Maxwell's Equations

James Clerk Maxwell

(1831 - 1879)

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$$\nabla \cdot \mathbf{B}(\mathbf{x},t) = 0, \qquad \nabla \times \mathbf{H}(\mathbf{x},t) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x},t) + \frac{1}{c}\frac{\partial \mathbf{D}(\mathbf{x},t)}{\partial t}$$
$$\nabla \times \mathbf{E}(\mathbf{x},t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t}, \qquad \nabla \cdot \mathbf{D}(\mathbf{x},t) = 4\pi\rho(\mathbf{x},t)$$

Contents

Our first task in this chapter is to put time into the equations of electromagnetism. There are traditionally two steps in this process. The first of them is to develop Faraday's Law of Induction which is the culmination of a series of experiments performed by Michael Faraday (1791 - 1867) around 1830. Faraday studied the current "induced" in one closed circuit when a second nearby current-carrying circuit either was moved or had its current varied as a function of time. He also did experiments in which the second circuit was replaced by a permanent magnet in motion. The general conclusion of these experiments is that if the "magnetic flux" through a closed loop or circuit changes with time, an induced voltage or electromotive force (abbreviated by $\mathcal{E}mf$) appears in the circuit.

1 Faraday's Law of Induction

Define the *magnetic flux* through, or "linking" a closed loop C as

$$F = \int_S d^2x \, \mathbf{B} \cdot \mathbf{n} \tag{1}$$

where S is an open surface that ends on the curve C and $\mathbf{n}$ is the usual unit right-hand normal (see below) to the surface. So long as $\nabla \cdot \mathbf{B} = 0$, this integral is the same for all such surfaces.

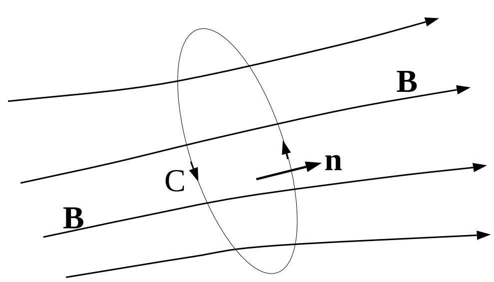


Figure 1: Orientation of n and C.

Define next the $\mathcal{E}mf$ in the loop as

$$\mathcal{E} = \oint_C d\mathbf{l} \cdot \mathbf{E}' \tag{2}$$

where $\mathbf{E}'$ is the electric field in that "frame of reference" in which the loop is at rest.[1] The path C is traversed in such a direction that the unit normal $\mathbf{n}$ is the right-hand normal relative to this direction. We may now write *Faraday's Law of Induction* as

$$\mathcal{E} = -k\frac{dF}{dt} \tag{3}$$

where t is the time and k is a positive constant.

Let us make two points in relation to this equation.

First, the minus sign has the consequence, when taken along with our definitions of the direction of $\mathbf{n}$, etc., that the induced electromotive force will try to drive a current through the loop in such a direction as to produce a flux through the loop that will be opposite in sign to the change in F that gave rise to the $\mathcal{E}mf$ in the first place. Thus if one tries to increase the flux F through the loop by manipulating some external currents or magnets, the resulting induced current will produce a $\mathbf{B}(\mathbf{x})$ which acts to counter the externally applied magnetic induction. This particular aspect of Faraday's Law is also known as *Lenz's Law.*

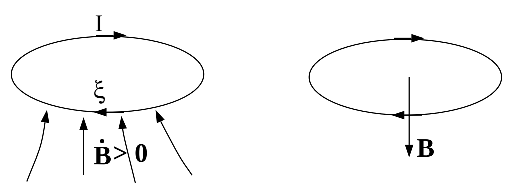


Figure 2: Induced current, and the resulting induced magnetic induction.

Second, in Gaussian units, the constant k has dimensions of T/L or inverse speed. It is in fact $1/c$ where c is the constant that appears in Ampère's Law and in the Lorentz force law. This is not a separate experimental fact but may be deduced from classical notions of relativity and the laws of electromagnetism as we currently understand them. We may demonstrate this claim; consider the time rate of change of flux through a loop C that is moving with some constant velocity $\mathbf{v}$ relative to the

[1]Notice that this definition cannot cover the case of a rotating loop since such a loop is not at rest in any one inertial (unaccelerated) frame.

(lab) frame in which we are measuring $\mathbf{x}$ and t. We have

$$\frac{dF}{dt} = \frac{d}{dt}\left(\int_S d^2x\, \mathbf{B}(\mathbf{x}, t) \cdot \mathbf{n}\right) = \int_S d^2x\, \frac{\partial \mathbf{B}(\mathbf{x}, t)}{\partial t} \cdot \mathbf{n} + \left(\frac{dF}{dt}\right)_2 \tag{4}$$

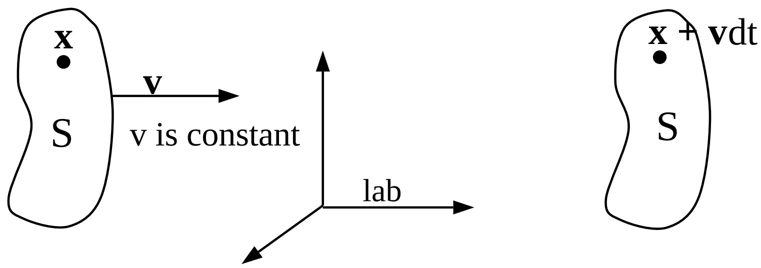


Figure 3: Loop moving relative to the lab frame

where the final term on the right accounts for the fact that the surface S over which we must integrate changes with time (it moves). To evaluate this term, we note that the distance the loop moves in time dt is $\mathbf{v}dt$ and that S is displaced by the same amount. A point on S initially at $\mathbf{x}$ goes to $\mathbf{x} + \mathbf{v}dt$ in time dt, so if $\mathbf{B}(\mathbf{x})$ is sampled before dt elapses, then $\mathbf{B}(\mathbf{x} + \mathbf{v}dt)$ is sampled afterwards. We can expand the latter as

$$\mathbf{B}(\mathbf{x} + \mathbf{v}dt) = \mathbf{B}(\mathbf{x}) + dt(\mathbf{v} \cdot \nabla)\mathbf{B}(\mathbf{x}) + ... \tag{5}$$

For an infinitesimal time element, we can ignore the higher-order terms. Hence, dF_2, which is the integral of the change $[\mathbf{B}(\mathbf{x}+\mathbf{v}dt) - \mathbf{B}(\mathbf{x})] \cdot \mathbf{n}$ over the surface S (at time t) is easy to formulate:

$$\left(\frac{dF}{dt}\right)_2 = \int_S d^2x\, (\mathbf{v} \cdot \nabla)[\mathbf{B}(\mathbf{x}) \cdot \mathbf{n}]. \tag{6}$$

Now recall the vector identity $\nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}(\nabla \cdot \mathbf{B}) - (\mathbf{v} \cdot \nabla)\mathbf{B}$. If we use the law $\nabla \cdot \mathbf{B}(\mathbf{x}) = 0$, we can use this identity to find that

$$\left(\frac{dF}{dt}\right)_2 = -\int_S d^2x\, \mathbf{n} \cdot [\nabla \times (\mathbf{v} \times \mathbf{B})] = -\oint_C d\mathbf{l} \cdot (\mathbf{v} \times \mathbf{B}), \tag{7}$$

the last step following from Stokes theorem. Using Eqs. (4) and (7) in Faraday's Law of Induction[2], we find

$$\oint_C d\mathbf{l} \cdot [\mathbf{E}' - k(\mathbf{v} \times \mathbf{B})] = -k \int_S d^2x \left(\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) \tag{8}$$

where $\mathbf{E}'$ is the electric field in the frame where the loop is at rest; this frame moves at velocity $\mathbf{v}$ relative to the one where $\mathbf{B}$ is measured (along with $\mathbf{x}$ and t).

Now consider a second circuit which is at rest in the frame where $\mathbf{B}$, $\mathbf{x}$, and t are measured and which coincides with the first loop at the particular time t.

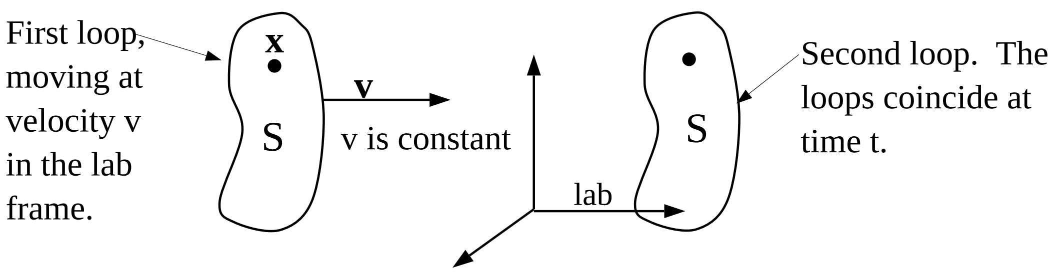


Figure 3b: Consider a second loop stationary in the lab frame

For this loop Faraday's Law says that

$$\oint_C d\mathbf{l} \cdot \mathbf{E} = -k \int_S d^2x \left(\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \right) \tag{9}$$

where $\mathbf{E}$ is the electric field in the lab frame. Comparing Eqs. (8) and (9), we see that

$$\oint_C d\mathbf{l} \cdot [\mathbf{E}' - k(\mathbf{v} \times \mathbf{B})] = \oint_C d\mathbf{l} \cdot \mathbf{E}. \tag{10}$$

This relation tells us that the integrands are equal, give or take a vector field $\mathbf{V}$ which has the property that the line integral of its component along the line is zero when taken around the loop C. This condition plus the arbitrariness of C tells us the $\nabla \times \mathbf{V} = 0$. This field can depend on $\mathbf{v}$, so we shall write it as $\mathbf{V}(\mathbf{v})$. Taking these statements together, we have the relation

$$\mathbf{E}' = \mathbf{E} + k(\mathbf{v} \times \mathbf{B}) + \mathbf{V}(\mathbf{v}). \tag{11}$$

[2] $\frac{dF}{dt} = -1/k \oint_C d\mathbf{l} \cdot \mathbf{E}'$

This relation gives us the transformation of the electric field from the lab frame to the rest frame of the moving circuit.

There is a second way to get this transformation. Consider a charge q following some trajectory $\mathbf{x}(t)$ under the influence of $\mathbf{E}$ and $\mathbf{B}$, as seen from the "lab" frame (the one where the unprimed quantities are measured).

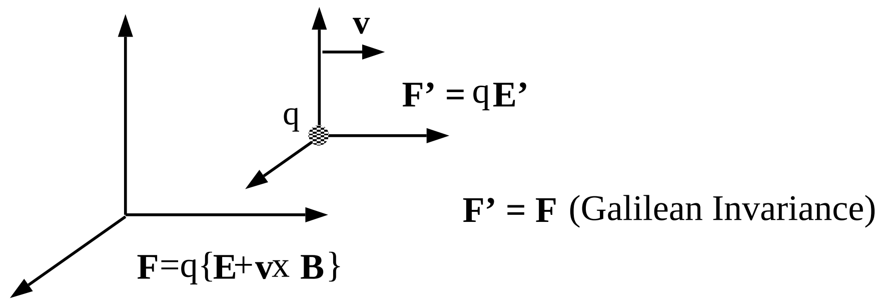


Figure 4: By Galilean invariance (see footnote) the forces F=F'

If at time t the particle is at some point $\mathbf{x}$ and has velocity $\mathbf{v}$, the same as the velocity of the moving circuit, then if feels a force $\mathbf{F}$ given by

$$\mathbf{F} = q[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})]. \tag{12}$$

But in its rest frame, where the electric field is $\mathbf{E}'$, it feels a force

$$\mathbf{F}' = q\mathbf{E}'. \tag{13}$$

Now, according to classical, or Galilean, relativity, $\mathbf{F}'$ is the same as $\mathbf{F}$, so, upon comparing the expressions for the force, we see that

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B}). \tag{14}$$

Comparison with Eq. (11) gives $k = 1/c$ and $\mathbf{V}(\mathbf{v}) = 0$. Using this result for k in Faraday's Law, we find that it is fully specified by

$$\mathcal{E} = -\frac{1}{c}\frac{dF}{dt} \text{ or } \oint_C d\mathbf{l} \cdot \mathbf{E}(\mathbf{x}) = -\frac{1}{c}\int_S d^2x \left(\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\right) \tag{15}$$

for a stationary path.[3] If we take the integral relation and apply Stokes theorem, we find

$$\int_S d^2x \left(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) \cdot \mathbf{n} = 0. \tag{16}$$

Because S is an arbitrary open surface, this relation must be true for all such surfaces. That can only be if the integrand is everywhere zero, leading us to the differential equation

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{x}, t)}{\partial t} \tag{17}$$

which is the differential equation statement of Faraday's Law.

