Static and Stationary Magnetic Fields

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(1775 - 1836)

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1 Introduction and Definitions

As far as anyone knows, there is no such thing as a free magnetic charge or *magnetic monopole*, although there are people who look for them (and occasionally claim to have found one); certainly they may exist. Because no known phenomena require their existence, we shall develop magnetostatics and eventually electrodynamics assuming that they do not exist. In this case there is a fundamental difference between electrostatics and magnetostatics, explaining in part why the two subjects developed independently and were regarded as distinct rather than different limits or aspects of one type of phenomenon (electromagnetic phenomena).

1.1 Magnetic Induction

In the absence of monopole moments, the most fundamental source of magnetic effects is the *magnetic dipole*. In the presence of other magnetic materials, a point dipole will experience some force. One defines the *magnetic flux density* or *magnetic induction* $\mathbf{B}$ in terms of the torque $\mathbf{N}$ exerted on the dipole. Given that the dipole moment is μ, the defining relation is

$$\mathbf{N} \equiv \mu \times \mathbf{B}. \tag{1}$$

Thus, as for electrostatics, the basic field of magnetostatics is defined by the effect produced on an elementary source.

1.2 Current Density and Conservation

Among the first known manifestations of magnetic phenomena were the forces observed to act on some materials (magnets) as a consequence of the earth's magnetic field. In 1819, Hans Christian Oersted (1777-1851) found that very similar effects could be produced by placing a magnet close to a current-carrying wire, indicating a connection between electrical current and magnetism. Much of what we have to say about magnetostatics will involve the use of currents as sources of $\mathbf{B}$, so let us say a few words about the properties of stationary, *i.e.,* time-independent, currents. We shall write the *current density* as $\mathbf{J}(\mathbf{x})$; it has dimensions of charge/area-time and is by definition such that a component J_i is the amount of charge that crosses unit area in unit time given that the normal to the surface is in the i-direction. Given a charge density $\rho(\mathbf{x})$ moving at a velocity $\mathbf{v}$, there is a current density

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{v}. \tag{2}$$

It is an experimental fact that charge is conserved. We may determine a continuity or conservation equation which expresses this fact.

Consider

$$\int_V d^3x \ \nabla \cdot \mathbf{J}(\mathbf{x}) = \int_S d^2x \ \mathbf{J} \cdot \mathbf{n}. \tag{3}$$

The surface integral gives the rate at which charge flows out of the domain V through the surface S. Because charge is conserved, this must be equal to the negative of the rate at which the total charge inside of V changes:

$$\int_S d^2x \ \mathbf{J}(\mathbf{x}) \cdot \mathbf{n} = -\frac{\partial}{\partial t} \left(\int_V d^3x \ \rho(\mathbf{x}) \right). \tag{4}$$

Assuming that V is independent of time, we may move the derivative with respect to time inside of the integral and so have

$$\int_V d^3x \ \nabla \cdot \mathbf{J}(\mathbf{x}) = \int_S d^2x \ \mathbf{J}(\mathbf{x}) \cdot \mathbf{n} = -\int_V d^3x \ \frac{\partial \rho(\mathbf{x})}{\partial t}. \tag{5}$$

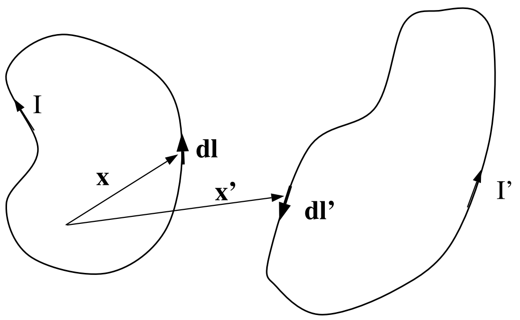
Now argue in the usual fashion: Because V is an arbitrary domain, this equation can only be true if the integrands on the two sides are the same everywhere. Hence we have

$$\nabla \cdot \mathbf{J}(\mathbf{x}) + \frac{\partial \rho(\mathbf{x})}{\partial t} = 0. \tag{6}$$

This equation is true so long as charge is *locally* conserved, meaning that the only way for charge to appear in V (or to disappear from V) is by flowing across the boundary. The equation has the typical form of a continuity equation which is that the divergence of the current density of some quantity plus the time derivative of the density of that quantity equals zero. If there are sources (or sinks) of the quantity in question, there is an additional term in the equation expressing the contribution of these sources.

2 Ampère's Law

Even as currents in wires produce forces on magnetic materials, so do they produce forces on other current loops. Félix Savart (1791-1841) and Jean-Baptiste Biot (1774-1862) began experiments on these forces soon after Oersted's discovery, as did André-Marie Ampère. Ampère continued his experiments for some years and published his collected results in 1825. The basic law emerging from Ampère's work deals with the forces acting between closed current loops. Suppose that we have a current I in one loop and a current I' in a second. Let $\mathbf{F}$ be the force acting on the loop carrying current I.



Then *Ampère's Law* may be expressed as follows:

$$\mathbf{F} = kII' \int\int \frac{d\mathbf{l} \times [d\mathbf{l}' \times (\mathbf{x} - \mathbf{x}')]}{|\mathbf{x} - \mathbf{x}'|^3} \quad (7)$$

where the integrals over $\mathbf{l}$ and $\mathbf{l}'$ are, respectively, around the loops carrying currents I and I'; $\mathbf{x}$ and $\mathbf{x}'$ are the position vectors of the integration points. The constant k depends on the units employed. For our units, with current expressed as *statcoul/sec*, or *statamp*, k has dimensions of (inverse speed)^2 (remember that charge has has dimensions of $M^{1/2}L^{3/2}/T$ in our units). Hence one writes $k \equiv 1/c^2$ where c is a speed. From appropriate experiments one may find that its numerical value in cgs units is $c = 2.998 \times 10^10\, cm/sec$. We recognize this as the speed of light, but that is, for the moment, not important.

At this juncture we may introduce the magnetic induction by writing the force as

$$\mathbf{F} \equiv \frac{I}{c} \int d\mathbf{l} \times \mathbf{B}(\mathbf{x}) \tag{8}$$

where $\mathbf{B}(\mathbf{x})$ is the magnetic induction produced by the other loop's current. It is not yet clear apparent this $\mathbf{B}$ is the same as the one introduced in Eq. (1); nevertheless, it is, as we shall see presently.

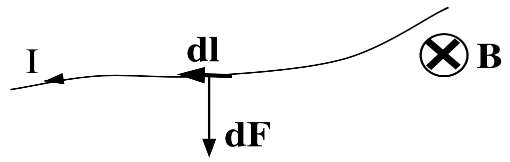
Comparison of Eq. (7) with Eq. (8) shows that the magnetic induction produced by the loop carrying current I' may be written as an integral over that loop,

$$\mathbf{B}(\mathbf{x}) = \frac{I'}{c} \int \frac{d\mathbf{l}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}; \tag{9}$$

this equation is often called the *Biot-Savart Law.*

One also writes these equations in differential form, although that may introduce some inaccuracies and even misunderstanding. First, the force acting on just an infinitesimal piece of the loop carrying current I is

$$d\mathbf{F}(\mathbf{x}) = \frac{I}{c} d\mathbf{l} \times \mathbf{B}(\mathbf{x}). \tag{10}$$



The correct interpretation of this equation is that it expresses the force on the element $d\mathbf{l}$ of the loop carrying current I which is produced by the current I' in the other loop; $\mathbf{B}(\mathbf{x})$ is the magnetic induction produced by this other loop. There will be additional forces on the element dl produced by the current in other parts of its own loop.

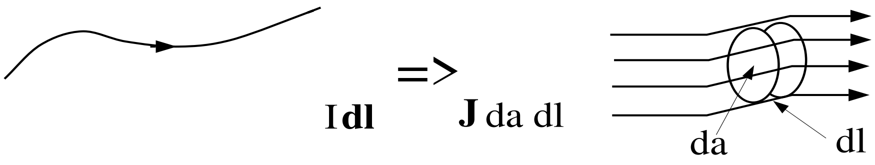
Another equation one frequently sees is an expression for the infinitesimal magnetic induction produced at a point $\mathbf{x}$ by an infinitesimal element of a source loop. Given that the source loop is, as above, the one carrying current I', this expression is

$$d\mathbf{B}(\mathbf{x}) = \frac{I'}{c} \frac{d\mathbf{l}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{11}$$

This is, however, not a correct statement in that the element $d\mathbf{l}'$ of this circuit acting alone does not produce such a magnetic induction. First, it is impossible to have such a source acting alone if there is no time dependence in the sources and fields; the flowing charge which gives the current I' in $d\mathbf{l}'$ has to come from somewhere and go somewhere and so if this element is the entire source current distribution, then there must be some time dependence associated with the accumulation and depletion of charge at the two ends of the element. When this time dependence is included, Eq. (11) will not give the magnetic induction correctly.

2.1 Induction of an Arbitrary Current Density

The preceding results are capable of generalization to arbitrary current distributions (not just filaments). One has to replace $Id\mathbf{l}$ by $\mathbf{J}(\mathbf{x})dadl$ where $\mathbf{J}(\mathbf{x})$ is the current density, da is the cross-sectional area of the filament, and dl is the magnitude of $d\mathbf{l}$. Then note that $dadl = d^3x$, a volume element.



