω calculate the diffraction pattern produced by an opaque screen with an aperture in the shape symmetric cross of inner dimension a and outer dimension b in the Fraunhofer limit of the scalar Kirchoff approximation. Very roughly tell what the diffraction pattern will look like.

2. The correct answer to the first part of this problem also yields the solution to the diffraction pattern involving a quite different set of apertures. What is this other set of apertures?



Solution. In the Fraunhoffer limit, we can use Eq. (197) of the notes.

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\frac{e^{ik(r+r_0)}}{rr_0}\int_{opening} d^2x'\, e^{i\mathbf{q}\cdot\mathbf{x}'}$$

where r_0 is the distance from the aperture to the source, r is the distance from the aperture to the observer, and $\mathbf{q}$ is the difference between the incident and scattered wavevector. We will take $\mathbf{q}$ to lie entirely in the plane of the aperture, as is appropriate in the small angle limit.

For the geometry in the figure, the integrals may be performed trivially

$$\psi(\mathbf{x}) = -\frac{ik}{2\pi}\frac{e^{ik(r+r_0)}}{rr_0}\left(\int_{-a}^{a} dx' \int_{-a}^{a} dy' + \int_{-a}^{a} dx' \left(\int_{-b}^{-a} dy' + \int_{a}^{b} dy'\right) + \int_{-a}^{a} dy' \left(\int_{-b}^{-a} dx' + \int_{a}^{b} dx'\right)\right) e^{iq_x x'}$$

so that

$$\psi(\mathbf{x}) = \frac{2ik}{\pi q_x q_y}\frac{e^{ik(r+r_0)}}{rr_0}\left[\sin(aq_x)\sin(bq_y) + \sin(aq_y)\sin(bq_x) - \sin(aq_x)\sin(aq_y)\right]$$

The corresponding intensity $|\psi(\mathbf{x})|^2$ will have a central maximum with $I \propto (2ab-a^2)^2$, and the whole pattern will be symmetric to rotations modulo $\pi/2$. By Babinet's principle we know that the diffraction from a cross-shaped screen has an amplitude, that when added to $\psi(\mathbf{x})$, is the same as if there was no diffraction at all.

7.4 Radiation from a Reciprocating Disk

A disc of radius a lies in the $z = 0$ plane and is centered at the origin. It is uniformly charged with surface charge density σ, and it rotates around the z-axis with an angular velocity $\Omega\cos(\omega t)$ where Ω and ω are constants. Assuming that the motion is nonrelativistic, find the fields in the radiation zone, the angular distribution of radiated power, and the total radiated power.

Solution. Here we will employ the methods developed to treat the radiation of harmonic current sources. The current of the rotation disc is (in cylindrical coordinates)

$$\mathbf{J}(\mathbf{x},t) = \delta(z)\sigma\Omega\cos(\omega t)\rho\hat{\phi}$$

or, in complex notation

$$\mathbf{J}(\mathbf{x},t) = \delta(z)\sigma\Omega e^{-i\omega t}\rho\hat{\phi} = \mathbf{J}(\mathbf{x})e^{-i\omega t}$$

Then,

$$\mathbf{f}(\theta,\phi) = \frac{1}{c}\int_V d^3x'\,\mathbf{J}(\mathbf{x}')e^{-ik\mathbf{n}\cdot\mathbf{x}'} = \frac{\sigma\Omega}{c}\int_0^a \rho' d\rho' \int_0^{2\pi} d\phi' \rho'\hat{\phi}' e^{-ik\mathbf{n}\cdot\mathbf{x}'}$$

We can use the azimuthal symmetry of the problem, and assume that $\mathbf{n}$ is in the xz plane. Then $\mathbf{n} = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}}$, $\mathbf{x}' = \rho'\left(\cos(\phi')\hat{\mathbf{x}} + \sin(\phi')\hat{\mathbf{y}}\right)$, and $\hat{\phi}' = \cos(\phi')\hat{\mathbf{y}} - \sin(\phi')\hat{\mathbf{x}}$. For non-relativistic motion $\Omega a/c \ll 1$, this allows us to keep only the first nonvanishing term in the exponential, then

$$\mathbf{f}(\theta,\phi) = -i\frac{\pi k a^4 \sigma\Omega}{4c}\sin\theta\hat{\phi}$$

Hence, in the radiation zone,

$$\mathbf{B} = ik\frac{e^{ikr}}{r}\left(\mathbf{n}\times\mathbf{f}\right) = -\frac{\pi k^2 a^4\sigma\Omega}{4c}\sin\theta\frac{e^{ikr}}{r}\hat{\theta}$$

and

$$\mathbf{E} = \mathbf{B}\times\mathbf{n} = \frac{\pi k^2 a^4\sigma\Omega}{4c}\sin\theta\frac{e^{ikr}}{r}\hat{\phi}$$

The power distribution is given by

$$\frac{dP}{d\Omega} = \frac{ck^2}{8\pi}\left[|\mathbf{f}|^2 - |\mathbf{n}\cdot\mathbf{f}|^2\right] = \frac{ck^2}{8\pi}\frac{\pi^2 k^2 a^8\sigma^2\Omega^2}{16c^2}\sin^2(\theta)$$

The total power is obtained by integrating over all solid angles

$$P = \frac{\pi^2 k^4\sigma^2\Omega^2 a^8}{48c}$$