2 Energy in the Magnetic Field

Given Faraday's Law, we are in a position to calculate the energy required to produce a certain current distribution $\mathbf{J}$ starting from a state with $\mathbf{J} = 0$ even though we do not as yet know all of the time-dependent terms in the field equations. In this section, we shall determine what is this energy. The mechanism that requires work to be done is as follows: If we attempt to make a change in any existing current distribution, there will be time-dependent sources (the current) with an accompanying time-dependent magnetic induction. The latter must in turn produce electromagnetic forces, or electric fields, against which work must be done not only in order to change the currents but also simply in order to maintain them. By examining this work, we can determine the change in the "magnetic energy" of the system.

To get started, consider a single loop or circuit carrying current I_1. Given a changing flux dF_1/dt through this loop, there is an $\mathcal{E}_1 = -c^{-1} dF_1/dt$. If we wish to maintain the current in the face of this electromotive force, we must counter the latter by introducing an

[3] It should be remarked that all of this makes sense only to order v/c since in the next order, v^2/c^2, Galilean relativity fails. However, the conclusion that $k = 1/c$ must remain valid since k is a constant (according to experiments), independent of the relative size of any velocities.

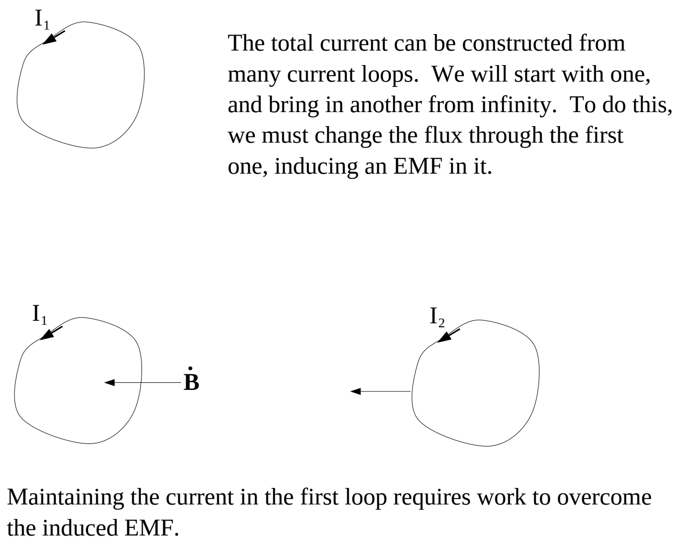


Figure 5: Energy required to construc a current distribution

external agent which maintains a voltage $V_1 = -\mathcal{E}_1 = \frac{1}{c}\frac{dF_1}{dt}$ around the loop. This agent thus does work at a rate $V_1 I_1 = -\mathcal{E}_1 I_1$. The total work that it does is

$$\delta W_1 = -\int dt I_1 \mathcal{E}_1 = \frac{I_1}{c}\int dt \frac{dF_1}{dt} = \frac{I_1 \delta F_1}{c} \tag{18}$$

where δF_1 is the total change of the flux through the loop. We may also express our result as

$$\delta W_1 = \frac{I_1}{c}\int_S d^2x\,(\delta \mathbf{B}\cdot\mathbf{n}) \tag{19}$$

where S and $\mathbf{n}$ are related to the loop in the usual way, and $\delta\mathbf{B}$ is the change in the magnetic induction. If we write $\delta\mathbf{B} = \nabla\times\delta\mathbf{A}$, which is possible so long as $\nabla\cdot\delta\mathbf{B} = 0$, we have

$$\delta W_1 = \frac{I_1}{c}\int_S d^2x\,(\nabla\times\delta\mathbf{A})\cdot\mathbf{n} = \frac{I_1}{c}\oint_C d\mathbf{l}\cdot\delta\mathbf{A} \tag{20}$$

where Stokes theorem has been invoked.

Now let us generalize. A current distribution $\mathbf{J}$ can be thought of as the sum of many infinitesimal loops. Given many such loops, the total work done by the external agent(s) will be the sum of the works done on each of the loops. The relation between the current in a loop, I, and $\mathbf{J}$ is $|\mathbf{J}|(d\sigma)$ where $d\sigma$ is the (infinitesimal) cross-sectional area of the loop. Thus $I d\mathbf{l} \to \mathbf{J} d\sigma dl$ or $\mathbf{J} d^3x$. Integrating over individual loops and summing over all loops is equivalent to integrating over all space, so we find that the change in energy accompanying a change $\delta\mathbf{A}(\mathbf{x})$ in the vector potential (reflecting an infinitesimal change $\delta\mathbf{B}(\mathbf{x})$ in the magnetic induction) is

$$\delta W = \frac{1}{c} \int d^3x \, [\mathbf{J}(\mathbf{x}) \cdot \delta\mathbf{A}(\mathbf{x})]. \tag{21}$$

The change in the magnetic induction has to be infinitesimal for this expression to be valid because we did not ask how the current density must be changed to produce it. Note that this form indicates that W is a natural thermodynamic function function of the potential or magnetic flux, rather than the sources

Now let us write the current density in terms of the fields. If we think we are doing macroscopic electromagnetism, then[4] $\mathbf{J} = c(\nabla \times \mathbf{H})/4\pi$ and we can proceed as follows:

$$\begin{aligned}\delta W = \frac{1}{4\pi} \int_V d^3x \, [(\nabla \times \mathbf{H}) \cdot \delta\mathbf{A}] &= \frac{1}{4\pi} \int_V d^3x \, [\nabla \cdot (\mathbf{H} \times \delta\mathbf{A}) + \mathbf{H} \cdot (\nabla \times \delta\mathbf{A})] \\ &= \frac{1}{4\pi} \oint_S d^2x \, (\mathbf{H} \times \delta\mathbf{A}) \cdot \mathbf{n} + \frac{1}{4\pi} \int_V d^3x \, (\mathbf{H} \cdot \delta\mathbf{B}) \\ &= \frac{1}{4\pi} \int d^3x \, (\mathbf{H} \cdot \delta\mathbf{B}). \end{aligned} \tag{22}$$

The final step in this argument is achieved by letting the domain of integration be all space and assuming that the fields $\mathbf{H}$ and $\delta\mathbf{A}$ fall off fast enough far away that the surface integral vanishes. This is in fact true for a set of localized sources in the limit that changes are made very slowly.

In the final step of our derivation we want to integrate $\delta\mathbf{B}$ up to some final $\mathbf{B}$ starting from zero magnetic induction or zero current density. We can only do this functional integral if we know how $\mathbf{H}$ depends on $\mathbf{B}$. For a **linear** medium or set of

[4]This relation is only true for static phenomena; since we are changing the fields with time, it is not valid. However, if the change is accomplished sufficiently slowly that $\frac{\partial \mathbf{D}}{\partial t}$ may be neglected, then the corrections to Ampère's Law are so small as to have negligible influence on our argument. Notice too that use of this equation demands that changes in $\mathbf{J}(\mathbf{x})$ be done in such a way that $\nabla \cdot \mathbf{J} = 0$.

media, we know that

$$\mathbf{H}\cdot\delta\mathbf{B} = \frac{1}{2}\delta(\mathbf{H}\cdot\mathbf{B}) \tag{23}$$

and so

$$\delta W = \frac{1}{8\pi}\int d^3x\,\delta(\mathbf{B}\cdot\mathbf{H}) \tag{24}$$

which integrates to

$$W = \frac{1}{8\pi}\int d^3x\,\mathbf{B}(\mathbf{x})\cdot\mathbf{H}(\mathbf{x}) \tag{25}$$

provided we define $W \equiv 0$ in the state with $\mathbf{B} \equiv 0$.

Our result, Eq. (25), can be written in other forms; a particularly useful one is obtained by writing $\mathbf{B} = \nabla\times\mathbf{A}$ and doing a parts integration in the familiar way. Assuming that one can discard the resulting surface term (valid for localized sources), we find

$$W = \frac{1}{2c}\int d^3x\,\mathbf{J}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x}). \tag{26}$$

In analogy with the electrostatic case, one conventionally defines the magnetic energy density to be

$$w \equiv \frac{1}{8\pi}(\mathbf{H}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})). \tag{27}$$

2.1 Example: Motion of a permeable Bit in a Fixed J

Let us look at a specific example of the use of the expression(s) for the energy. Suppose that there is some initial current distribution $\mathbf{J}_0$ which produces fields $\mathbf{H}_0$ and $\mathbf{B}_0$ and energy W_0. Then we have

$$W_0 = \frac{1}{8\pi}\int d^3x\,\mathbf{B}_0\cdot\mathbf{H}_0 = \frac{1}{2c}\int d^3x\,\mathbf{J}_0\cdot\mathbf{A}_o. \tag{28}$$

Now move some permeable materials around without changing the macroscopic current density J_0.

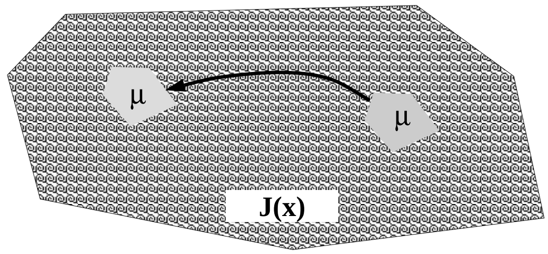


Figure 6: Motion of a permeable bit in a fixed current distribution

After doing so we have a new energy which is

$$W_1 = \frac{1}{8\pi}\int d^3x\,\mathbf{B}_1\cdot\mathbf{H}_1 = \frac{1}{2c}\int d^3x\,\mathbf{J}_0\cdot\mathbf{A}_1 \tag{29}$$

where the new fields, energy, etc., are designated by the subscript '1'. The change in energy may be written as

$$\begin{aligned} W_1 - W_0 &= \frac{1}{2c}\int d^3x\,\mathbf{J}_0\cdot(\mathbf{A}_1-\mathbf{A}_0) = \frac{1}{8\pi}\int d^3x\,[(\nabla\times\mathbf{H}_0)\cdot\mathbf{A}_1 - (\nabla\times\mathbf{H}_1)\cdot\mathbf{A}_0] \\ &= \left(\frac{1}{8\pi}\int d^3x\,\nabla\cdot[\mathbf{H}_0\times(\mathbf{A}_1-\mathbf{A}_0)] + \int d^3x\,[\mathbf{H}_0\cdot(\nabla\times\mathbf{A}_1) - \mathbf{H}_1\cdot(\nabla\times\mathbf{A}_0)]\right) \\ &= \frac{1}{8\pi}\left(\oint_S d^2x\,\mathbf{n}\cdot[\mathbf{H}_0\times(\mathbf{A}_1-\mathbf{A}_0)] + \int d^3x\,[\mathbf{H}_0\cdot\mathbf{B}_1 - \mathbf{H}_1\cdot\mathbf{B}_0]\right) \end{aligned} \tag{30}$$

The surface term may be discarded for localized sources. Assuming further that all materials are isotropic so that $\mathbf{B} = \mu\mathbf{H}$, we find

$$W_1 - W_0 = \frac{1}{8\pi}\int d^3x\,(\mu_1-\mu_0)(\mathbf{H}_0\cdot\mathbf{H}_1) \tag{31}$$

where μ_1 is the final value of the permeability (a function of position) and μ_0 is the initial value. If in addition $\mu_0 = 1$, a value appropriate for non-permeable materials or empty space, and $\mu_1 \neq 1$ only in some particular domain V, then

$$W_1 - W_0 = \frac{1}{2}\int_V d^3x\,(\mathbf{M}_1\cdot\mathbf{B}_0) \tag{32}$$

where we have made use of the facts that $\mathbf{M}_1 = (\mu_1 - 1)\mathbf{H}_1/4\pi$ and $\mathbf{B}_0 = \mathbf{H}_0$.

It is instructive (maybe it's just confusing) to compare the answer with other things that we have seen. First, we calculated previously the change in the electrostatic energy when a piece of dielectric is introduced into a previously empty space in the presence of some fixed charges. We found

$$W_1 - W_0 = -\frac{1}{2}\int_V d^3x\,\mathbf{P}_1\cdot\mathbf{E}_0. \tag{33}$$

The difference in sign between the electrostatic and magnetostatic energies is a reflection of the fact that in the magnetic system we maintained fixed currents and in the electrostatic system, fixed charges. In the former case external agents must do work to maintain the currents, and in the latter one, no work need be done to maintain the fixed charges. Hence in the latter case, the force on the dielectric may be found by using the argument that in a conservative system, the force is the negative gradient of the energy (which is given above) with respect to the displacement of the dielectric. **If**, in the magnetic system, the work done on the system by the current-maintaining external agents turns out to be precisely twice as large as the change in the system's energy, then the force on the permeable material will be the (positive) gradient of the energy with respect to the displacement of the material. This is, in fact, the case, as we shall see below.

2.2 Energy of a Current distribution in an External Field

As a second example, consider the energy of a current distribution in an external field.

$$W = \frac{1}{c}\int d^3x\,\mathbf{J}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x}). \tag{34}$$

where $\mathbf{A}(\mathbf{x})$ is due to sources other than $\mathbf{J}(\mathbf{x})$, which do not overlap the region where $\mathbf{J}(\mathbf{x})$ is finite. (note the lack of a factor of $\frac{1}{2}$, why?)

Figure 7: The source of the vector potential is far removed from the current distribution.