Hence one finds that the flux produced by an extended current distribution is

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \tag{12}$$

while the force on an extended current distribution $\mathbf{J}(\mathbf{x})$ produced by some externally applied $\mathbf{B}(\mathbf{x})$ is

$$\mathbf{F} = \frac{1}{c}\int d^3x\ \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}). \tag{13}$$

The corresponding force density at point $\mathbf{x}$ is

$$d\mathbf{F} = \frac{1}{c} d^3x\, \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}), \tag{14}$$

and the expression for $d\mathbf{B}$ (not to be trusted) becomes

$$d\mathbf{B}(\mathbf{x}) = \frac{1}{c} d^3x\, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{15}$$

Finally, let's add to our arsenal of equations one for the torque $\mathbf{N}$ felt by a current distribution $\mathbf{J}(\mathbf{x})$ acted upon by an externally applied magnetic induction $\mathbf{B}(\mathbf{x})$. Given an object experiencing a force $\mathbf{F}$, the torque relative to a point O is just $\mathbf{x} \times \mathbf{F}$ where $\mathbf{x}$ is the location of the object relative to O. Hence the torque on the current distribution is

$$\mathbf{N} = \frac{1}{c}\int d^3x\, [\mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}))]. \tag{16}$$

2.2 An Alternate Form of Ampère's law

Before going on to additional formalism, let us express Ampère's law, Eq. (7), in a more symmetric form by applying some vector manipulations. First, we will make use of the general identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ to have

$$\mathbf{F} = \frac{II'}{c^2}\int\int \frac{[d\mathbf{l} \cdot (\mathbf{x} - \mathbf{x}')]d\mathbf{l}' - [d\mathbf{l} \cdot d\mathbf{l}'](\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{17}$$

According to Newton, the force of the loop carrying current I should be equal and opposite to the force on the other loop, implying certain symmetries in the integrand above. The second term in the numerator changes sign under interchange of $\mathbf{x}$ and $\mathbf{x}'$, but the first does not. Let us study the latter more closely. Consider

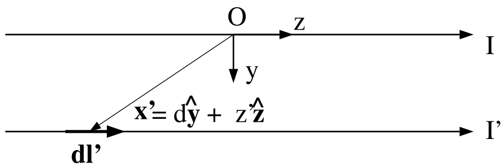
$$\int d\mathbf{l}' \int d\mathbf{l} \cdot \left[-\nabla \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \right] = -\int d\mathbf{l}' \int d \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \equiv 0 \tag{18}$$

where the last step follows from the fact we integrate over a closed path[1]. Thus we find that the force $\mathbf{F}$ may be written simply as

$$\mathbf{F} = -\frac{II'}{c^2} \int\int \frac{d\mathbf{l} \cdot d\mathbf{l}'}{|\mathbf{x}-\mathbf{x}'|^3} (\mathbf{x}-\mathbf{x}'). \tag{19}$$

2.3 Example: Force Between Parallel Wires

Let us look at an example. Suppose we have two parallel wires a distance d apart carrying currents I and I' and wish to find the force per unit length acting on one of them. At a point $\mathbf{x}$ on the one carrying current I, there is an induction B from the other one which is directed perpendicular to the plane containing the wires. This field is given by the integral over the sources in the other wire,



[1] For example, $\oint d\phi = \phi(x) - \phi(x)$, where x is an arbitrary point along the contour

$$|\mathbf{B}| = B = \frac{I'}{c}\int \frac{d\mathbf{l}' \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} = \frac{I'd}{c}\int_{-\infty}^{\infty} \frac{dz'}{(d^2+z'^2)^{3/2}} = \frac{I'}{cd}\int_{-\infty}^{\infty} \frac{du}{(1+u^2)^{3/2}} = \frac{2I'}{cd}. \tag{20}$$

The consequent force on the current I in length dz is

$$|d\mathbf{F}| = \frac{I}{c}\,dz\,B = \frac{2II'}{c^2 d}. \tag{21}$$

The direction of $\mathbf{F}$ is such that the wires are attracted toward each other if the currents are in the same direction and they are repelled if the currents are in opposite directions. In doing this calculations we have conveniently ignored the fact that we didn't deal with closed current loops; in principal the wires have to be bent into closed loops somewhere far away from where we have calculated the force. With a little work one can convince himself that for sufficiently large loops, the contributions from the part that we ignored will be as small as desired.

3 Differential Equations of Magnetostatics

Even as we found equations for the divergence and curl of the electric field, so can we find such equations for the magnetic induction. Our starting point is the integral expression for the induction produced by a source distribution $\mathbf{J}(\mathbf{x})$,

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c}\int d^3x'\,\mathbf{J}(\mathbf{x}') \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{22}$$

or, proceeding in the same manner as we have done before,

$$\mathbf{B}(\mathbf{x}) = -\frac{1}{c}\int d^3x'\,\mathbf{J}(\mathbf{x}')\times\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{23}$$

Now apply the identity $\nabla\times(f\mathbf{A}) = (\nabla\times\mathbf{A})f + \nabla f\times\mathbf{A}$:

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) = (\nabla\times\mathbf{J}(\mathbf{x}'))\frac{1}{|\mathbf{x}-\mathbf{x}'|} + \nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\mathbf{J}(\mathbf{x}'). \tag{24}$$

The curl (taken with respect to components of $\mathbf{x}$) of $\mathbf{J}(\mathbf{x}')$ is zero, so, upon substituting into Eq. (24), we find

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c}\int d^3x'\,\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{1}{c}\nabla\times\left(\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{25}$$

Because $\mathbf{B}(\mathbf{x})$ is the curl of a vector field, its divergence must be zero,

$$\nabla\cdot\mathbf{B}(\mathbf{x}) = 0. \tag{26}$$

This is our equation for the divergence of the magnetic induction. It tells us that there are no magnetic charges.

To find a curl equation for $\mathbf{B}(\mathbf{x})$, we take the curl of Eq. (25),

$$\nabla\times\mathbf{B}(\mathbf{x}) = \frac{1}{c}\nabla\times\left[\nabla\times\left(\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)\right]. \tag{27}$$

Now employ the identity

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}, \tag{28}$$

valid for any vector field $\mathbf{A}(\mathbf{x})$; the last term on the right-hand side of this identity is to be interpreted as

$$\nabla^2\mathbf{A} \equiv (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}; \tag{29}$$

it is important that this relation is written in Cartesian coordinates. Using the identity, we find

$$\nabla\times\left[\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)\right]=\nabla\left[\mathbf{J}(\mathbf{x}')\cdot\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]-\mathbf{J}(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{30}$$

The second term is just $4\pi\mathbf{J}(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')$, so we have

$$\nabla\times\mathbf{B}(\mathbf{x})=\frac{4\pi}{c}\mathbf{J}(\mathbf{x})+\frac{1}{c}\nabla\left[\int d^3x'\,\mathbf{J}(\mathbf{x}')\cdot\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]. \tag{31}$$

The remaining integral may be completed as follows:

$$\begin{aligned}\nabla\int d^3x'\,\mathbf{J}(\mathbf{x}')\cdot\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)&=-\nabla\int d^3x'\,\mathbf{J}(\mathbf{x}')\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\&=-\nabla\int d^3x'\,\nabla'\cdot\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)+\nabla\int d^3x'\,\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned} \tag{32}$$

The first term in the final expression can be converted to a surface integral which then vanishes for a localized current distribution which lies totally within the domain enclosed by the surface. The integrand of the second term involves $\nabla'\cdot\mathbf{J}(\mathbf{x}')=-\partial\rho(\mathbf{x}')/\partial t\equiv 0$ for a steady-state current distribution. Consequently we have the curl equation

$$\nabla\times\mathbf{B}(\mathbf{x})=\frac{4\pi}{c}\mathbf{J}(\mathbf{x})\text{ Sometimes called Ampere's Law} \tag{33}$$

The curl and divergence equations, $\nabla\cdot\mathbf{B}(\mathbf{x})=0$ and $\nabla\times\mathbf{B}(\mathbf{x})=(4\pi/c)\mathbf{J}(\mathbf{x})$ plus an appropriate statement about the behavior of $\mathbf{B}(\mathbf{x})$ on a boundary tell us all we need to know to find the magnetic induction for a given set of sources $\mathbf{J}(\mathbf{x})$. There are, of course, also integral versions of these differential equations. One finds the most common forms of them from the Stokes theorem and the divergence theorem.

$$\int_V d^3x\, \nabla \cdot \mathbf{B}(\mathbf{x}) = \int_S d^2x\, \mathbf{B}(\mathbf{x}) \cdot \mathbf{n} = 0 \tag{34}$$

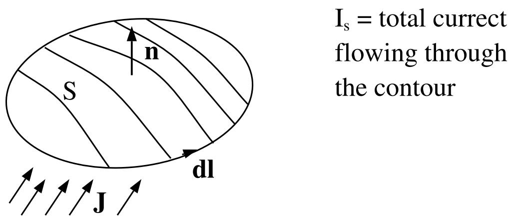
and

$$\int_S d^2x\, (\nabla \times \mathbf{B}(\mathbf{x})) \cdot \mathbf{n} = \int_C d\mathbf{l} \cdot \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c} \int_S d^2x\, \mathbf{J}(\mathbf{x}) \cdot \mathbf{n}. \tag{35}$$

The last of these is also commonly written as

$$\int_C d\mathbf{l} \cdot \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c} I_s \tag{36}$$

where I_s is the total current passing through the surface S in the direction of a right-hand normal relative to the direction in which the line integral around C is done.