Now assume that $\mathbf{A}(\mathbf{x})$ changes little over the region where $\mathbf{J}(\mathbf{x})$ is finite, and thus expand $\mathbf{A}(\mathbf{x})$ about the origin of the current distribution.

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(0) + \mathbf{x} \cdot \nabla \mathbf{A}(\mathbf{x})\big]_{\mathbf{x}=0} + \cdots$$

through the now familiar manipulations, we end up with

$$W = \mathbf{m} \cdot \mathbf{B}(0) + \cdots$$

where $\mathbf{m}$ is the dipole moment of the current distribution

This appears to make no sense at all!! Recall last quarter, we found that the force on a permanent dipole is

$$\mathbf{F} = \nabla(\mathbf{M} \cdot \mathbf{B}_0)$$

which acts to minimize the potential $-\mathbf{m} \cdot \mathbf{B}$, rather than $+\mathbf{m} \cdot \mathbf{B}$!

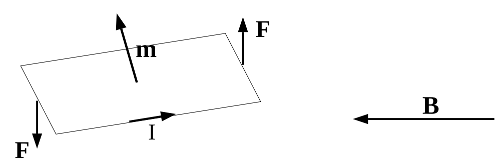


Figure 8: Forces on a current loop.

Consider a current loop Clearly the force tries to make $\mathbf{m}$ and $\mathbf{B}$ parallel. Have we misplaced a "-"-sign? Our expression for W is correct; however, we do not have a conservative situation, since in calculating the force on the loop, we assumed that the current I is constant. However, rotating the loop changes the flux through it which induces an $\mathcal{E}mf$ which opposes the changing flux. Thus I will not be constant unless external work is done to make it so. In such a non-conservative situation, the force is not the negative gradient of the energy. For this particular situation, it must be that the the force is given by the *positive* gradient of the energy.

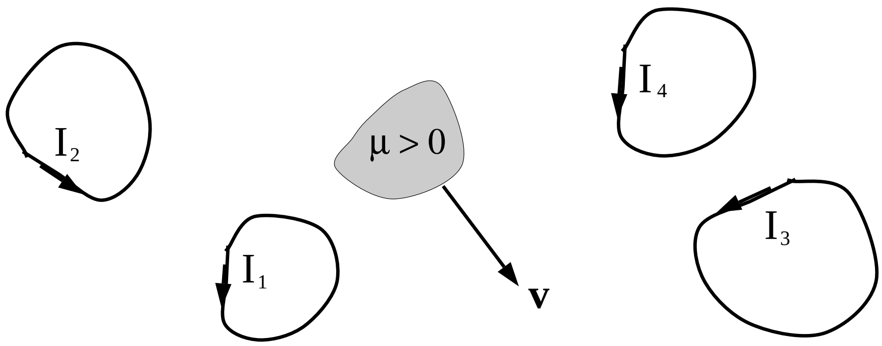
To demonstrate that the last statement above is correct, we consider a set of circuits C_i, $i = 1, 2, ..., n$, with currents I_i. The energy of this system is

$$W = \frac{1}{2c} \int d^3x \, \mathbf{J} \cdot \mathbf{A} = \frac{1}{2c} \sum_i I_i \oint_{C_i} d\mathbf{l} \cdot \mathbf{A} = \frac{1}{2c} \sum_i \int_{S_i} d^2x \, \mathbf{n} \cdot (\nabla \times \mathbf{A})$$

$$= \frac{1}{2c} \sum_i I_i \int_{S_i} d^2x \, (\mathbf{n} \cdot \mathbf{B}) = \frac{1}{2c} \sum_i I_i F_i \tag{35}$$

where F_i is the flux linking the i^{th} circuit.

Now suppose that a piece of permeable material moves through the system at velocity $\mathbf{v}$.



It will experience a force which we may calculate by demanding total energy conservation. The various energies that must be considered are as follows:

1. The energy transferred to the moving object from the magnetic field. In time dt, this energy is
$$dW_m = \mathbf{F} \cdot \mathbf{v} dt = F_\eta v dt \tag{36}$$
given that $\mathbf{v}$ is in the η-direction.

2. The field energy; for fixed currents in the circuits, this energy changes by
$$dW = \frac{1}{2c} \sum_i I_i dF_i \tag{37}$$
in time dt.

3. The energy transferred to the magnetic field (or circuits) by some external agents whose business it is to maintain the currents. The $\mathcal{E}mf$'s in the circuits are $\mathcal{E}_i = -c^{-1} dF_i/dt$ and so the external agents do work on the i^{th} circuit at a rate $-\mathcal{E}_i I_i$ in order to maintain the currents. The work done on these external

agents in time dt is therefore

$$dW_e = \sum_i \mathcal{E}_i I_i dt = -\frac{1}{c}\sum_i I_i dF_i = -2dW \tag{38}$$

Now invoke conservation of energy which demands that the sum of all of the preceding infinitesimal energy changes must be zero

$$dW_m + dW + dW_e = 0 \tag{39}$$

or, using Eqs. (36) and (38),

$$F_\eta v dt = dW. \tag{40}$$

Further, $vdt = d\eta$, so we find that

$$F_\eta = +\left(\frac{\partial W}{\partial \eta}\right)_{\mathbf{J}} \tag{41}$$

as suggested earlier.

What would the force be if the flux F was held constant?

3 Maxwell's Displacement Current; Maxwell's Equations

Let us summarize the equations of electromagnetism as we now have them:

$$\begin{aligned}\nabla\cdot\mathbf{B} &= 0\\ \nabla\cdot\mathbf{D} &= 4\pi\rho\\ \nabla\times\mathbf{H} &= \frac{4\pi}{c}\mathbf{J}\\ \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}.\end{aligned} \tag{42}$$

These are not internally consistent. Consider the divergence of Ampère's Law:

$$\nabla\cdot(\nabla\times\mathbf{H}) = 0 = \frac{4\pi}{c}(\nabla\cdot\mathbf{J}) \tag{43}$$

which implies that $\nabla \cdot \mathbf{J} = 0$. We know, however, that for general time-dependent phenomena the divergence of the current density is not necessarily zero; indeed, we have seen that charge conservation requires

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{44}$$

In consequence of this requirement, we must, at the very least, add a (time-dependent) term to Ampère's Law which will give, in distinction to Eq. (43),

$$\nabla \cdot (\nabla \times \mathbf{H}) = \frac{4\pi}{c}\left[\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right]. \tag{45}$$

This new term which must be a vector field $\mathbf{X}$, has to be such that

$$\nabla \cdot \mathbf{X} = \frac{4\pi}{c}\frac{\partial \rho}{\partial t}. \tag{46}$$

We need not look beyond the things we have already learned to find a plausible candidate for this term. We have a (static) equation which reads

$$\rho = \frac{1}{4\pi}\nabla \cdot \mathbf{D}; \tag{47}$$

if we accept this as correct, we have

$$\frac{4\pi}{c}\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{D} = \nabla \cdot \left(\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right). \tag{48}$$

The simplest possible resolution of the inconsistency in the field equations is thus to choose $\mathbf{X}$ to be $c^{-1}\partial \mathbf{D}/\partial t$, which would turn Ampère's Law into the relation

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}. \tag{49}$$

This adjustment was made[5] by J. C. Maxwell in 1864, and the resulting set of differential field equations has since become known as *Maxwell's Equations*:

$$\nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0, \tag{50}$$

[5]He delivered a paper containing this statement in 1864, but he'd had the idea at least as early as 1861.

$$\nabla \cdot \mathbf{D}(\mathbf{x}, t) = 4\pi\rho(\mathbf{x}, t), \tag{51}$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{x}, t)}{\partial t}, \tag{52}$$

and

$$\nabla \times \mathbf{H}(\mathbf{x}, t) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}, t) + \frac{1}{c}\frac{\partial \mathbf{D}(\mathbf{x}, t)}{\partial t}. \tag{53}$$

The term that Maxwell added was called by him the *displacement current* $\mathbf{J}_D$:

$$\mathbf{J}_D \equiv \frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}. \tag{54}$$

This term has the appearance of an additional current entering Ampère's Law so that the latter reads

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}(\mathbf{J} + \mathbf{J}_D). \tag{55}$$

Notice that $\nabla \cdot (\mathbf{J} + \mathbf{J}_D) = 0$. We shall not emphasize the "current" interpretation of $\mathbf{J}_D$ because it is misleading; the displacement current does not describe a flow of charge and is not a true current density.

We close this section with two comments. **First**, the Maxwell equations must be regarded as empirically justified. In the years since Maxwell's final adjustment of the field equations of electromagnetism, they have been subjected to extensive experimental tests and have been found to be correct for classical phenomena (no quantum effects or general relativistic effects); with proper interpretation, they even have considerable validity within the realm of quantum phenomena. **Second**, the equations as we have written them are for macroscopic electromagnetism. The more fundamental version of these equations has **B** in place of **H** and **E** in place of **D**; then the sources ρ and **J** are the total charge and current densities.

4 Vector and Scalar Potentials

For time-dependent phenomena, which are fully described by the Maxwell equations, one can still write the fields **E** and **B** in terms of a scalar potential and a vector potential. Because the divergence of the magnetic induction is still zero, one continues to be able to find a vector potential $\mathbf{A}(\mathbf{x}, t)$ which has the property that

$$\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t); \tag{56}$$

this potential is not unique. Further, since the curl of the electric field is not zero in general, we cannot write $\mathbf{E}(\mathbf{x}, t)$ as the gradient of a scalar; however, from Faraday's Law, Eq. (52), and from Eq. (56), we have

$$\nabla \times \left(\mathbf{E}(\mathbf{x}, t) + \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{x}, t)}{\partial t} \right) = 0 \tag{57}$$

and so we can write the combination of fields in parentheses as the gradient of a scalar function $\Phi(\mathbf{x}, t)$:

$$\mathbf{E}(\mathbf{x}, t) + \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{x}, t)}{\partial t} = -\nabla \Phi(\mathbf{x}, t) \tag{58}$$

or

$$\mathbf{E}(\mathbf{x}, t) = -\nabla \Phi(\mathbf{x}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{x}, t)}{\partial t}. \tag{59}$$

Equations (56) and (59) tell us how to find $\mathbf{B}$ and $\mathbf{E}$ from potentials; these potentials must themselves satisfy certain field equations that can be derived from the Maxwell equations involving the sources ρ and $\mathbf{J}$; the "homogeneous" or source-free equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = c^{-1} \partial \mathbf{B} / \partial t$ are then automatically satisfied. Letting $\mathbf{H} = \mathbf{B}$ and $\mathbf{D} = \mathbf{E}$ for simplicity, we have, upon substituting Eqs. (56) and (59) into Eqs. (51) and (53),

$$\nabla^2 \Phi + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -4\pi\rho \tag{60}$$

and

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla \left(\frac{1}{c} \frac{\partial \Phi}{\partial t} \right) = \frac{4\pi}{c} \mathbf{J}. \tag{61}$$

These equations clearly do not have particularly simple pleasing or symmetric forms. However, we have some flexibility left in the choice of the potentials because we can choose the vector potential's divergence in an arbitrary fashion.

5 Gauge Transformations

Suppose that we have some $\mathbf{A}$ and Φ which give $\mathbf{E}$ and $\mathbf{B}$ correctly. Let us add to $\mathbf{A}$ the gradient of a scalar function $\chi(\mathbf{x}, t)$, thereby obtaining $\mathbf{A}'$:

$$\mathbf{A}' \equiv \mathbf{A} + \nabla\chi. \tag{62}$$

The field $\mathbf{A}'$ has the same curl as $\mathbf{A}$, and hence $\mathbf{B} = \nabla \times \mathbf{A}'$. However, the electric field is **not** given by $-\nabla\Phi - c^{-1}\partial\mathbf{A}'/\partial t$; we must therefore change Φ to Φ' where Φ' is chosen to that

$$\mathbf{E} = -\nabla\Phi' - \frac{1}{c}\frac{\partial\mathbf{A}'}{\partial t}. \tag{63}$$

To this end we consider $\Phi' = \Phi + \psi$ where ψ is some scalar function of $\mathbf{x}$ and t. Our requirement, Eq. (63), is that

$$\mathbf{E} = -\nabla\Phi - \nabla\psi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \frac{1}{c}\frac{\partial(\nabla\chi)}{\partial t}. \tag{64}$$

However, we know that $\mathbf{E}$ is given according to Eq. (59), so, combining this relation and Eq. (64), we find that ψ must satisfy the equation

$$\nabla\psi = -\frac{1}{c}\frac{\partial(\nabla\chi)}{\partial t}. \tag{65}$$

A clear possible choice of ψ is $\psi = -c^{-1}\partial\chi/\partial t$.

What we have learned is that, given potentials $\mathbf{A}$ and Φ, we may make a *gauge transformation* to equally acceptable potentials $\mathbf{A}'$ and Φ' given by

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\chi \\ \Phi' &= \Phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \end{aligned} \tag{66}$$

where $\chi(\mathbf{x}, t)$ is an arbitrary scalar function of position and time.