This relation is frequently called Ampère's law. One interesting feature of this equation is that the result is independent of the actual surface S employed so long as it is an open surface that ends on the path C. The amount of charge passing through all such surfaces per unit time *i.e.*, I_s, is independent of S because $\nabla \cdot \mathbf{J}(\mathbf{x}) = 0$.

4 Vector and Scalar Potentials

In all electromagnetic systems it is possible to devise a potential which is a vector field for $\mathbf{B}(\mathbf{x})$; it is called a *vector potential*. Sometimes, it is also possible to devise a scalar potential for $\mathbf{B}(\mathbf{x})$. We shall consider the latter first.

4.1 Scalar Potential

As we have seen in the study of electrostatics, one can write a vector field as the gradient of a scalar if the vector field has zero divergence. This is the case for $\mathbf{B}(\mathbf{x})$ in those regions of space where $\mathbf{J}(\mathbf{x}) = 0$. Thus, in a source-free domain, we can write

$$\mathbf{B}(\mathbf{x}) = -\nabla\Phi_M(\mathbf{x}). \tag{37}$$

The *magnetic scalar potential* satisfies a differential equation which follows from the requirement that $\nabla \cdot \mathbf{B}(\mathbf{x}) = 0$; it is

$$\nabla^2\Phi_M(\mathbf{x}) = 0. \tag{38}$$

Thus, wherever there is a magnetic scalar potential, it satisfies the Laplace equation. In order to solve for it, we may apply any of the techniques we learned for finding the electrostatic potential in charge-free regions of space. Hence no more will be said about the magnetic scalar potential in general in this chapter.

4.2 Vector Potential and Gauge Invariance

Consider now the vector potential for a magnetostatic field. Because $\nabla \cdot \mathbf{B}(\mathbf{x}) = 0$, it is possible to write the magnetic induction as the curl of another vector field (since $\nabla \cdot (\nabla \times \mathbf{A})$). We have in fact already constructed such a vector field because in Eq. (25) we have written

$$\mathbf{B}(\mathbf{x}) = \nabla \times \left(\frac{1}{c} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right). \tag{39}$$

The field within the parentheses is a vector potential for $\mathbf{B}(\mathbf{x})$. We shall write it as $\mathbf{A}(\mathbf{x})$. It is, in contrast to the electrostatic scalar potential, not unique because one can always add to it the gradient of any scalar field $\chi(\mathbf{x})$ and have a vector field whose curl is still $\mathbf{B}(\mathbf{x})$. That is, given

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \tag{40}$$

which is such that $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$, we can write

$$\mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \nabla\chi(\mathbf{x}), \tag{41}$$

where $\chi(\mathbf{x})$ is arbitrary. Then it is true that $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}'(\mathbf{x})$ because the curl of the gradient of a scalar field is zero.

By writing $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$, we have automatically satisfied the requirement that $\nabla \cdot \mathbf{B}(\mathbf{x}) = 0$; Hence we may find a single (vector) equation for the vector potential by substituting $\mathbf{B} = \nabla \times \mathbf{A}$ into Ampère's law:

$$\nabla \times \mathbf{B}(\mathbf{x}) = \nabla \times (\nabla \times \mathbf{A}(\mathbf{x})) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}), \tag{42}$$

or, using $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$,

$$\nabla^2 \mathbf{A}(\mathbf{x}) - \nabla(\nabla \cdot \mathbf{A}(\mathbf{x})) = -\frac{4\pi}{c}\mathbf{J}(\mathbf{x}). \tag{43}$$

This equation would be considerably simpler if we could make the divergence of $\mathbf{A}(\mathbf{x})$ vanish. It is possible to do this by using a vector potential constructed with an appropriate choice of $\chi(\mathbf{x})$. The underlying mathematical point is the following: so far, the only condition we have placed on the vector potential is that its curl should be the magnetic induction. We are free to choose it so that its divergence conforms to our wishes because a vector field is sufficiently flexible that its curl and divergence can both be specified arbitrarily. Consider then the divergence of the most general vector potential. We write this potential as

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \nabla\chi(\mathbf{x}). \tag{44}$$

Then

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \mathbf{J}(\mathbf{x}') \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) + \nabla^2\chi(\mathbf{x}). \tag{45}$$

Now change the ∇ operator to $-\nabla'$ and then do an integration by parts. The surface integral may be discarded if the volume integral is over all space and $\mathbf{J}(\mathbf{x}')$ is localized (so that it vanishes on the surface of the volume of integration). What one then has left is

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{\nabla' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \nabla^2\chi(\mathbf{x}) = \nabla^2\chi(\mathbf{x}), \tag{46}$$

where the last step follows from the fact that a time-independent set of sources give a current distribution having zero divergence.

From this result we can see two things: first, the vector potential has zero divergence if we forget about $\chi(\mathbf{x})$, *i.e.,* if we set it equal to zero. Second, we can make the divergence be any scalar function $f(\mathbf{x})$ we want by setting $\nabla^2\chi(\mathbf{x}) = f(\mathbf{x})$. We would then have to solve for $\chi(\mathbf{x})$ which is in principle straightforward because the differential equation for χ is just the Poisson equation and we know how to solve that. The solution is

$$\chi(\mathbf{x}) = -\frac{1}{4\pi}\int d^3x'\,\frac{f(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{47}$$

Specifying the divergence of $\mathbf{A}(\mathbf{x})$ is called choosing the *gauge* of the vector potential, and changing the function $\chi(\mathbf{x})$ is called making a *gauge transformation.* Notice that one can make a transformation without changing the gauge (*i.e.,* without changing $\nabla\cdot\mathbf{A}$); to do this one must change χ by a function which satisfies the Laplace equation. The particular gauge specified by $\nabla\cdot\mathbf{A}(\mathbf{x}) = 0$ is called the *Coulomb gauge.*

Returning to our original point, we have found that we can pick a vector potential with zero divergence. In particular,

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{48}$$

has this property. In this, the Coulomb gauge, the vector potential satisfies the equations

$$\nabla^2 \mathbf{A}(\mathbf{x}) = -\frac{4\pi}{c}\mathbf{J}(\mathbf{x}), \tag{49}$$

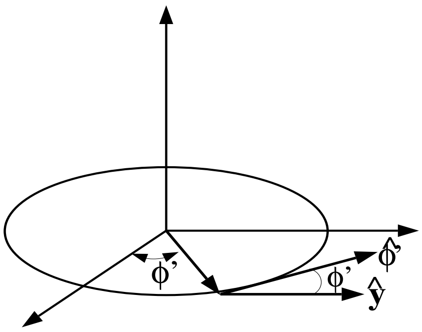
which **must** be interpreted as

$$\nabla^2 A_i(\mathbf{x}) = -\frac{4\pi}{c} J_i(\mathbf{x}) \tag{50}$$

where the subscript i denotes any Cartesian component of $\mathbf{A}$.

4.3 Example: A Circular Current Loop

Consider a circular loop of radius a carrying a current I. Let the loop lie in the $z = 0$ plane and be centered at the origin of coordinates.



Then we may write $\mathbf{J}(\mathbf{x}) = J_\phi(\mathbf{x})\hat{\phi}$ with

$$\mathbf{J}_\phi(\mathbf{x}) = \frac{I}{a}\delta(\cos\theta)\delta(r-a). \tag{51}$$

It is a natural temptation to do the following INCORRECT thing: write $\mathbf{A}(\mathbf{x}) = A_\phi(\mathbf{x})\hat{\phi}$ (correct so far) with

$$A_\phi(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{J_\phi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{52}$$

This integral is not correct because the unit vector $\hat{\phi}$ at the field point $\mathbf{x}$ is not the same as the unit vector $\hat{\phi}'$ at the field point $\mathbf{x}'$.

What we can do instead is to make use of the azimuthal symmetry of the system and trivially generalize to all values of ϕ after having first evaluated the vector potential at some particular value of ϕ. Let us look at the potential at $\phi = 0$, *i.e.*, in the x-z plane. Here, the vector potential will be in the y direction, so we can say that

$$A_\phi(r,\theta) = A_y(r,\theta,\phi=0) = \frac{1}{c}\int d^3x' \, \frac{J_y(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{53}$$

at $\phi = 0$. Now, $J_y(\mathbf{x}') = J_{\phi'}\cos\phi'$, so

$$A_\phi(r,\phi) = \frac{I}{ac}\int d^3x' \, \frac{\delta(r'-a)\delta(\cos\theta')\cos\phi'}{(r^2+a^2-2ar\cos\gamma)^{1/2}} \tag{54}$$

where γ is the angle between the directions of $\mathbf{x}$ and $\mathbf{x}'$, $\cos\gamma = \sin\theta\cos\phi'$.