5.1 Lorentz Gauge

Now let's look again at the differential equations, Eqs. (60) and (61), for the potentials; these may be written as

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{J} \tag{67}$$

where we have used the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$. These have a much more pleasing form if $\mathbf{A}$ and Φ satisfy the *Lorentz condition*

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0. \tag{68}$$

Supposing for the moment that such a choice is possible, we make use of Eq. (68) in Eq. (67) and find

$$\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{J}. \end{aligned} \tag{69}$$

These are very pleasing in that there are distinct equations for Φ and $\mathbf{A}$, driven by ρ and $\mathbf{J}$, respectively; furthermore, all equations have the form of the classical wave equation,

$$\Box^2 \psi(\mathbf{x}, t) = -4\pi f(\mathbf{x}, t) \tag{70}$$

where

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{71}$$

is known as the *D'Alembertian* operator.

Consider next whether it is generally possible to find potentials that satisfy the Lorentz condition. Suppose we have some potentials $\mathbf{A}_0$ and Φ_0 for a given set of sources $\mathbf{J}$ and ρ. We make a gauge transformation to new potentials $\mathbf{A}$ and Φ,

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_0 + \nabla\chi \\ \Phi &= \Phi_0 - \frac{1}{c}\frac{\partial \chi}{\partial t}, \end{aligned} \tag{72}$$

where the gauge function χ is to be chosen so that the Lorentz condition is satisfied by the new potentials. The condition on χ is thus

$$0 = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = \nabla \cdot \mathbf{A}_0 + \nabla^2\chi + \frac{1}{c}\frac{\partial \Phi_0}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}, \tag{73}$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\chi = -\nabla\cdot\mathbf{A}_0 - \frac{1}{c}\frac{\partial\Phi_0}{\partial t}. \tag{74}$$

The function that we seek is thus itself a solution of the classical wave equation with a "source" which is, aside from some constant factor, just $\nabla\cdot\mathbf{A}_0 + c^{-1}(\partial\Phi_0/\partial t)$. Such a function always exists. In fact, there are many solutions to this wave equation which means that there are many sets of potentials Φ and $\mathbf{A}$ which satisfy the Lorentz condition. Potentials satisfying the Lorentz condition are said to be in the *Lorentz gauge*.

5.2 Coulomb Transverse Gauge

Another gauge which can be useful is the *Coulomb* or *transverse gauge*. It is defined by the condition that $\nabla\cdot\mathbf{A} = 0$. The beauty of this gauge is that in it the scalar potential satisfies the Poisson equation,

$$\nabla^2\Phi(\mathbf{x},t) = -4\pi\rho(\mathbf{x},t). \tag{75}$$

We know the solution of this equation:

$$\Phi(\mathbf{x},t) = \int d^3x'\,\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}. \tag{76}$$

Notice that the time is the same at both the source point $\mathbf{x}'$ and the field point $\mathbf{x}$. There is nothing unacceptable about this because the scalar potential is not a measurable quantity.

The vector potential in the Coulomb gauge is less satisfying; it obeys the wave equation

$$\Box^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\Phi). \tag{77}$$

In practice one may solve for the potentials in the transverse gauge, given the sources ρ and $\mathbf{J}$, by first finding the scalar potential from the integral Eq. (76) and then using the result in Eq. (77) and solving the wave equation (see the following section). One should not be surprised to learn that the "source" term in Eq. (77) involving the scalar potential can be made to look like a current; consider

$$\frac{1}{c}\frac{\partial}{\partial t}(\nabla\Phi) = \frac{1}{c}\frac{\partial}{\partial t}\left[\nabla\left(\int d^3x'\,\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right)\right] = -\nabla\int d^3x'\,\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$$
$$= -\frac{4\pi}{c}\left(\frac{1}{4\pi}\nabla\int d^3x'\,\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{78}$$

The negative of the quantity within the parentheses (...) is called the *longitudinal current density*. More generally, the *longitudinal* and *transverse* components $\mathbf{J}_l$ and $\mathbf{J}_t$ of a vector field such as $\mathbf{J}$ are defined by the conditions[6]

$$\mathbf{J}_l + \mathbf{J}_t = \mathbf{J},\;\; \nabla\times\mathbf{J}_l = 0,\;\; \text{and}\;\; \nabla\cdot\mathbf{J}_t = 0. \tag{79}$$

In other words, J_l and J_t satisfy the equations

$$\begin{array}{cc} \nabla\cdot\mathbf{J}_l = \nabla\cdot\mathbf{J} & \nabla\cdot\mathbf{J}_t = 0 \\ \nabla\times\mathbf{J}_l = 0 & \nabla\times\mathbf{J}_t = \nabla\times\mathbf{J}, \end{array} \tag{80}$$

which means that $\mathbf{J}_l$ carries all of the divergence of the current density and $\mathbf{J}_t$ has all of the curl.

From these equations and our knowledge of electrostatics we can see immediately that

$$\mathbf{J}_l(\mathbf{x},t) = -\frac{1}{4\pi}\nabla\left(\int d^3x'\,\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{81}$$

which may also be written as

$$\mathbf{J}_l(\mathbf{x},t) = -\frac{1}{4\pi}\nabla\left(\nabla\cdot\int d^3x\,\frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right), \tag{82}$$

and from that of magnetostatics we see that

$$\mathbf{J}_t(\mathbf{x},t) = \nabla\times\left(\frac{1}{4\pi}\int d^3x'\,\frac{\nabla'\times\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{83}$$

[6] In the appendix it is shown that such a decomposition, into longitudinal and transverse parts, of a vector function of position is always possible

which may also be written as

$$\mathbf{J}_t(\mathbf{x},t) = \nabla \times \left[\frac{1}{4\pi}\nabla \times \left(\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right)\right]. \tag{84}$$

Comparison of Eq. (81) with Eqs. (77) and (76) shows that the wave equation for $\mathbf{A}$ in the Coulomb or transverse gauge is

$$\Box^2\mathbf{A} = -\frac{4\pi}{c}(\mathbf{J}-\mathbf{J}_l) = -\frac{4\pi}{c}\mathbf{J}_t. \tag{85}$$

This equation lends justification to the term "transverse gauge." In this gauge, the vector potential is driven by the transverse part of the current in the same sense that the vector potential is driven by the entire current in the Lorentz gauge.

6 Green's Functions for the Wave Equation

We're going to spend quite a lot of time looking for solutions of the classical wave equation

$$\Box^2\psi(\mathbf{x},t) = -4\pi f(\mathbf{x},t); \tag{86}$$

Therefore, it could be useful to have Green's functions for the D'Alembertian operator, meaning functions $G(\mathbf{x},t;\mathbf{x}',t')$ which satisfy the equation

$$\Box^2 G(\mathbf{x},t;\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{87}$$

subject to some boundary conditions.

Compare this equation with that for the Green's function for the Laplacian operator, $\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$. The solution for the latter is, as we know, $G(\mathbf{x},\mathbf{x}') = 1/|\mathbf{x}-\mathbf{x}'|$ in an infinite space; physically, it is the scalar potential at $\mathbf{x}$ produced by a unit point charge located at $\mathbf{x}'$. The function $G(\mathbf{x},t;\mathbf{x}',t')$, by contrast has dimensions $1/LT$ and may be thought of as the "response" at the space-time point $(\mathbf{x},t)$ to a unit "source" at the space-time point $(\mathbf{x}',t')$, that is, to a source which exists only for a moment at a single space point. If this source is a pulse of current or an evanescent point charge, then the Green's function would be the vector or scalar potential in the Lorentz gauge produced by that pulse.

As for the question of boundary conditions, we shall keep things as simple as possible by considering an infinite system and applying the boundary condition that $G(\mathbf{x},t;\mathbf{x}',t') \to 0$ for $|\mathbf{x}| \to \infty$. There is also a boundary condition on the behavior of $G(\mathbf{x},t;\mathbf{x}',t')$ as a function of t; this is usually called an initial condition. We shall not make any specific statement now of the initial conditions but will keep them in mind as our derivation progresses.

We shall solve the wave equation for $G(\mathbf{x},t;\mathbf{x}',t')$ by appealing to Fourier analysis. Assuming that the Fourier transform $q(\mathbf{k},\omega)$ of a function $f(\mathbf{x},t)$ exists, then one can write that function in terms of the transform as

$$f(\mathbf{x},t) = \int d^3k\, d\omega e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} q(\mathbf{k},\omega). \tag{88}$$

This expression may be thought of as an expansion of $f(\mathbf{x},t)$ using basis functions $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. These form a complete orthonormal set; the orthonormality condition is

$$\int d^3x\, dt\, e^{-i(\mathbf{k}'\cdot\mathbf{x}-\omega' t)} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = (2\pi)^4\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega') \tag{89}$$

and the completeness relation is much the same,

$$\int d^3k\, d\omega e^{-i(\mathbf{k}\cdot\mathbf{x}'-\omega t')} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = (2\pi)^4\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{90}$$

If we substitute this last relation into the right-hand side of Eq. (87) and also expand $G(\mathbf{x},t;\mathbf{x}',t')$ on the left-hand side as[7]

$$G(\mathbf{x},t;\mathbf{x}',t') = \int d^3k\, d\omega g(\mathbf{k},\omega) e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]}, \tag{91}$$

then one finds that

$$\begin{aligned} \Box^2 G(\mathbf{x},t;\mathbf{x}',t') &= \int d^3k\, d\omega \left(-k^2+\frac{\omega^2}{c^2}\right) g(\mathbf{k},\omega) e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]} \\ &= -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') = -\frac{4\pi}{(2\pi)^4}\int d^3k\, d\omega e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')]}. \end{aligned} \tag{92}$$

[7]From Eq. (85) one may i infer that $G(\mathbf{x},t;\mathbf{x}',t')$ can be written as a function of $\mathbf{x}-\mathbf{x}'$ and of $t-t'$ in an infinite space.

The orthonormality of the basis functions may now be used to argue that the integrands on the two sides of this equation must be equal; hence,

$$g(\mathbf{k},\omega) = \frac{4\pi}{(2\pi)^4}\frac{1}{k^2-\omega^2/c^2}. \tag{93}$$

Now we have only to do the Fourier transform to find $G(\mathbf{x},t;\mathbf{x}',t')$. Consider first the frequency integral,

$$I(\mathbf{k},t-t') = \int_{-\infty}^{\infty} d\omega \left(\frac{e^{-i\omega(t-t')}}{k^2-\omega^2/c^2}\right). \tag{94}$$

This integral can have different values depending on how we handle the poles in the integrand at $\omega = \pm ck$. Let's see what are the possibilities: If $t < t'$, we can extend the path of integration at $|\omega| \to \infty$ around a semi-circle in the upper half-plane where $\Re\,\omega > 0$ without getting an additional non-zero contribution to the integral.

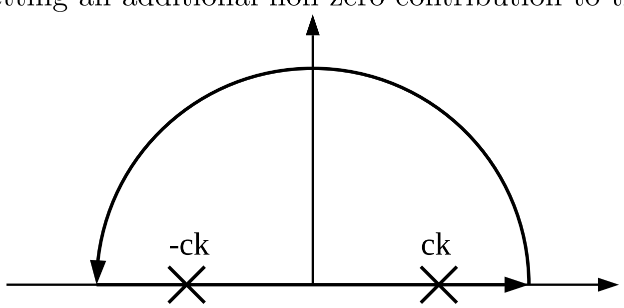


Figure 10: Appropriate Contour when t < t'

Hence, in this case the integral in Eq. (94) is just $2\pi i$ times the sum of the residues of the poles of the integrand in the upper half-plane. The only poles in the integrand as it stands are the ones at $\omega = \pm ck$. The path specified in the integral in fact runs right across both of them. However, if we make infinitesimal deformations of the path so that the poles are either inside of or outside of the contour, we will still obtain a function $G(\mathbf{x},t;\mathbf{x}',t')$ which is a solution of Eq. (87). Thus the solution that we are obtaining is not unique.

Why not? The answer lies in the fact that we are dealing with a differential equation which is second-order in time, and there are two independent solutions which we can discriminate by specifying some initial condition. Consider what happens if we deform the contour so as to go just above the poles on the real-ω axis. Then, for $t < t'$, we close the contour in the upper half-plane and find that the integral around the closed contour is zero (there are no poles inside of it). This means that $G(\mathbf{x}, t; \mathbf{x}', t') = 0$ for $t < t'$. But if $t > t'$, then we must close the contour in the lower half-plane and will find that $G(\mathbf{x}, t; \mathbf{x}', t') \neq 0$. This Green's function is called the *retarded Green's function* and its essential character is that it is non-zero only for times t which are **later** than the time t' of the source pulse which "produces" it. Instead of deforming the contour, we may change the integrand in such a way as to put the poles an infinitesimal distance below the real-ω axis and so have

$$I(\mathbf{k}, t - t') = \int_{-\infty}^{\infty} d\omega \left(\frac{e^{-i\omega(t-t')}}{k^2 - (\omega + i\epsilon)^2/c^2} \right) \tag{95}$$

where the integral is to be evaluated in the limit that ϵ is a positive infinitesimal constant. In this way we obtain a Green's function that vanishes at all times t earlier than the time t'.