The integral is an elliptic integral; rather than deal with its arcane properties, we will expand it in the usual way:

$$\begin{aligned}
A_\phi(r,\theta) &= \frac{I}{ca}\int d^3x' \, \frac{\delta(r'-a)\delta(\cos\theta')\cos\phi'}{|\mathbf{x}-\mathbf{x}'|} \\
&= Re\left\{\frac{I}{ca}\int d^3x' \, \frac{\delta(r'-a)\delta(\cos\theta')}{|\mathbf{x}-\mathbf{x}'|}e^{i\phi'}\right\} \\
&= Re\left\{\frac{I}{ca}\int d^3x' \, \delta(r'-a)\delta(\cos\theta')e^{i\phi'}\sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l,m}^*(\theta',\phi')Y_{l,m}(\theta,\phi)|_{\phi=0}\right\} \\
&= Re\left\{\frac{Ia}{c}\sum_{l,m}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(0)\frac{r_<^l}{r_>^{l+1}}\int_0^{2\pi} d\phi' e^{i\phi'}e^{-im\phi'}\right\}
\end{aligned}$$

$$= \frac{2\pi I a}{c} \sum_{l=1}^{\infty} \frac{P_l^1(0) P_l^1(\cos\theta)}{l(l+1)} \frac{r_<^l}{r_>^{l+1}} \quad (55)$$

where $r_<$ and $r_>$ refer to the smaller and larger of r and a. Next one notes that $P_l^1(0) = 0$ for l even and

$$P_l^1(0) = (-1)^{(l+1)/2} l!/[(l-1)!!]^2 \quad (56)$$

for l odd; set $l = 2n+1$ and have

$$A_\phi(r,\theta) = \frac{2\pi I a}{c} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{(2n+1)(2n+2)[(2n)!!]^2} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^1(\cos\theta)$$
$$= \frac{\pi I a}{c} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)!!}{2^n(n+1)!} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^1(\cos\theta). \quad (57)$$

At large r, *i.e.*, $r >> a$, we can keep just the leading term (n=0) and find

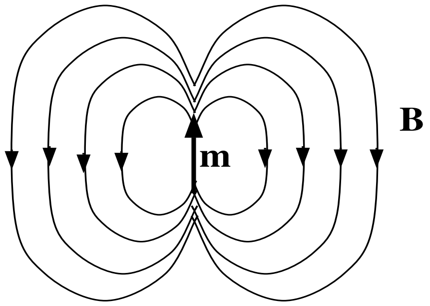
$$A_\phi(r,\theta) = \frac{\pi I a}{c} \frac{a}{r^2} \sin\theta. \quad (58)$$

The corresponding components of the magnetic induction are

$$B_r(r,\theta) = \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_\phi) = \frac{2\pi I a^2}{c r^3} \cos\theta, \quad (59)$$

and

$$B_\theta(r,\theta) = -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) = \frac{\pi I a^2}{c r^3} \sin\theta. \quad (60)$$



We recognize these as having the same form as the field of an electric powers of a/r, the current loop is treated as a *magnetic dipole* with *magnetic dipole moment* $\mathbf{m} \equiv (\pi I a^2/c)\hat{\mathbf{z}}$; the corresponding vector potential, Eq. (58), is

$$\mathbf{A}(\mathbf{x}) = A_\phi(r,\theta)\hat{\phi} = \frac{|\mathbf{m}|\sin\theta}{r^2}\hat{\phi} = \frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}. \tag{61}$$

We may compare this potential to that of an electric dipole which is

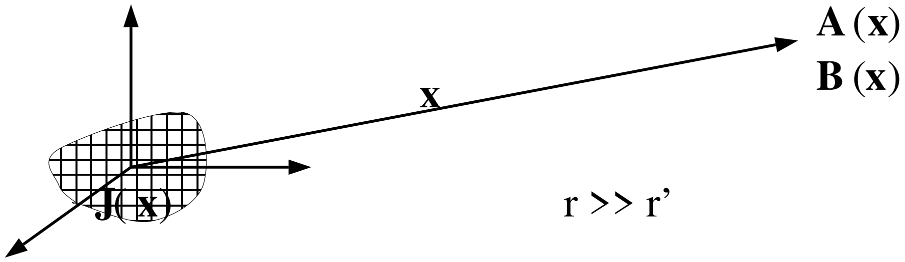
$$\Phi(\mathbf{x}) = \frac{\mathbf{p}\cdot\mathbf{x}}{|\mathbf{x}|^3}. \tag{62}$$

5 The Field of a Localized Current Distribution

The example of the preceding section is a special case of the field of a localized current distribution. Let us now suppose that we have some such distribution $\mathbf{J}(\mathbf{x}')$ around the origin of coordinates. We shall find the potential of this distribution in the dipole approximation. To this end we must expand the function

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{1}{|\mathbf{x}|}\left[1+\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^2}+...\right] \tag{63}$$

in powers of r'/r.



Of particular interest to us is the first-order term in the brackets. Using this expansion, we find that the vector potential is

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x'\,\mathbf{J}(\mathbf{x}')\left[\frac{1}{|\mathbf{x}|}+\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3}+...\right]. \tag{64}$$

The first term $\sim \int d^3x'\,\mathbf{J}(\mathbf{x}')$ is zero for a localized steady current distribution (Why?)[2] ; the second one is

$$\mathbf{A}_d(\mathbf{x}) = \frac{1}{c|\mathbf{x}|^3}\int d^3x'\,(\mathbf{x}\cdot\mathbf{x}')\mathbf{J}(\mathbf{x}'). \tag{65}$$

However, $(\mathbf{x}\cdot\mathbf{x}')\mathbf{J} = -\mathbf{x}\times(\mathbf{x}'\times\mathbf{J}) + (\mathbf{x}\cdot\mathbf{J})\mathbf{x}'$, so

$$\mathbf{A}_d(\mathbf{x}) = \frac{1}{c|\mathbf{x}|^3}\left\{\int d^3x'\,(\mathbf{x}\cdot\mathbf{J}(\mathbf{x}'))\mathbf{x}' - \mathbf{x}\times\int d^3x'\,(\mathbf{x}'\times\mathbf{J}(\mathbf{x}'))\right\}. \tag{66}$$

Consider the j^{th} component of the first integral:

$$\int d^3x'\,x'_j(\mathbf{x}\cdot\mathbf{J}(\mathbf{x}')) = \sum_{i=1}^{3}\int d^3x'\,x'_j x_i J_i(\mathbf{x}'). \tag{67}$$

Now, it is the case that

$$\nabla'\cdot(x'_i\mathbf{J}(\mathbf{x}')) = (\nabla' x'_i)\cdot\mathbf{J}(\mathbf{x}') + x'_i(\nabla'\cdot\mathbf{J}(\mathbf{x}')) = J_i(\mathbf{x}'), \tag{68}$$

since the divergence of $\mathbf{J}$ is zero. This allows us to write

$$\int d^3x'\,(\mathbf{x}\cdot\mathbf{J}(\mathbf{x}'))x'_j = \sum_{i=1}^{3} x_i\int d^3x'\,[\nabla'\cdot(x'_i\mathbf{J}(\mathbf{x}'))]x'_j$$

[2] The short answer is just that in lieu of sources or sinks of charge the total current density in a current distribution must be zero. More formally, consider the 3-vector with components U_i

$$U_i = 0 = \int_V d^3x\,x_i\nabla\cdot\mathbf{J} = \int_V d^3x\,[\nabla\cdot(x_i\mathbf{J}) - \mathbf{J}\cdot\nabla x_i]$$

Using the divergence theorem, this becomes

$$\int_S d^2x\,x_i\mathbf{J}\cdot\mathbf{n} - \int_V d^3x\,\mathbf{J}\cdot\hat{\mathbf{x}}_i$$

which follows since $\nabla x_i = \hat{\mathbf{x}}_i$. Then if we take the surface to infinity, where there is no current density, we obtain the desired result

$$0 = -\int_V d^3x\,J_i$$

Parts integration:

$$= -\sum_{i=1}^{3} x_i \int d^3x' \, (x'_i \mathbf{J}(\mathbf{x}')) \cdot (\nabla' x'_j) = -\int d^3x' \, (\mathbf{x} \cdot \mathbf{x}') J_j(\mathbf{x}'). \qquad (69)$$

Generalizing now to all three components of this integral, we find

$$\int d^3x' \, (\mathbf{x} \cdot \mathbf{J}(\mathbf{x}'))\mathbf{x}' = -\int d^3x' \, (\mathbf{x} \cdot \mathbf{x}')\mathbf{J}(\mathbf{x}'). \qquad (70)$$

Comparison with the expressions for $\mathbf{A}_d$ shows that

$$\mathbf{A}_d(\mathbf{x}) = -\frac{1}{2c}\frac{1}{|\mathbf{x}|^3}\mathbf{x} \times \left[\int d^3x' \, (\mathbf{x}' \times \mathbf{J}(\mathbf{x}'))\right] \equiv \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3} = \nabla \times \left(\frac{\mathbf{m}}{|\mathbf{x}|}\right) \qquad (71)$$

where the *magnetic dipole moment* of the current distribution is defined as

$$\mathbf{m} \equiv \frac{1}{2c} \int d^3x' \, (\mathbf{x}' \times \mathbf{J}(\mathbf{x}')). \qquad (72)$$

The *magnetic moment density* of the distribution is defined by

$$\mathcal{M}(\mathbf{x}) \equiv \frac{1}{2c}\mathbf{x} \times \mathbf{J}(\mathbf{x}) \qquad (73)$$

so that the magnetic moment is the integral of this density over all space.