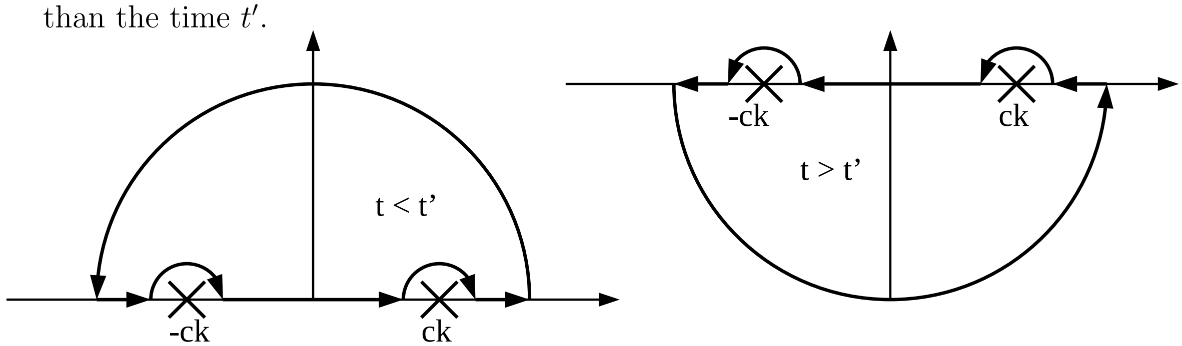


Figure 11: Contours for the retarded Green's function.

The second possibility is that we displace the poles so that they lie slightly above the real-ω axis. Then we will get a Green's function that will vanish for all $t > t'$ but not for times t earlier than t'; this one is called the *advanced Green's function* and to have an expression for it, we need only change $+\epsilon$ to $-\epsilon$ in Eq. (95).

Now that we have determined how to handle the poles in the integrand, let us proceed with the evaluation of the retarded Green's function in particular. For $t < t'$, it is identically zero, and for $t > t'$, we close the contour in the lower half-plane and find

$$I(\mathbf{k}, t-t') = -c^2 \oint_C d\omega \frac{e^{-i\omega(t-t')}}{[\omega - (ck - i\epsilon)][\omega - (-ck - i\epsilon)]}$$

$$= -c^2(-2\pi i)\left[\frac{e^{-ick(t-t')}}{2ck} + \frac{e^{ick(t-t')}}{-2ck}\right] = \frac{2\pi c}{k}\sin[ck(t-t')]. \tag{96}$$

Now we may complete the determination of $G(\mathbf{x}, t; \mathbf{x}', t')$ by doing the integration over the wavevector:

$$G(R, \tau) = \frac{1}{4\pi^3}\int d^3k\, \frac{2\pi c}{k}\sin(ck\tau)e^{i\mathbf{k}\cdot\mathbf{R}} \tag{97}$$

where $\tau \equiv t - t'$ and $\mathbf{R} \equiv (\mathbf{x} - \mathbf{x}')$. Thus,

$$G(R,\tau) = \frac{c}{2\pi^2}\int_0^\infty kdk \int_{-1}^1 du \int_0^{2\pi} d\phi \sin(ck\tau)e^{ikRu}$$

$$= \frac{c}{\pi}\int_0^\infty kdk\left(\frac{e^{ikR} - e^{-ikR}}{ikR}\right)\sin(ck\tau)$$

$$= -\frac{c}{4\pi R}\int_{-\infty}^\infty dk\left(e^{ikR} - e^{-ikR}\right)\left(e^{ick\tau} - e^{-ick\tau}\right)$$

$$= -\frac{c}{2R}[\delta(R + c\tau) + \delta(-R - c\tau) - \delta(R - c\tau) - \delta(-R + c\tau)] = \frac{\delta(\tau - R/c)}{R}; \tag{98}$$

the final step here follows from the fact that this expression is only correct for $\tau > 0$; for $\tau < 0$ $G \equiv 0$. In more detail, our result for the retarded Green's function is

$$G^{(+)}(\mathbf{x}, t; \mathbf{x}', t') = \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|}. \tag{99}$$

By similar manipulations one may show that the advanced Green's function is

$$G^{(-)}(\mathbf{x}, t; \mathbf{x}', t') = \frac{\delta(t' - t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|}. \tag{100}$$

Given the appropriate Green's function, we can write down a solution to the classical wave equation with matching initial conditions and the appropriate boundary conditions as $|\mathbf{x}| \to \infty$. Suppose that the wave equation is

$$\Box^2\psi(\mathbf{x}, t) = -4\pi f(\mathbf{x}, t); \tag{101}$$

the solution is

$$\psi(\mathbf{x},t) = \int d^3x'dt'G(\mathbf{x},t;\mathbf{x}',t')f(\mathbf{x}',t') \tag{102}$$

as may be seen by operating on this equation with $\Box^2$:

$$\Box^2\psi(\mathbf{x},t) = \int d^3x'dt'\Box^2G(\mathbf{x},t;\mathbf{x}',t')f(\mathbf{x}',t')$$

$$= \int d^3x'dt'[-4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')]f(\mathbf{x}',t') = -4\pi f(\mathbf{x},t). \tag{103}$$

The fact that $G(\mathbf{x},t;\mathbf{x}',t')$ is proportional to a delta function means that one can always complete the integration over time trivially. For the retarded Green's function in particular, one finds that

$$\psi(\mathbf{x},t) = \int d^3x' \, \frac{f(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \tag{104}$$

which has a fairly obvious interpretation.

Finally, it is worth pointing out that one may always add to $\psi(\mathbf{x},t)$ a solution of the homogeneous wave equation.

7 Derivation of Macroscopic Electromagnetism

Regretfully omitted because of time constraints.

8 Poynting's Theorem; Energy and Momentum Conservation

A conservation law is a statement to the effect that some quantity q (such as charge) is a constant for an isolated system. Often, it is possible to express the law mathematically in the form

$$\nabla\cdot\mathbf{J}_q + \frac{\partial\rho_q}{\partial t} = \left(\frac{d\rho_q}{dt}\right)_e. \tag{105}$$

Here, the density of the conserved quantity is ρ_q, its current density is $\mathbf{J}_q$, and the term on the right-hand side of the equation represents the contribution to the density at a particular space-time point arising from external sources or sinks. As we have already seen for the particular case of electrical charge, the divergence term represents the flow of the conserved quantity away from a point in space while the partial time derivative represents the rate at which its density is changing at that point. For electrical charge, there are no (known) sources or sinks and so the term on the right-hand side is zero.

8.1 Energy Conservation

It is possible to derive several such conservation laws from the Maxwell equations. These include the charge conservation law[8] and also ones that are interpreted as energy, momentum, and angular momentum conservation. To get started, consider the rate per unit volume at which the fields transfer energy to the charged particles, or sources. The magnetic induction does no work since it is directed normal to the velocity of a particle, and so we have only the electric field which does work a the rate $\mathbf{J} \cdot \mathbf{E}$ per unit volume. This is, in the context of Eq. (105), the term $-(d\rho/dt)$ representing the transfer of energy to external (to the fields) sources or sinks.

The total power $\mathcal{P}_m$ transferred to the sources within some domain V is found by integrating over that domain,

$$\mathcal{P}_m = \int_V d^3x \, (\mathbf{J} \cdot \mathbf{E}). \tag{106}$$

Now what we would like to do is perform some manipulations on $\mathbf{J} \cdot \mathbf{E}$ designed to remove all reference to the current density, leaving only electromagnetic fields; further, we want to make the expression look like the left-hand side of Eq. (105).

[8]Not surprising since the equations were explicitly constructed so as to be consistent with charge conservation.

That is not so hard to do, starting with the generalization of Ampère's law:

$$\begin{aligned}\mathbf{J}\cdot\mathbf{E} &= \frac{1}{4\pi}\left[c\mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right] \\ &= \frac{1}{4\pi}\left[-c\nabla\cdot(\mathbf{E}\times\mathbf{H}) + c\mathbf{H}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right] \\ &= -\frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{H}) - \frac{1}{4\pi}\left(\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right).\end{aligned} \tag{107}$$

In the last step, Faraday's law has been employed. The result is promising; there is a divergence and the remainder is almost a time derivative. It becomes a time derivative if the materials in the system have linear properties so that

$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{H}) \text{ and } \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}). \tag{108}$$

Then Eq. (107) becomes

$$-\mathbf{J}\cdot\mathbf{E} = \frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \frac{1}{8\pi}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \tag{109}$$

which is of the very form we seek. The interpretation of this equation is that the *Poynting vector* $\mathbf{S}$, defined by

$$\mathbf{S} \equiv \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}) \tag{110}$$

is the energy current density of the electromagnetic field and that the *energy density* u, defined by

$$u \equiv \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \tag{111}$$

is, indeed, the energy density of the electromagnetic field. In terms of these quantities the conservation law is

$$-\mathbf{J}\cdot\mathbf{E} = \nabla\cdot\mathbf{S} + \frac{\partial u}{\partial t}. \tag{112}$$

Notice, however, that the energy current density is not unique. Because only its divergence enters into the conservation law, we may add to $\mathbf{S}$ the curl of any vector field and would still have Eq. (112). The distinction is not important for measurable quantities because one always measures the rate of energy flow through a closed surface (the surface of the detector),

$$\oint_S d^2x\,(\mathbf{n}\cdot\mathbf{S}) \equiv \int_V d^3x\,(\nabla\cdot\mathbf{S}), \tag{113}$$

so that only the divergence of **S** is important. Finally, note that if the conservation law is integrated over some volume V, it can be expressed as

$$\int_V d^3x\,\frac{\partial u}{\partial t} = -\oint_S d^2x\,(\mathbf{n}\cdot\mathbf{S}) - \int_V d^3x\,(\mathbf{J}\cdot\mathbf{E}) \tag{114}$$

with the interpretation that rate of change of field energy within the domain V is equal to the rate at which field energy flows in through the surface of the domain plus the rate at which the sources within V transfer energy to the field.

8.2 Momentum Conservation

We can find a similar-looking momentum conservation law. We start from an expression for the rate per unit volume at which the fields transfer momentum to the sources; this is just the force density **f**,

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}(\mathbf{J}\times\mathbf{B}). \tag{115}$$

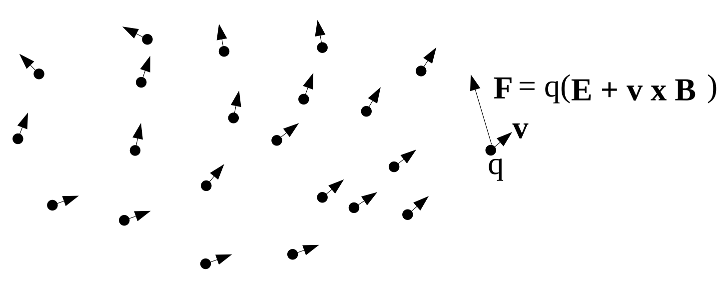


Figure 12: A swarm of charged particles.

Now use the Maxwell equations to remove all appearance of the sources ρ and $\mathbf{J}$:

$$\mathbf{f} = \left(\frac{\nabla\cdot\mathbf{D}}{4\pi}\right)\mathbf{E} + \frac{1}{c}\left(\frac{c}{4\pi}\nabla\times\mathbf{H} - \frac{1}{4\pi}\frac{\partial\mathbf{D}}{\partial t}\right)\times\mathbf{B}$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\left[\mathbf{E}(\nabla\cdot\mathbf{D}) + (\nabla\times\mathbf{H})\times\mathbf{B} - \frac{1}{c}\left(\frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}\right)\right] \\
&= \frac{1}{4\pi}\left[\mathbf{E}(\nabla\cdot\mathbf{D}) - \mathbf{B}\times(\nabla\times\mathbf{H}) - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \frac{1}{c}\left(\mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}\right)\right] \\
&= -\frac{1}{4\pi}\left\{\frac{1}{c}\frac{\partial(\mathbf{D}\times\mathbf{B})}{\partial t} + [\mathbf{B}\times(\nabla\times\mathbf{H}) - \mathbf{H}(\nabla\cdot\mathbf{B}) + \mathbf{D}\times(\nabla\times\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{D})]\right\}
\end{aligned} \tag{116}
$$

where we've used Faraday's law and the field equation $\nabla\cdot\mathbf{B} = 0$. Now we have to get all of the terms in the square brackets [...] in the final expression to look like a divergence. There is one complication which is that the equation is a vector equation, so we need a divergence which yields a vector, not a scalar.