The field produced by the source in the magnetic dipole approximation is

$$\mathbf{B}_d(\mathbf{x}) = \nabla \times \mathbf{A}_d(\mathbf{x}) = \nabla \times \left(\frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3}\right). \qquad (74)$$

Using the identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}, \qquad (75)$$

we can write this equation as

$$\mathbf{B}_d(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3} \tag{76}$$

where $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$.

This is, as was the case for the electric dipole, only the field outside of the current distribution. For a point magnetic dipole, it would be the field away from the location of the dipole (the origin), and at the origin there is a delta-function field as for the point electric dipole. We may find the magnitude and direction of this singular field by a more careful analysis of what happens as $r \to 0$. To this end it is useful to write the vector potential of the dipole as

$$\mathbf{A}_d(\mathbf{x}) = \nabla\times\left(\frac{\mathbf{m}}{|\mathbf{x}|}\right). \tag{77}$$

Then we can write

$$\mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x}) = \nabla\times\left[\nabla\times\left(\frac{\mathbf{m}}{r}\right)\right] = \nabla\left[\nabla\cdot\left(\frac{\mathbf{m}}{r}\right)\right] - \nabla^2\left(\frac{\mathbf{m}}{r}\right). \tag{78}$$

The last term on the right-hand side is just $4\pi\mathbf{m}\delta(\mathbf{x})$. The first one we have seen before, as it is the same as the electric field of an electric dipole; we already know what singularity is contained therein but will figure it out again as an exercise. Start by integrating this term over a small sphere of radius ϵ centered at the origin and then take the limit as $\epsilon \to 0$:

$$\int_{r<\epsilon} d^3x \nabla \left[\nabla \cdot \left(\frac{\mathbf{m}}{r}\right)\right] = \int_{r=\epsilon} d^2x \mathbf{n} \left[\nabla \cdot \left(\frac{\mathbf{m}}{r}\right)\right], \tag{79}$$

where we have used the identity, valid for any scalar function $f(\mathbf{x})$:

$$\int d^3x \, \nabla f(\mathbf{x}) = \int d^2x \, \mathbf{n} f(\mathbf{x}); \tag{80}$$

S is the surface enclosing the domain V and $\mathbf{n}$ is the usual outward unit normal. Continuing, we have

$$\int_{r=\epsilon} d^2x \mathbf{n} \left[\nabla \cdot \left(\frac{\mathbf{m}}{r}\right)\right] = -\int_{r=\epsilon} d^2x \, \hat{\mathbf{r}} \left(\frac{\mathbf{m} \cdot \mathbf{x}}{r^3}\right) = -\frac{4\pi}{3}\mathbf{m}. \tag{81}$$

Hence we find that $\nabla[\nabla\cdot(\mathbf{m}/r)]$ contains the singular piece $(4\pi/3)\mathbf{m}\delta(\mathbf{x})$. Putting it into Eq. (78), we conclude that the delta-function piece of the magnetic field is $(8\pi/3)\mathbf{m}\delta(\mathbf{x})$, and hence the total field of the magnetic dipole is

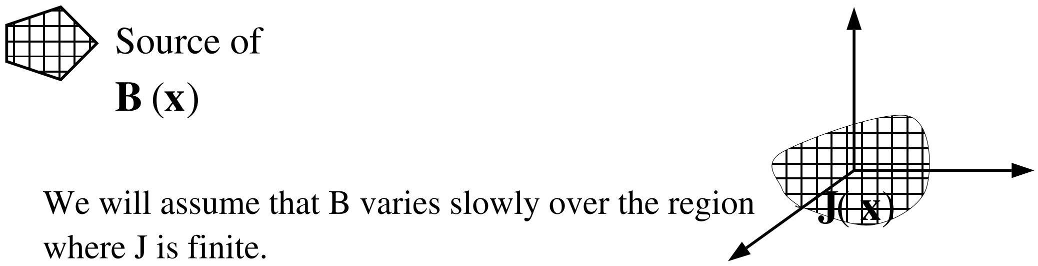
$$\mathbf{B}(\mathbf{x}) = \frac{3(\mathbf{n} \cdot \mathbf{m})\mathbf{n} - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{x}). \tag{82}$$

The consequences of the presence of the delta-function piece are observed in atomic hydrogen where the magnetic moment of the electron interacts with that of the nucleus, or proton. Without this interaction, all total-spin states of the atom would be degenerate. As a consequence of the interaction, the “triplet” or “spin-one” states are raised slightly in energy relative to the “singlet” or “spin-zero” state. The splitting is small even on the scale of atomic energies, being about $10^{-17} erg$ or

10^{-5} ev. A photon which has the same energy as this splitting, and which is thus produced or absorbed by a transition between the two atomic energy levels, has a wavelength of 21 cm. The delta-function part of the field also plays an important role in the scattering of neutrons from magnetic materials.

6 Forces on a Localized Current Distribution

We shall look next at the interaction of a localized current distribution with an externally applied field, again using procedures reminiscent of the multipole expansion for a localized charge distribution. First, we expand the applied field around some suitably chosen origin (at the center of the current distribution),



$$\mathbf{B}(\mathbf{x}) = \mathbf{B}(0) + (\mathbf{x} \cdot \nabla')\mathbf{B}(\mathbf{x}')|_{\mathbf{x}'=0} + \ldots; \tag{83}$$

The force that the field exerts on a localized current distribution located around the origin is then expanded as follows:

$$\mathbf{F} = \frac{1}{c}\int d^3x\,(\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}))$$
$$= -\frac{1}{c}\mathbf{B}(0) \times \int d^3x\,\mathbf{J}(\mathbf{x}) + \frac{1}{c}\int d^3x\,\mathbf{J}(\mathbf{x}) \times [(\mathbf{x}\cdot\nabla')\mathbf{B}(\mathbf{x}')|_{\mathbf{x}'=0}] + ... \quad (84)$$

Now, the first integral in the last line vanishes for a localized steady-state current distribution (there can't be any net flow of charge in any direction), and we can manipulate the integrand in the final integral as follows:

$$(\mathbf{x}\cdot\nabla')\mathbf{B}(\mathbf{x}') = \nabla'(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}')) - \mathbf{x} \times (\nabla' \times \mathbf{B}(\mathbf{x}')), \qquad (85)$$

and we may suppose that $\mathbf{B}$ is due entirely to external sources so that $\nabla' \times \mathbf{B}(\mathbf{x}') = 0$ for $\mathbf{x}'$ around the origin. Thus we find that the force is

$$\begin{aligned}\mathbf{F} &= \frac{1}{c}\int d^3x\,\mathbf{J}(\mathbf{x}) \times [\nabla'(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))|_{\mathbf{x}'=0}] \\ &= -\frac{1}{c}\int d^3x\,\nabla' \times [(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))|_{\mathbf{x}'=0}\mathbf{J}(\mathbf{x})] \\ &= -\frac{1}{c}\nabla' \times \int d^3x\,[\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))|_{\mathbf{x}'=0}\mathbf{J}(\mathbf{x})]. \end{aligned} \qquad (86)$$

Now, we can write the last integral as

$$\int d^3x\,(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))\mathbf{J}(\mathbf{x}) = -\int d^3x\,[\mathbf{B}(\mathbf{x}') \times (\mathbf{x} \times \mathbf{J}(\mathbf{x})) - \mathbf{x}(\mathbf{B}(\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}))]. \qquad (87)$$

Further, it is true that

$$\int d^3x\,\mathbf{x}(\mathbf{B}(\mathbf{x}')\cdot\mathbf{J}(\mathbf{x})) = -\int d^3x\,(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))\mathbf{J}(\mathbf{x}); \qquad (88)$$

we can demonstrate this fact by considering the i^{th} component of the integral.

$$-\int d^3x\,(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))J_i(\mathbf{x}) = -\int d^3x\,(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))\nabla\cdot(x_i\mathbf{J}(\mathbf{x}))$$
$$= \int d^3x\,[\nabla(\mathbf{x}\cdot\mathbf{B}(\mathbf{x}'))]\cdot(x_i\mathbf{J}(\mathbf{x})) = \int d^3x\,\mathbf{B}(\mathbf{x}')\cdot(x_i\mathbf{J}(\mathbf{x}))$$
$$= \int d^3x\,x_i(\mathbf{B}(\mathbf{x}')\cdot\mathbf{J}(\mathbf{x})). \qquad (89)$$

Hence, from Eqs. (84), (85), and (86) we find that

$$\mathbf{F} = \frac{1}{2c}\nabla'\times\left\{\int d^3x\,[\mathbf{B}(\mathbf{x}')\times(\mathbf{x}\times\mathbf{J}(\mathbf{x}))]\right\}|_{\mathbf{x}'=0}$$
$$= \frac{1}{2c}\nabla'\times\left\{\mathbf{B}(\mathbf{x}')\times\int d^3x\,(\mathbf{x}\times\mathbf{J}(\mathbf{x}))\right\}|_{\mathbf{x}'=0}$$
$$= \nabla'\times(\mathbf{B}(\mathbf{x}')\times\mathbf{m})|_{\mathbf{x}'=0} = (\mathbf{m}\cdot\nabla')\mathbf{B}(\mathbf{x}')|_{\mathbf{x}'=0} - \mathbf{m}(\nabla'\cdot\mathbf{B}(\mathbf{x}'))|_{\mathbf{x}'=0}$$
$$= \nabla'(\mathbf{m}\cdot\mathbf{B}(\mathbf{x}'))|_{\mathbf{x}'=0} - \mathbf{m}\times(\nabla'\times\mathbf{B}(\mathbf{x}'))|_{\mathbf{x}'=0}. \qquad (90)$$

Along the way in this derivation we have made use of the facts that the divergence and curl of $\mathbf{B}$ are zero in the region near the origin. The final result has the form of the gradient of a scalar function,

$$\mathbf{F} = -\nabla(-\mathbf{m}\cdot\mathbf{B}(\mathbf{x})) \qquad (91)$$

where the gradient is to be evaluated at the center of the current distribution. Notice in particular that there is no force if the applied magnetic induction is uniform. More generally, the force is in the direction of the gradient of the component of $\mathbf{B}$ in the direction of $\mathbf{m}$.

The torque $\mathbf{N}$ on the current distribution may be found using similar manipulations.

$$\mathbf{N} = \frac{1}{c}\int d^3x \; \mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})) \approx \frac{1}{c}\int d^3x \; \mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(0))$$
$$= \frac{1}{c}\int d^3x \; [(\mathbf{x}\cdot\mathbf{B})\mathbf{J}(\mathbf{x}) - (\mathbf{x}\cdot\mathbf{J}(\mathbf{x}))\mathbf{B}] = -\frac{1}{2c}\int d^3x \, \mathbf{B} \times (\mathbf{x} \times \mathbf{J}(\mathbf{x}))$$
$$-\frac{1}{c}\mathbf{B}\int d^3x \, (\mathbf{x}\cdot\mathbf{J}(\mathbf{x})) = \mathbf{m} \times \mathbf{B} - \frac{1}{c}\mathbf{B}\int d^3x \, (\mathbf{x}\cdot\mathbf{J}(\mathbf{x})) \tag{92}$$

However,

$$\nabla \cdot (r^2\mathbf{J}(\mathbf{x})) = 2\mathbf{x}\cdot\mathbf{J}(\mathbf{x}) + r^2(\nabla\cdot\mathbf{J}(\mathbf{x})); \tag{93}$$

the final term here is zero, so it is the case that

$$\int d^3x \, (\mathbf{x}\cdot\mathbf{J}(\mathbf{x})) = \frac{1}{2}\int d^3x \, \nabla\cdot(r^2\mathbf{J}(\mathbf{x})) = 0, \tag{94}$$

the final step following from the fact that the current distribution is localized. Thus the torque on the localized source reduces to

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}, \tag{95}$$

in the dipole approximation. Compare this result with Eq. (1).

7 Macroscopic Magnetostatics

If one has a macroscopic object with lots of microscopic currents flowing around in it (electrons on molecules give many small current loops), then it is best to carry out suitable averages over volumes small compared to anything macroscopic but large compared to molecules and to calculate only the average quantities. The elementary procedure, in which many complications are blissfully ignored, is to break the sources up into free and bound:

1. the *macroscopic current density*, written as $\mathbf{J}(\mathbf{x})$

2. the *magnetization* $\mathbf{M}(\mathbf{x})$ which is defined as the magnetic dipole moment per unit volume.

The latter comes from the molecules, atoms, ions, etc., in a material medium. The macroscopic current density comes from moving charges which are not bound on molecules, *i.e.,* not localized. Notice that within this approach we can accommodate sources which cannot actually be described as the motion of charges, or currents; that is, the spin magnetic moments of "elementary" particles such as the electron. These may be treated as point dipoles and so simply contribute to the magnetization.

Given these sources, use of Eqs. (48) and (71) tell us that the (macroscopic) vector potential is

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \left[\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{c\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]. \tag{96}$$

We can manipulate the term involving the magnetization in a manner which should be familiar:

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \left\{\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{c\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right\}$$

$$= \frac{1}{c}\int d^3x' \left\{ \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + c\mathbf{M}(\mathbf{x}') \times \nabla' \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \right\}$$

$$= \frac{1}{c}\int d^3x' \left\{ \frac{\mathbf{J}(\mathbf{x}') + c(\nabla' \times \mathbf{M}(\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|} \right\} - \int d^3x' \, \nabla' \times \left(\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right) \quad (97)$$

However, there is an identity which allows us to convert the integral of a curl to a surface integral:

$$\int_V d^3x \, \nabla \times \mathbf{V} = \oint_S d^2x \, \mathbf{n} \times \mathbf{V}. \quad (98)$$

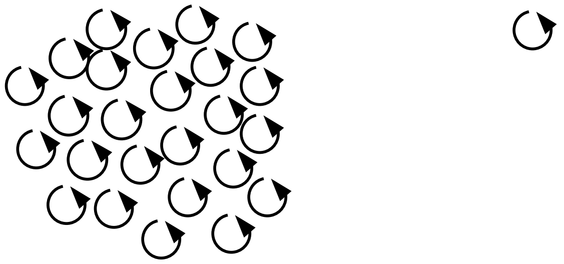
In the present instance, when the last integral in Eq. (97) is converted to a surface integral, it will vanish because if the surface encloses all of the magnetic materials, $\mathbf{M}$ will be zero on the boundary. Hence we have the result

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}') + c[\nabla' \times \mathbf{M}(\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|} \quad (99)$$

for the macroscopic vector potential.

7.1 Magnetization Current Density

The form of this equation implies that $c\nabla \times \mathbf{M}(\mathbf{x})$ is a current density associated with spatial variations of the magnetization. It is possible to imagine how the curl of the magnetization yields a current source, if we consider each magnetic dipole as originating from a vanishingly small current loop.



This may or may not really be the case. If the magnetization is the consequence of the motions of bound charges on molecules, then it is indeed a current. But if the magnetization is the consequence of the intrinsic dipole moments of elementary particles such as the electron, then it is not reasonable to think of the curl of the magnetization as a current density. Nevertheless, we shall define the *magnetization current density* $\mathbf{J}_M(\mathbf{x})$ as[3]

$$\mathbf{J}_M(\mathbf{x}) = c\nabla \times \mathbf{M}(\mathbf{x}). \tag{100}$$

In terms of the magnetization current density, we have

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}') + \mathbf{J}_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{101}$$

The differential field equations obeyed by the *macroscopic magnetic induction*, which is the curl of the macroscopic vector potential,

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}), \tag{102}$$

[3]Notice that this object has zero divergence, as a steady-state current density should.

are easy to write down by referring to our earlier derivation; they are

$$\nabla \times \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c}(\mathbf{J}(\mathbf{x}) + \mathbf{J}_M(\mathbf{x})) \tag{103}$$

and, of course,

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0. \tag{104}$$

7.2 Magnetic Field

The difficulty one faces in solving these equations is much the same as faced in macroscopic electrostatics. One generally does not know magnetization until after solving for the induction. Further, the relation between $\mathbf{B}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x})$ depends on the material. It is customary to define an additional macroscopic field $\mathbf{H}(\mathbf{x})$, called the *magnetic field*,

$$\mathbf{H}(\mathbf{x}) \equiv \mathbf{B}(\mathbf{x}) - 4\pi\mathbf{M}(\mathbf{x}). \tag{105}$$

Then Eq. (103) can be rewritten as

$$\nabla \times \mathbf{B}(\mathbf{x}) = \frac{4\pi}{c}(\mathbf{J}(\mathbf{x}) + c\nabla \times \mathbf{M}(\mathbf{x})) \tag{106}$$

or

$$\nabla \times (\mathbf{B}(\mathbf{x}) - 4\pi\mathbf{M}(\mathbf{x})) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}) \tag{107}$$

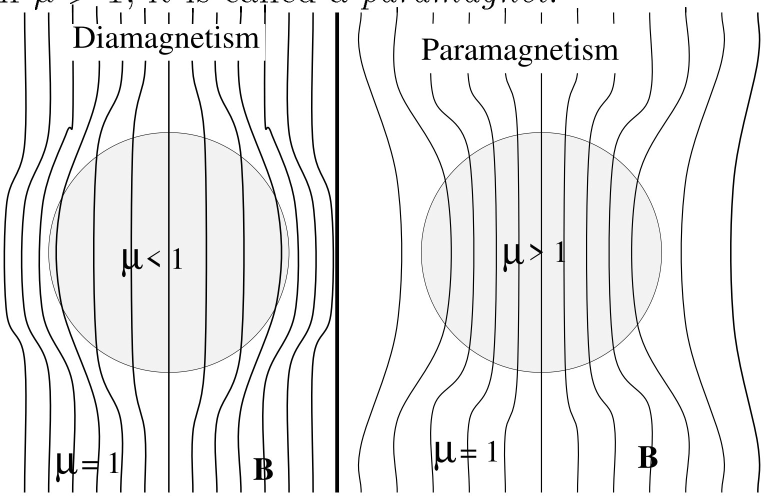
or

$$\nabla \times \mathbf{H}(\mathbf{x}) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}). \tag{108}$$

We haven't resolved anything by making this definition, of course; we have simply phrased the problem in a somewhat different form. The advantage in introducing the magnetic field is that it obeys a relatively simple differential equation and that, in many materials, there is a simple approximate relation between **B** and **H**. The simplest materials are, like dielectrics, linear, isotropic, and uniform, in which case the relation between **B** and **H** is

$$\mathbf{H}(\mathbf{x}) = \frac{1}{\mu}\mathbf{B}(\mathbf{x}) \tag{109}$$

where μ is a constant known as the *magnetic permeability*. This positive constant can be smaller or larger than unity, leading to two classifications of magnetic materials: If $\mu < 1$, the material is said to be a *diamagnet*; if $\mu > 1$, it is called a *paramagnet*.