Alternatively, we can look at Eq. (116) as three scalar equations, one for each Cartesian component of the force density. Then the term in brackets contains three components U_i and we want to write each component as the divergence of some vector field $\mathbf{V}_i$, $U_i = \nabla\cdot\mathbf{V}_i$. If we expand the vector fields as

$$
\mathbf{V}_i = \sum_{j=1}^{3} V_{ji}\boldsymbol{\epsilon_j}, \tag{117}
$$

with

$$
U_i = \nabla\cdot\mathbf{V}_i = \sum_j \frac{\partial V_{ji}}{\partial x_j}, \tag{118}
$$

then we have nine scalar functions V_{ji} which can be put into a 3×3 matrix. Let us go a step further and define a *dyadic* $\bar{\mathbf{V}}$ by its inner products with the complete set of basis vectors ϵ_i:

$$
\boldsymbol{\epsilon_j}\cdot\bar{\mathbf{V}}\cdot\boldsymbol{\epsilon_i} \equiv V_{ji}. \tag{119}
$$

A convenient way to write $\bar{\mathbf{V}}$ in terms of its components is

$$
\bar{\mathbf{V}} = \sum_{i,j=1}^{3} \boldsymbol{\epsilon_j} V_{ji}\boldsymbol{\epsilon_i}, \tag{120}
$$

with the understanding that when an inner product is taken of $\bar{\mathbf{V}}$ with a vector, the dot product is taken with the left or right $\boldsymbol{\epsilon_i}$ depending on whether the other vector lies to the left or right of $\bar{\mathbf{V}}$. In particular,

$$\nabla \cdot \bar{\mathbf{V}} = \sum_k \boldsymbol{\epsilon_k} \frac{\partial}{\partial x_k} \sum_{i,j} \boldsymbol{\epsilon_j} V_{ji} \boldsymbol{\epsilon_i} = \sum_{i,j} \frac{\partial V_{ji}}{\partial x_j} \boldsymbol{\epsilon_i} = \sum_i U_i \boldsymbol{\epsilon_i}. \tag{121}$$

Now what we need, referring back to Eq. (116), is some $\bar{\mathbf{V}}$ such that

$$\nabla \cdot \bar{\mathbf{V}} = \mathbf{B} \times (\nabla \times \mathbf{H}) - \mathbf{H}(\nabla \cdot \mathbf{B}) + \mathbf{D} \times (\nabla \times \mathbf{E}) - \mathbf{E}(\nabla \cdot \mathbf{D}) \tag{122}$$

where we shall restrict ourselves to the case of vacuum, or at least constant μ and ϵ. First, we have an identity,

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \frac{1}{2} \nabla (\mathbf{B} \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla) \mathbf{B} \tag{123}$$

which allows us to write

$$\begin{aligned}
\mathbf{B} \times (\nabla \times \mathbf{H}) - \mathbf{H}(\nabla \cdot \mathbf{B}) &= \frac{1}{\mu} \left[\frac{1}{2} \nabla (\mathbf{B} \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla) \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{B}) \right] \\
&= \frac{1}{\mu} \sum_{i,j} \left[\frac{1}{2} \delta_{ij} \boldsymbol{\epsilon_j} \frac{\partial B^2}{\partial x_i} - B_i \frac{\partial B_j}{\partial x_i} \boldsymbol{\epsilon_j} - B_j \boldsymbol{\epsilon_j} \frac{\partial B_i}{\partial x_i} \right] \\
&= \frac{1}{\mu} \sum_k \left[\boldsymbol{\epsilon_k} \frac{\partial}{\partial x_k} \right] \cdot \left[\sum_{i,j} \boldsymbol{\epsilon_i} \left(\frac{1}{2} \delta_{ij} B^2 - B_i B_j \right) \boldsymbol{\epsilon_j} \right] \\
&= \nabla \cdot \left[\sum_{i,j} \frac{1}{\mu} \boldsymbol{\epsilon_i} \left(\frac{1}{2} \delta_{ij} B^2 - B_i B_j \right) \boldsymbol{\epsilon_j} \right]
\end{aligned} \tag{124}$$

which is indeed the divergence of a dyadic. By similar means one may demonstrate that

$$\mathbf{D} \times (\nabla \times \mathbf{E}) - \mathbf{E}(\nabla \cdot \mathbf{D}) = \nabla \cdot \left[\sum_{i,j} \epsilon \boldsymbol{\epsilon_i} \left(\frac{1}{2} \delta_{ij} E^2 - E_i E_j \right) \boldsymbol{\epsilon_j} \right]. \tag{125}$$

Putting these back into Eq. (124), we may write

$$-\mathbf{f} = \frac{\epsilon}{4\pi c} \frac{\partial}{\partial t} \left[(\mathbf{E} \times \mathbf{B}) \right] - \nabla \cdot \bar{\mathbf{T}} \tag{126}$$

where the components T_{ij} of the *Maxwell stress tensor* are

$$T_{ij} \equiv \frac{1}{4\pi} \left[\epsilon E_i E_j + \frac{1}{\mu} B_i B_j - \frac{1}{2} \delta_{ij} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right) \right]. \tag{127}$$

The Maxwell stress tensor is a symmetric tensor (in a uniform, isotropic, linear medium) and has the physical interpretation that $-T_{ij}$ is the j-component of the current density of the i-component of momentum. Note, however, that $\bar{\mathbf{T}}$ is not unique in that if we redefine it to include the curl of another dyadic, Eq. (116) would still hold.

Let us define also a vector field $\mathbf{g}$,

$$\mathbf{g} \equiv \frac{1}{4\pi c}(\mathbf{D} \times \mathbf{B}) \tag{128}$$

which is interpreted as the momentum density of the electromagnetic field, or the momentum per unit volume. Notice that for $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$,

$$\mathbf{g} = \frac{\epsilon\mu}{4\pi c}(\mathbf{E} \times \mathbf{H}) = \frac{\mu\epsilon}{c^2}\mathbf{S}; \tag{129}$$

there is thus a simple connection between the energy current density and the momentum density of the field. Our conservation law is now simply written as

$$-\mathbf{f} = \frac{\partial \mathbf{g}}{\partial t} - \nabla \cdot \bar{\mathbf{T}}; \tag{130}$$

if integrated over some domain V, it may also be expressed as

$$-\int_V d^3x\,\mathbf{f} = \int_V d^3x\,\frac{\partial \mathbf{g}}{\partial t} - \int_S d^2x\,(\mathbf{n} \cdot \bar{\mathbf{T}}). \tag{131}$$

Finally, let us define

$$\frac{d\mathbf{P}_m}{dt} \equiv \int_V d^3x\,\mathbf{f} \qquad \text{and} \qquad \frac{d\mathbf{P}_f}{dt} \equiv \int_V d^3x\,\frac{\partial \mathbf{g}}{\partial t}; \tag{132}$$

these are rather obviously meant to be the time rate of change of the mechanical momentum[9] and of the field momentum in V. Now we can write the conservation law as

$$\frac{d\mathbf{P}_m}{dt} + \frac{d\mathbf{P}_f}{dt} = \int_S d^2x\,\mathbf{n} \cdot \bar{\mathbf{T}}. \tag{133}$$

[9]Note, however, that it includes only the rate of change of mechanical momentum as a consequence of the forces applied to the particles by the electromagnetic field and does not include the change in mechanical momentum which comes about because particles are entering and leaving the domain V.

8.3 Need for Field Momentum

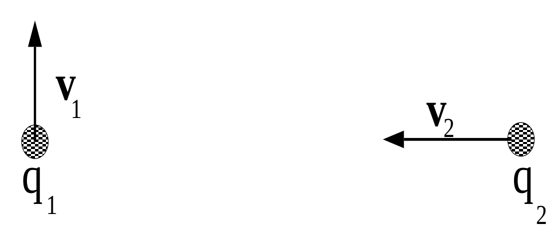
We defined our fields, the electric field and the magnetic induction, in terms of the force and torque, respectively, that they inflict upon an elementary source. However, as we have seen above, this definition is incomplete since the fields also carry energy and momentum. From a quantum point of view, this is obvious since we can think of the fields as resulting from the exchange of real and virtual photons each with momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$. However, from a 19'th century perspective the need of the field to carry energy and momentum (especially momentum) is less obvious.

However, Newton's law of action and reaction requires that the field carry momentum. First consider two completely isolated particles in free space. If the only force exerted on either particle is from its counterpart, then the net momentum $\mathbf{P}$ is conserved when the forces are equal and opposite.

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}\left(\mathbf{P}_1 + \mathbf{P}_2\right) = m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} = \mathbf{F}_1 + \mathbf{F}_2$$

I.e. the momentum is conserved when $\mathbf{F}_1 = -\mathbf{F}_2$

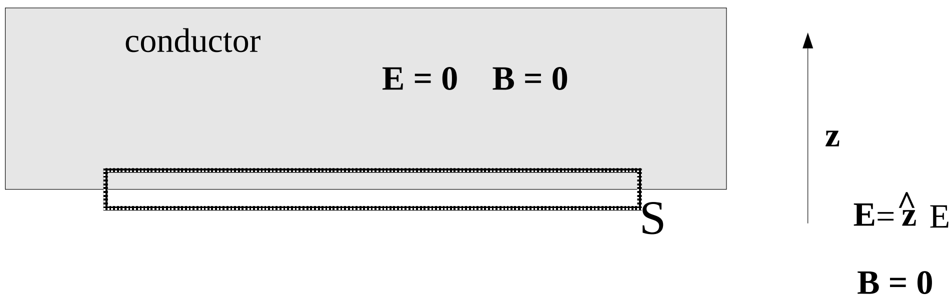
Now consider the situation where the particles are charged, and have trajectories as shown in the figure below.



Clearly the forces due to the electric field are equal and opposite, but those due to the magnetic fields are not. In fact, charge 1 exerts a magnetic force on charge 2, but charge 2 does not exert a magnetic force on charge 1. Momentum is not conserved by the particles (and the forces on the particles) alone. The excess momentum must be carried by the field! Thus, the field is required to have a momentum density by Newtonian law.

8.4 Example: Force on a Conductor

As an example, consider a conductor in the presence of an external electric field ($\mathbf{B} = 0$).



Since $\mathbf{B} = 0$ everywhere, $\mathbf{E} \times \mathbf{B} = 0$, and there is no electromagnetic momentum density. By conservation of momentum, the force on the conducting surface is then given

$$\left(\frac{d\mathbf{P}}{dt}\right)_i = F_i = \sum_j \int_S d^2x\, n_i T_{ij} \tag{134}$$

If we take the surface of integration as shown in the figure, then we need only consider the surface in the xy plane for which $\mathbf{n}$ is in the $-\hat{\mathbf{z}}$ direction. Thus we only need T_{ij} when $i = z$. Since $\mathbf{E} = \hat{\mathbf{z}}E$, and since

$$T_{ij} = \frac{1}{4\pi}\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) \tag{135}$$

when $\mathbf{B} = 0$, we have

$$T_{zx} = T_{zy} = 0 \qquad T_{zz} = \frac{1}{8\pi}E^2 \,. \tag{136}$$

Thus,

$$F_z = -\int_S d^2x\, \frac{1}{8\pi}E^2 = -\frac{A}{8\pi}E^2 \tag{137}$$

where A is the area of the conductor presented to the E-field. Thus the force on the conductor is downward

9 Conservation Laws for Macroscopic Systems

Regretfully omitted.

10 Poynting's Theorem for Harmonic Fields; Impedance, Admittance, etc.

Poynting's theorem has many important applications at the practical (electrical engineering) level. In this section we briefly make a foray into such an application. We shall restrict attention to harmonic fields which means ones that are harmonic (sine) functions of time. To this end we write the time-dependent part of the fields, as well as the sources, as $e^{-i\omega t}$ where ω is the angular frequency, and the physical field or source so represented is the real part of the complex mathematical function that we are using. For example,

$$\mathbf{E}(\mathbf{x},t) = \Re\left[\mathbf{E}(\mathbf{x})e^{-i\omega t}\right] \equiv \frac{1}{2}\left[\mathbf{E}(\mathbf{x})e^{-i\omega t} + \mathbf{E}^*(\mathbf{x})e^{i\omega t}\right]. \tag{138}$$

Similarly,

$$\begin{aligned}\mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t) &= \frac{1}{4}\left[\mathbf{J}(\mathbf{x})e^{-i\omega t} + \mathbf{J}^*(\mathbf{x})e^{i\omega t}\right]\cdot\left[\mathbf{E}(\mathbf{x})e^{-i\omega t} + \mathbf{E}^*(\mathbf{x})e^{i\omega t}\right] \\ &= \frac{1}{2}\Re\left[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})e^{-2i\omega t}\right].\end{aligned} \tag{139}$$

The time-average of the product is particularly simple; the term which varies as $e^{-2i\omega t}$ has a zero time-average and so we are left with

$$< \mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t) >= \frac{1}{2}\Re\left[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})\right]. \tag{140}$$

One can easily see that there is a general rule for the time-average of the product of two harmonic functions. Given $Q(\mathbf{x},t)$ and $R(\mathbf{x},t)$,

$$Q(\mathbf{x},t) = \Re\left[Q(\mathbf{x})e^{-i\omega t}\right] \qquad R(\mathbf{x},t) = \Re\left[R(\mathbf{x})e^{-i\omega t}\right], \tag{141}$$

the time average of the product may be expressed as

$$< Q(\mathbf{x},t)R(\mathbf{x},t) >= \frac{1}{2}\Re\left[R^*(\mathbf{x})Q(\mathbf{x})\right] = \frac{1}{2}\Re\left[R(\mathbf{x})Q^*(\mathbf{x}).\right] \tag{142}$$

We start by supposing that all fields are harmonic. Then they all have the complex form $\mathbf{F}(\mathbf{x})e^{-i\omega t}$ and they all have time derivatives give by

$$\frac{\partial(\mathbf{F}(\mathbf{x})e^{-i\omega t})}{\partial t} = -i\omega\mathbf{F}(\mathbf{x})e^{-i\omega t} \tag{143}$$

so that the Maxwell equations for the complex amplitudes (just the position-dependent parts of the fields) are

$$\begin{aligned} \nabla\cdot\mathbf{B}(\mathbf{x}) = 0 \qquad\qquad & \nabla\times\mathbf{E}(\mathbf{x}) - i\frac{\omega}{c}\mathbf{B}(\mathbf{x}) \\ \nabla\cdot\mathbf{D}(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \qquad\qquad & \nabla\times\mathbf{H}(\mathbf{x}) + i\frac{\omega}{c}\mathbf{D}(\mathbf{x}) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}). \end{aligned} \tag{144}$$

Notice that there is no problem in generalizing the Maxwell equations to complex fields because the equations involve linear combinations, with real coefficients, of complex objects. One thus has two sets of equations, one for the real parts of these objects and one for the imaginary parts. The set for the real parts comprises the "true" Maxwell equations.