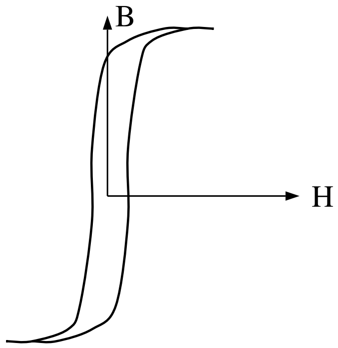
For linear, isotropic materials, it is also common to introduce the *mag-*

netic susceptibility χ_m,

$$\chi_m \equiv \frac{\mu - 1}{4\pi} \tag{110}$$

so that $\mathbf{M} = \chi_m \mathbf{H}$.

Many materials have dramatically different relations among $\mathbf{B}$, $\mathbf{M}$, and $\mathbf{H}$. *Ferromagnets* are a prime example; they can have a finite magnetic induction with zero magnetic field as well as the converse. In addition, as shown in the figure below the relation between these two fields in ferromagnets is usually not single-valued; the particular magnetic field one finds for a given value of $\mathbf{B}$ depends on the "history" of the sample, meaning it depends on what external fields it was subjected to prior to determining $\mathbf{B}$ and $\mathbf{H}$. We will not concern ourselves with the origins of this behavior.



7.3 Boundary Conditions

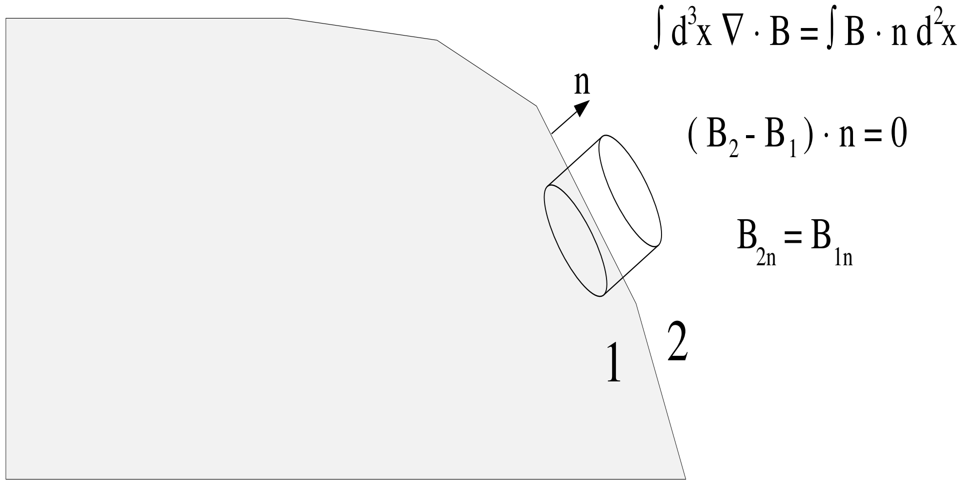
We have the general differential equations of magnetostatics

$$\nabla \times \mathbf{H}(\mathbf{x}) = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}) \tag{111}$$

and

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0. \tag{112}$$

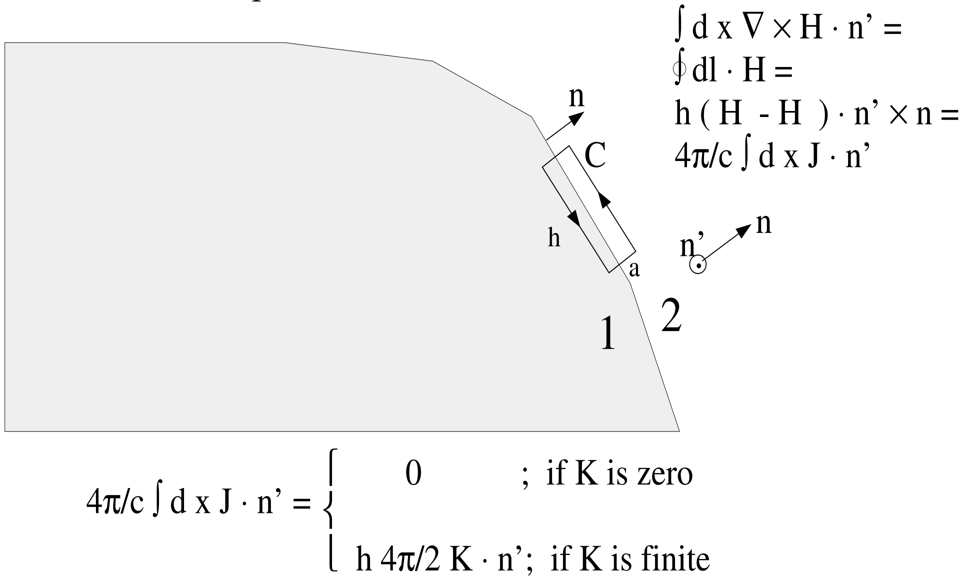
These are valid for any system, independent of the particular relation between the magnetic induction and the magnetic field. From them we can derive general boundary or continuity conditions. From the divergence equation, we infer that the normal component of the magnetic induction is continuous at an interface between two materials,



$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0 \tag{113}$$

where the subscripts 1 and 2 refer to the induction in each of two materials meeting at an interface, and $\mathbf{n}$ is a unit vector normal to the interface and pointing into medium 2.

From the curl equation, one finds that the tangential components of $\mathbf{H}$ can be discontinuous. The discontinuity is related to the current flowing along (parallel to) the interface. Consider Stokes' theorem as applied to the curl equation.



Thus,

$$\int_S d^2x\,[\nabla \times \mathbf{H}(\mathbf{x})] \cdot \mathbf{n}' = \oint_C d\mathbf{l} \cdot \mathbf{H}(\mathbf{x}) = \frac{4\pi}{c} \int_S d^2x\,\mathbf{J}(\mathbf{x}) \cdot \mathbf{n}' \qquad (114)$$

where the unit vector $\mathbf{n}'$ is directed normal to the surface S over which the integration is done. Using a rectangle of dimensions a by h, where $a << h$ and a is directed perpendicular to the interface while h is parallel to it, we find that the line integral comes down to

$$\oint_C d\mathbf{l} \cdot \mathbf{H}(\mathbf{x}) = h(\mathbf{H}_2 - \mathbf{H}_1) \cdot (\mathbf{n}' \times \mathbf{n}) \qquad (115)$$

which is to say, we find the discontinuity in a particular tangential component of **H** across the interface. The integral over the current, on the other hand, gives, for a finite, slowly varying current,

$$\frac{4\pi}{c}\int_S d^2x\,\mathbf{J}(\mathbf{x})\cdot\mathbf{n}' \sim ah\mathbf{J}\cdot\mathbf{n}'; \tag{116}$$

this is proportional to a and so is insignificant for $a << h$ assuming a finite well-behaved current density. In this case, there is no discontinuity in the tangential components of **H**,

$$(\mathbf{H}_2 - \mathbf{H}_1)\times\mathbf{n} = 0. \tag{117}$$

There is also the possibility of a singular term in $\mathbf{J}(\mathbf{x})$ which would be of the form

$$\mathbf{J}_s(\mathbf{x}) = \mathbf{K}(\mathbf{x})\delta(\xi) \tag{118}$$

where ξ is the distance from the interface and the vector **K** points in a direction parallel to the interface. This vector is a *surface-current density* and has dimensions of charge per unit length per unit time. This expression is, of course, an idealization. When currents run along the surface of a material, they typically are not localized precisely at the surface (*i.e.,* , within an atomic size) but are spread over a surface layer of thickness ranging from a few hundred Angstroms to some microns. If the length a is significantly larger than this layer's thickness, then it is reasonable to talk about a surface-current density and quite acceptable to regard it as being localized at the interface. Proceeding on this basis, we find that the continuity condition becomes

$$h(\mathbf{H}_2 - \mathbf{H}_1) \cdot (\mathbf{n}' \times \mathbf{n}) = h\frac{4\pi}{c}\mathbf{K} \cdot \mathbf{n}'. \tag{119}$$

This relation can be written in the more general form

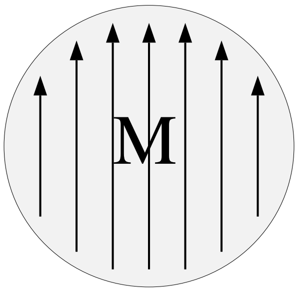
$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \frac{4\pi}{c}\mathbf{K} \tag{120}$$

as may be shown by (1) realizing that $\mathbf{K} \cdot \mathbf{n} = 0$ and (2) taking the inner product of Eq. (120) with any vector (such as $\mathbf{n}'$) lying in the plane of the interface.