For the remainder of this section, the symbols $\mathbf{E}$, $\mathbf{B}$, etc, stand for the complex amplitudes $\mathbf{E}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, ... We can rederive the Poynting theorem for these by starting from the inner product $\mathbf{J}^*\cdot\mathbf{E}$ and proceeding as in the original derivation. The result is

$$\mathbf{J}^*\cdot\mathbf{E} = \frac{c}{4\pi}\left[-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) - i\frac{\omega}{c}(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)\right]. \tag{145}$$

Define now the (complex) Poynting vector and energy densities

$$\mathbf{S} \equiv \frac{c}{8\pi}(\mathbf{E}\times\mathbf{H}^*), \tag{146}$$

and

$$w_e \equiv \frac{1}{16\pi}(\mathbf{E}\cdot\mathbf{D}^*) \qquad\qquad w_m \equiv \frac{1}{16\pi}(\mathbf{B}\cdot\mathbf{H}^*). \tag{147}$$

Notice that the real part of $\mathbf{S}$ is just the time-averaged real Poynting vector while the real parts of the energy densities are the time-averaged energy densities. More generally, the energy densities can be complex functions, depending on the relations between $\mathbf{D}$ and $\mathbf{E}$ and between $\mathbf{B}$ and $\mathbf{H}$. If the two members of each pair of fields are in phase with one another, then the corresponding energy density is real. Similarly, if $\mathbf{E}$ and $\mathbf{H}$ are in phase, then $\mathbf{S}$ is also real.

In terms of our densities, Poynting's theorem for harmonic fields becomes

$$\frac{1}{2}\mathbf{J}^* \cdot \mathbf{E} = -\nabla \cdot \mathbf{S} - 2i\omega(w_e - w_m). \tag{148}$$

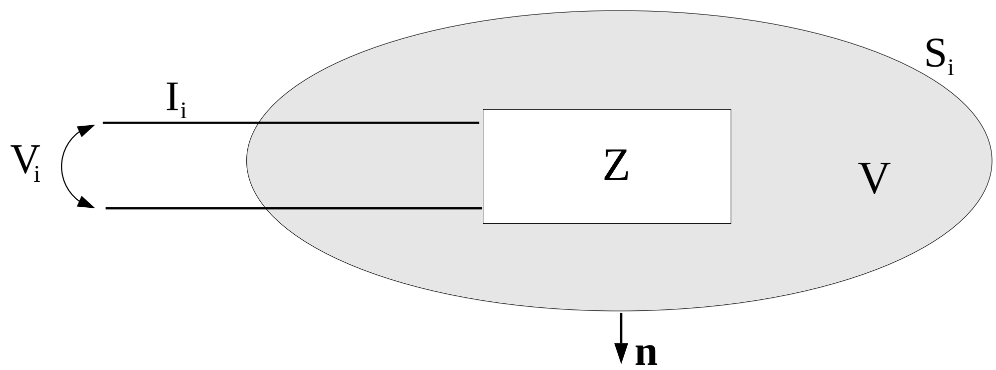
If we integrate this expression over some domain V and apply the divergence theorem to the term involving the Poynting vector, we find the relation

$$\frac{1}{2}\int_V d^3x\, \mathbf{J}^* \cdot \mathbf{E} + 2i\omega \int_V d^3x\, (w_e - w_m) + \oint_S d^2x\, \mathbf{S} \cdot \mathbf{n} = 0. \tag{149}$$

The interpretation of this equation is that the real part expresses the time-averaged conservation of energy. The imaginary part also has a meaning in connection with energy and its flow.

Consider first the simplest case of real w_e and w_m. Then the energy densities drop out of the real part of this equation and what it (the real part) tells us is that the time-average rate of doing work on the sources in V is equal to the time-averaged flow of energy (expressed by the Poynting vector) into V through the surface S. If the energy densities are not real, then there is an additional real piece in Eq. (149) so that the work done on the sources in V is not equal to the energy that comes in through S; this case corresponds to having "lossy" materials within V which dissipate additional energy.

Now let's suppose that there is some electromagnetic device within V, i.e., surrounded by S. Let it have two input terminals which are its only material communication with the rest of the world. At these terminals there are some input current I_i and voltage V_i which we suppose are harmonic and which may also be written in the form Eq. (138).



Then the (complex) input power is $I_i^* V_i/2$, meaning that the time-averaged input power is the real part of this quantity. Using our interpretation of the Poynting vector, we can express the input power in terms of a surface integral of the normal component of $\mathbf{S}$,

$$\frac{1}{2} I_i^* V_i = -\int_{S_i} d^2x\, \mathbf{S} \cdot \mathbf{n} \tag{150}$$

where the surface integral is done just over the cross-section of the (presumed) coaxial cable feeding power into the device; it is assumed that for such a cable, the input fields are confined to the region within the shield on the cable and so the integral over the remainder of the surface S surrounding the device has no contribution from the incident fields.

If we now combine this equation with Eq. (149), we find that we can write

$$\frac{1}{2} I_i^* V_i = \frac{1}{2} \int_V d^3x\, \mathbf{J}^* \cdot \mathbf{E} + 2i\omega \int_V d^3x\, (w_e - w_m) + \int_{S-S_i} d^2x\, \mathbf{S} \cdot \mathbf{n}. \tag{151}$$

The surface integral in this expression gives the power passing through the surface S, excluding the part through which the input power comes. The real part of this integral is the power radiated by the device.

Now let us define the *input impedance* Z of the device,

$$V_i \equiv Z I_i; \tag{152}$$

the impedance is complex and so can be written as

$$Z \equiv R - iX \tag{153}$$

where the *resistance* R and the *reactance* X are real. From Eq. (151) we find expressions for these:

$$R = \frac{1}{|I_i|^2}\left\{Re\left[\int_V d^3x\,\mathbf{J}^*\cdot\mathbf{E} + 2\int_{S-S_i} d^2x\,\mathbf{S}\cdot\mathbf{n}\right] + 4\omega Im\left[\int_V d^3x\,(w_m - w_e)\right]\right\} \tag{154}$$

and

$$X = \frac{1}{|I_i|^2}\left\{4\omega Re\left[\int_V d^3x\,(w_m - w_e)\right] - Im\left[\int_V d^3x\,\mathbf{J}^*\cdot\mathbf{E} + 2\int_{S-S_i} d^2x\,\mathbf{S}\cdot\mathbf{n}\right]\right\}. \tag{155}$$

By deforming the surface so that it lies far away from the device, one may make the integral over $\mathbf{S}\cdot\mathbf{n}$ purely real so that it does not contribute to the reactance; then it is only a part of the resistance and is the so-called "radiation resistance" which will be present if the device radiates a significant amount of power.

Our result has a simple and pleasing form at low frequencies. Then radiation is negligible and so the contributions of the surface integral may be ignored. Also, we may drop the term in the resistance proportional to ω. Then, assuming the current density and electric field are related by $\mathbf{J} = \sigma\mathbf{E}$ where σ is the (real) *electrical conductivity*, and assuming real energy densities, we find

$$R = \frac{1}{|I_i|^2}\int_V d^3x\,\sigma|\mathbf{E}|^2 \tag{156}$$

and

$$X = \frac{4\omega}{|I_i|^2}\int_V d^3x\,(w_m - w_e) \tag{157}$$

The last equation may be used to established contact between our expressions, based on the electromagnetic field equations, and some standard and fundamental relations in elementary circuit theory. If there is an inductance (magnetic energy-storing device) in the "black-box," then the integral of the magnetic energy may be expressed (see the first two or three problems at the end of Chap. 6 of Jackson) as Ł$|I_i|^2/4$, and so we find the familiar (if one knows anything about circuits) result that $X = L\omega$. But if there is a capacitor, the energy becomes $|Q_i|^2/4C$ where the charge Q_i is obtained by integrating the current over time; that gives $|Q_i|^2 = |I_i|^2/\omega^2$ and so $X = -1/\omega C$, another familiar tenet of elementary circuit theory.

11 Transformations: Reflection, Rotation, and Time Reversal

Before entering into discussion of the specific transformations of interest, we give a brief review of orthogonal transformations. Introduce a 3×3 matrix a with components a_{ij} and use it to transform a position vector $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ into a new vector $\mathbf{x}'$:

$$x_i' = \sum_j a_{ij} x_j. \tag{158}$$

An orthogonal transformation is one that leaves the length of the vector unchanged,

$$\sum_i (x_i')^2 = \sum_i x_i^2. \tag{159}$$

Using this condition, one may show that a must satisfy the conditions

$$\sum_i a_{ij} a_{ik} = \delta_{jk} \tag{160}$$

and

$$det(a) = \pm 1. \tag{161}$$

Orthogonal transformations with $det(a) = +1$ are simple rotations. The other ones are combinations of a rotation and an inversion[10]; these are called *improper rotations.*

It is common to refer to a collection of three objects $\psi_i,\ i = 1, 2, 3,$ which transform, under orthogonal transformations, in the same way as the components of $\mathbf{x}$, as a *vector*, a *polar vector*, or a *rank-one tensor.* A collection of nine objects $q_{ij},\ i, j = 1, 2, 3,$ which transform in the same way as the nine objects $x_i x_j$ is called a *rank-two tensor.* And so on. An object which is *invariant*, that is, which is unchanged under an orthogonal transformation, is called a *scalar* or *rank-zero tensor*; the length of a vector is such an object.

One also defines *pseudotensors* of each rank. A *rank-p pseudotensor* comprises a set of 3^p objects which transform in the same way as a rank-p tensor under ordinary

[10] An inversion is a transformation $\mathbf{x}' = -\mathbf{x}$.

rotations but which transform with an extra change of sign relative to a rank-p tensor under improper rotations. One also uses the terms *pseudoscalar* for a rank-0 pseudotensor and *pseudovector* or *axial vector* for a rank-1 pseudotensor. Notice that under inversion, for which a is just the negative of the unit 3×3 matrix, a vector changes sign, $\mathbf{x}' = -\mathbf{x}$, while a pseudovector is invariant. This statement can be generalized: Under inversion, a tensor T of rank n transforms to T' with

$$T' = (-1)^n T. \tag{162}$$

A pseudotensor P of the same rank, on the other hand, transforms according to

$$P' = (-1)^{(n+1)} P \tag{163}$$

under inversion.

11.1 Transformation Properties of Physical Quantities

It is important to realize that objects which we are accustomed to referring to as "vectors," such as $\mathbf{B}$, are not necessarily vectors in the sense introduced here; indeed, it is one of our tasks in this section to find out just what sorts of tensor are the various physical quantities we have been studying. Consider for example the charge density. Suppose that we have a system with a certain $\rho(\mathbf{x})$ and that we rotate it; then ρ becomes ρ' and $\mathbf{x}$ becomes $\mathbf{x}'$.

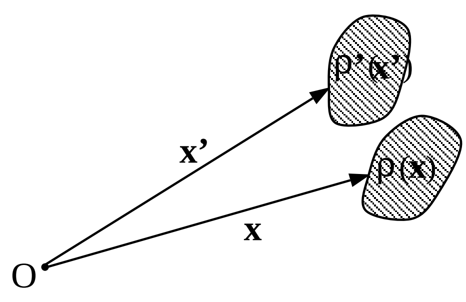


Figure 14: Under a rotation $\rho(\mathbf{x}) = \rho'(\mathbf{x}')$

The question is, how is $\rho'(\mathbf{x}')$ related to $\rho(\mathbf{x})$? It is easy to see, since $\mathbf{x}'$ is what $\mathbf{x}$ becomes as a consequence of the rotation, that $\rho'(\mathbf{x}') = \rho(\mathbf{x})$. Under an inversion also, this relation is true. Hence we conclude that the charge density is a scalar or rank-0 tensor.

An example of a vector or rank-1 tensor is, of course, $\mathbf{x}$. Similarly, from this fact one may show that the operator ∇ is a rank-1 tensor (Differential operators can also be tensors or components of tensors);

$$\frac{\partial}{\partial x_i'} = \sum_j a_{ij} \frac{\partial}{\partial x_j}. \tag{164}$$

What then is $\nabla \rho$? From the (known) transformation properties of ρ and of ∇, it is easy to show that it is a rank-1 tensor. The gradient of any scalar function is a rank-1 tensor. Similarly, one may show that the inner product of two rank-1 tensors, or vectors, is a scalar as is the inner product of two rank-1 pseudotensors; the inner product of a rank-1 tensor and a rank-1 pseudotensor is a pseudoscalar; and the gradient of a pseudoscalar is a rank-1 pseudotensor.

All of the foregoing are quite easy to demonstrate. A little harder is the crossproduct of two vectors (rank-1 tensors). Consider that $\mathbf{b}$ and $\mathbf{c}$ are rank-1 tensors. Their cross product may be written as

$$\mathbf{u} = \mathbf{b} \times \mathbf{c} \tag{165}$$

with a Cartesian component given by

$$u_i = \sum_{j,k} \epsilon_{ijk} b_j c_k \tag{166}$$

where

$$\epsilon_{ijk} \equiv \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1 & \text{if } (i,j,k) = (2,1,3), (1,3,2), (3,2,1) \\ 0 & \qquad\qquad \text{otherwise.} \end{cases} \tag{167}$$

If we **define** ϵ_{ijk} to be given by this equation in all frames, then we can **show** that it is a rank-3 pseudotensor. Alternatively, we can use Eq. (164) to specify it in a single frame, **define** it to be a rank-3 pseudotensor, and then **show** that it is given by Eq. (164) in **any** frame. However one chooses to do it, one can use this object, called

Table 1: Rotation, inversion, and time-reversal properties of some common mechanical quantities.

Function	Rank	Inversion Symmetry	Time-reversal Symmetry
$\mathbf{x}$	1	$-$ (vector)	$+$
$\mathbf{v} = d\mathbf{x}/dt$	1	$-$ (vector)	$-$
$\mathbf{p} = m\mathbf{v}$	1	$-$ (vector)	$-$
$\mathbf{L} = \mathbf{x} \times m\mathbf{v}$	1	$+$ (pseudovector)	$-$
$\mathbf{F} = d\mathbf{p}/dt$	1	$-$ (vector)	$+$
$\mathbf{N} = \mathbf{x} \times \mathbf{F}$	1	$+$ (pseudovector)	$+$
$T = p^2/2m$	0	$+$ (scalar)	$+$
V	0	$+$ (scalar)	$+$

the *completely antisymmetric unit rank-3 pseudotensor*, and the assumed transformation properties of $\mathbf{b}$ and $\mathbf{c}$ (rank-1 tensors) to determine the transformation properties of the crossproduct. What one finds is that

$$u_i' = det(a) \sum_j a_{ij} u_j \tag{168}$$

which means that $\mathbf{u}$ is a pseudovector or a rank-1 pseudotensor.

The transformations considered so far have all dealt with space; to them we wish to add the time-reversal transformation. The question to ask of a given entity is how it changes if time is reversed. Imagine making a videotape of the entity's behavior and then running the tape backwards. If, in this viewing, the quantity is the same at a given point on the tape as when the tape is running forward, then the quantity is *even* or *invariant* under time reversal. If its sign has been reversed, then it is *odd* under time reversal. For example, the position $\mathbf{x}(t)$ of an object is even under time reversal; the velocity of the object, however, is odd.

In Table 1, we catalog some familiar mechanical functions according to their rotation, inversion, and time-reversal symmetries.

Table 2: Rotation, inversion and time-reversal properties of some common electromagnetic quantities.

function	rank	inversion symmetry	time-reversal symmetry
ρ	0	+ (scalar)	+
$\mathbf{J}$	1	− (vector)	−
$\mathbf{E}, \mathbf{D}, \mathbf{P}$	1	− (vector)	+
$\mathbf{B}, \mathbf{H}, \mathbf{M}$	1	+ (pseudovector)	−
$\mathbf{S}, \mathbf{g}$	1	− (vector)	−
$\bar{\mathbf{T}}$	2	+ (tensor)	+

We may make the same sort of table for various electromagnetic quantities, basing our analysis on the Maxwell equations, which we assume to be the correct equations of electromagnetism. Given that ρ is a scalar and that ∇ is a vector, the equation $\nabla \cdot \mathbf{E} = 4\pi\rho$ tells us that the electric field is a vector; further, it is even under time reversal (since ρ and ∇ are both even). Similarly, $\mathbf{D}$ and $\mathbf{P}$ must be vectors and even under time reversal. Moving on to Faraday's Law, $\nabla \times \mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t$, from our knowledge of the properties of the gradient, the cross product, and the electric field, we see that $\mathbf{B}$ is a pseudovector and that it is odd under time reversal; $\mathbf{H}$ and $\mathbf{M}$ have the same properties. Finally, Ampère's Law, $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J} + c^{-1}\partial\mathbf{E}/\partial t$ is consistent with these determinations and with the statement that $\mathbf{J}$ is a vector, odd under time reversal, which follows from the fact that $\mathbf{J} = \rho\mathbf{v}$. Finally, $\mathbf{S}$ and $\mathbf{g}$ are vectors with odd time-reversal symmetry while the Maxwell stress tensor is a rank-2 tensor, even under time reversal. These properties are summarized in Table 2.

The usefulness of these expressions lies in the belief that acceptable equations of physics should be invariant under various symmetry operations. The Maxwell equations and the classical equations of mechanics (Newton's Laws), for example, are invariant under time reversal and under orthogonal transformations, meaning that each term in any given equation transforms in the same way as all of the other terms in that equation[11]. If we believe that this should be true of all elementary equations of classical physics, then there are certain implied constraints on the form of the equations. Consider as an example the relation between **P** and **E**. Supposing that one can make an expansion of a component of **P** in terms of the components of **E**, we have

$$P_i = \sum_j \alpha_{ij} E_j + \sum_{jk} \beta_{ijk} E_j E_k + \sum_{jkl} \gamma_{ijkl} E_j E_k E_l + ... \tag{169}$$

where, since **P** and **E** are both rank-1 tensors, invariant under time reversal, it follows, using the invariance argument, that the coefficients α_{ij} are the components of a rank-2 tensor, invariant under time reversal; the β_{ijk} are components of a rank-3 tensor, invariant under time reversal; and the γ_{ijkl} are components of a rank-4 tensor, also invariant under time reversal.

If we now add some statement about the properties of the medium, we can get further conditions. In the simplest case of an isotropic material, it must be the case that each of these tensors is invariant under orthogonal transformations. This condition severely limits their forms; in particular, it means that $\alpha_{ij} = \alpha\delta_{ij}$. We can see this by appealing to the transformation properties of second rank tensors. Thus, α_{ij} must transform like $x_i x_j$, or

$$\alpha'_{nm} = a_{ni} a_{mj} \alpha_{ij} \tag{170}$$

Since the medium is isotropic, we require that $\alpha'_{ij} = \alpha_{ij}$. The only way to satisfy both of these conditions of transformation is if

$$\alpha'_{nm} = \alpha a_{ni} a_{mi} = \alpha\delta_{nm} \tag{171}$$

The same type of thing cannot be done with β, so that $\beta_{ijk} = 0$, bu we can perform similar manipulations on γ so the coefficients γ_{ijkl} are such as to produce

$$\sum_{jkl} \gamma_{ijkl} E_i E_j E_l = \gamma(\mathbf{E} \cdot \mathbf{E})\mathbf{E} \tag{172}$$

[11] Of course there are some equations, like Ohm's Law, which describe truly irreversible processes for which time reversal invariance does not hold.

where γ is a constant. Thus, through third-order terms, the expansion of $\mathbf{P}$ in terms of $\mathbf{E}$ must have the form

$$\mathbf{P} = \alpha\mathbf{E} + \gamma E^2\mathbf{E}. \tag{173}$$

The general forms of many other less obvious relations may be determined by similar considerations.

12 Do Maxwell's Equations Allow Magnetic Monopoles?

The answer is yes, but only in a restricted, and trivial, sense. If there were magnetic charges of density ρ_m and an associated magnetic current density $\mathbf{J}_m$, with a corresponding conservation law

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \mathbf{J}_m = 0, \tag{174}$$

then the field equations would read

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 4\pi\rho_m & \qquad \nabla \times \mathbf{H} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} &= 4\pi\rho & \qquad \nabla \times \mathbf{E} &= -\frac{4\pi}{c}\mathbf{J}_m - \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \tag{175}$$

In fact, the Maxwell equations as we understand them can be put into this form by making a particular kind of transformation, called a *duality transformation* of the fields and sources. Introduce

$$\begin{aligned} \mathbf{E} &= \mathbf{E}'\cos\eta + \mathbf{H}'\sin\eta & \qquad \mathbf{D} &= \mathbf{D}'\cos\eta + \mathbf{B}'\sin\eta \\ \mathbf{H} &= -\mathbf{E}'\sin\eta + \mathbf{H}'\cos\eta & \qquad \mathbf{B} &= -\mathbf{D}'\sin\eta + \mathbf{B}'\cos\eta \\ \rho &= \rho'\cos\eta + \rho'_m\sin\eta & \qquad \mathbf{J} &= \mathbf{J}'\cos\eta + \mathbf{J}'_m\sin\eta \\ \rho_m &= -\rho'\sin\eta + \rho'_m\cos\eta & \qquad \mathbf{J}_m &= -\mathbf{J}'\sin\eta + \mathbf{J}'_m\cos\eta. \end{aligned} \tag{176}$$

where η is an arbitrary real constant.

If one now substitutes these into the generalized field equations, one finds, upon separating the coefficients of $\sin\eta$ from those of $\cos\eta$ (These must be independent because η is arbitrary), that the primed fields and sources obey an identical set of field equations. What this means is that the Maxwell equations (with no magnetic sources) may be thought of as a special case of the generalized field equations, one in which η is chosen so that ρ_m and $\mathbf{J}_m$ are equal to zero. From the form of the transformations for the sources, we see that this is possible if the ratio of ρ to ρ_m for each source (particle)is the same as that for all of the other sources (particles). Hence it is meaningless to say that there are no magnetic monopoles; the real question is whether all elementary particles have the same ratio of electric to magnetic charge. If they do, then Maxwell's equations are correct and correspond, as stated above, to a particular choice of η in the more general field equations.

If one subjects the electron and proton to scrutiny regarding the question of whether they have the same ratio of electric to magnetic charge, one finds that if one defines (by choice of η) the magnetic charge of the electron to be zero, then experimentally the magnetic charge of the proton is known to be smaller than 10^{-24} of its electric charge. That's pretty good evidence for its being zero.

But there remains the question whether there are other kinds of particles, not yet discovered, which have a different ratio ρ/ρ_m than do electrons and protons. Dirac, for example, has given a simple and clever argument which shows that the quantization of the electric charge follows from the mere existence of an electrically uncharged magnetic monopole. Moreover, the argument gives the magnetic charge g of the monopole as $g = nhc/4\pi e$ where n is any integer and h is Planck's constant. This is, in comparison with the electric charge, very large so that it ought to be in principle easy to detect a "Dirac monopole" should there by any of them around. So far, none has been reliably detected.

A Helmholtz' Theorem

Any vector function of position $\mathbf{C}(x)$ can be written as the sum of two vector functions such that the divergence vanishes for one and the curl vanishes for the other. In other words, the decomposition

$$\mathbf{C}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) + \mathbf{F}(\mathbf{x}) \tag{177}$$

is always possible, where

$$\nabla \cdot \mathbf{D} = 0 \quad \nabla \times \mathbf{F} = 0 \tag{178}$$

Proof. We may satisfy the two conditions for $\mathbf{F}$ and $\mathbf{D}$, by writing

$$\mathbf{D} = \nabla \times \mathbf{A} \qquad \mathbf{F} = -\nabla \Phi \,. \tag{179}$$

Then taking the curl and divergence of these equations respectively, we can write

$$\nabla^2 \Phi = -\nabla \cdot \mathbf{C} \qquad \nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{C} \,. \tag{180}$$

We already know how to solve these solutions (at least in Cartesian coordinates).

$$\Phi(\mathbf{x}') = \frac{1}{4\pi} \int_V d^3x \, \frac{\nabla \cdot \mathbf{C}(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} \qquad \mathbf{A}(\mathbf{x}') = \frac{1}{4\pi} \int_V d^3x \, \frac{\nabla \times \mathbf{C}(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} \tag{181}$$

Since $\mathbf{D}$ and $\mathbf{F}$ can now be found from these potentials, we have demonstrated the decomposition claimed by Helmholtz' Theorem, and thus proved it.

An interesting corollary of this theorem is that a vector function is completely determined if its curl and divergence are known everywhere. The field $\mathbf{F} = -\nabla \Phi$, where produced by a point source, is longitudinal to the vector from the source to the point where the field is evaluated. The field $\mathbf{D} = \nabla \times \mathbf{A}$ is transverse to the vector from the source to the field point. Thus $\mathbf{F}$ is typically called the longitudinal, and $\mathbf{D}$ the transverse, part of $\mathbf{C}$.