8 Examples of Boundary-Value Problems in Magnetostatics

8.1 Uniformly Magnetized Sphere

The principal example is a uniformly magnetized sphere meaning a material that maintains a constant magnetization $\mathbf{M}_0$ in the absence of any applied field.



We shall let the direction of $\mathbf{M}$ be the z-direction and so have

$$\mathbf{M}(\mathbf{x}) = \begin{cases} M_0\hat{\mathbf{z}} & r < a \\ 0 & r > a. \end{cases} \tag{121}$$

In the region $r > a$, $\mathbf{B} = \mathbf{H}$. For $r < a$, $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$. However, $\mathbf{M}$ is constant here and so has no curl or divergence. This means that the curls of both $\mathbf{B}$ and $\mathbf{H}$ are zero everywhere except at the boundary of the sphere.

Consider the magnetization current density, $\mathbf{J}_M = c\nabla\times\mathbf{M}$; it is non-zero only at $r = a$ where it is singular. By applying Stokes' theorem to this equation in the same manner as was done to find the continuity condition on the tangential components of $\mathbf{H}$ at an interface, one finds that there is a magnetization surface-current density $\mathbf{K}_M$ given by

$$\mathbf{K}_M = c\mathbf{n} \times (\mathbf{M}_2 - \mathbf{M}_1) \tag{122}$$

where $\mathbf{n}$ is the unit normal at the surface pointing into material 2. In the present application, $\mathbf{M}_1 = \mathbf{M}_0$ and $\mathbf{M}_2 = 0$. Since $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\sin\theta\hat{\phi}$, we have $\mathbf{K}_M = K_M\hat{\phi}$ where $K_M = cM_0\sin\theta$.

8.1.1 Scalar Potential for the Induction

Now that we have identified the sources, let's look at some methods of solution. First, consider a scalar potential approach. Because the curl of $\mathbf{B}$ is zero for $r < a$ and $r > a$, we can devise scalar potentials for the magnetic induction in these two regions. Because $\nabla \cdot \mathbf{B}(\mathbf{x}) = 0$, the potentials satisfy the Laplace equation. Further, the system is invariant under rotation around the z-axis, implying that the potential is independent of ϕ. Hence it must be possible to write

$$\Phi_<(\mathbf{x}) = M_0 a \sum_l A_l \left(\frac{r}{a}\right)^l P_l(\cos\theta) \text{ for } r < a \tag{123}$$

and

$$\Phi_>(\mathbf{x}) = M_0 a \sum_l C_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta) \text{ for } r > a. \tag{124}$$

Given that $\mathbf{B} = -\nabla\Phi(\mathbf{x})$, the condition that the normal component of $\mathbf{B}$ is continuous at $r = a$ becomes (making use of the orthogonality of the Legendre polynomials in the usual way)

$$lA_l = -(l+1)C_l \tag{125}$$

for all l. The other boundary condition is that the tangential component of $\mathbf{H}$, or H_θ is continuous. Since $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, and since the θ-component of the magnetization is $-M_0 \sin\theta$ (and $\sin\theta = -dP_1(\cos\theta)/d\theta$, this second condition leads to

$$A_l = C_l \text{ for } l \text{ not equal to } 1 \tag{126}$$

and

$$C_1 = A_1 + 4\pi. \tag{127}$$

¿From these equations it is easy to see that $A_l = C_l = 0$, $l \neq 1$, while $C_1 = 4\pi/3$ and $A_1 = -8\pi/3$.

Having found the potential, we may compute the fields. The magnetic induction at $r > a$ has the familiar dipolar form; the field within the sphere is a constant:

$$\mathbf{B}_< = \frac{8\pi}{3}\mathbf{M}_0 \text{ and } \mathbf{H}_< = -\frac{4\pi}{3}\mathbf{M}_0. \tag{128}$$

8.1.2 Scalar Potential for the Field

A second approach to this problem is to devise a scalar potential for the magnetic field. Since $\mathbf{J}(\mathbf{x}) = 0$ everywhere, the curl of $\mathbf{H}$ is zero everywhere and so there is a potential Φ_H for **H everywhere** such that $\mathbf{H}(\mathbf{x}) = -\nabla\Phi_H(\mathbf{x})$. The divergence of $\mathbf{H}$ obeys

$$\nabla\cdot\mathbf{H}(\mathbf{x}) = -\nabla^2\Phi_H(\mathbf{x}) = \nabla\cdot\mathbf{B}(\mathbf{x}) - 4\pi\nabla\cdot\mathbf{M}(\mathbf{x}) = -4\pi\nabla\cdot\mathbf{M}(\mathbf{x}). \tag{129}$$

Looking upon this as a Poisson equation, we can immediately see that the solution for the potential is

$$\Phi_H(\mathbf{x}) = -\int d^3x' \, \frac{\nabla'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = -\int d^3x' \left[\nabla'\cdot\left(\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) - \mathbf{M}(\mathbf{x}')\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right] \tag{130}$$

The first term in the final expression may be converted to a surface integral which is seen to be zero from the fact that the magnetization is non-zero only within the sphere. Hence

$$\Phi_H(\mathbf{x}) = M_0\hat{\mathbf{z}}\cdot\int_{r'<a} d^3x'\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\nabla\cdot\left[M_0\hat{\mathbf{z}}\int_{r'<a}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\right]$$
$$= -\nabla\cdot\left[M_0\hat{\mathbf{z}}\int_0^a r'^2 dr' 4\pi\frac{1}{r_>}\right] \tag{131}$$

where $r_>$ is the larger of a and r. For $r > a$, we find

$$\Phi_H(\mathbf{x}) = -\nabla \cdot \left(M_0 \hat{\mathbf{z}} 4\pi \frac{a^3}{3r} \right) = \frac{4\pi}{3} a^3 M_0 \frac{\cos\theta}{r^2}, \tag{132}$$

and for $r < a$,

$$\Phi_H(\mathbf{x}) = -\nabla \cdot \left[M_0 \hat{\mathbf{z}} 4\pi \left(\frac{a^2}{2} - \frac{r^2}{6} \right) \right] = \frac{4\pi}{3} M_0 z. \tag{133}$$

Notice how the preceding calculation avoids having to think about what happens at the surface of the sphere; the integration by parts leaves us with a simpler integration over the magnetization. It is, of course, possible to evaluate the divergence of the magnetization and complete the integral directly without using the integration by parts. To this end, write the magnetization as

$$\mathbf{M}(\mathbf{x}) = M_0 \hat{\mathbf{z}} \theta(a - r) \tag{134}$$

where the θ-function is a step function,

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases} \tag{135}$$

Then

$$\nabla \cdot \mathbf{M}(\mathbf{x}) = M_0 \frac{\partial}{\partial z} \theta(a - r) = -M_0 \delta(a - r) \frac{\partial r}{\partial z} \tag{136}$$

since

$$\frac{d\theta(x)}{dx} = \delta(x). \tag{137}$$

Hence

$$\Phi_H(\mathbf{x}) = \int d^3x' \, \frac{\nabla' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = M_0 a^2 \int d\Omega' \frac{\cos\theta'}{|\mathbf{x}-\mathbf{x}'|} \tag{138}$$

$$= M_0 a^2 \frac{4\pi}{3}\frac{3}{4\pi}\frac{4\pi}{3}\frac{r_<}{r_>^2}\cos\theta = \frac{4\pi}{3} M_0 a^2 \frac{r_<}{r_>^2}\cos\theta. \tag{139}$$

Here, $\mathbf{x}'$, after integrating over r', has the magnitude a.

8.1.3 Direct Calculation of B

Another approach to this problem is to calculate the vector potential for $\mathbf{B}(\mathbf{x})$ directly. Since $\mathbf{J}(\mathbf{x}) = 0$ everywhere, only the curl of the magnetization acts as a source of this potential. Thus we have

$$\begin{aligned}\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{c\nabla' \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} &= \int d^3x' \left[\nabla' \times \left(\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) - \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \times \mathbf{M}(\mathbf{x}')\right] \\ &= \oint_S d^2x' \, \mathbf{n}' \times \left(\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) + \nabla \times \left(\int d^3x' \, \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) \\ &= \nabla \times M_0 \hat{\mathbf{z}} 4\pi \begin{cases} a^3/3r, & r > a \\ a^2/2 - r^2/6, & r < a. \end{cases}\end{aligned} \tag{140}$$

One may easily work out the curl to find

$$\mathbf{A}(\mathbf{x}) = \frac{4\pi}{3} M_0 a^3 \frac{\sin\theta}{r^2}\hat{\phi} \text{ for } r > a \tag{141}$$

and

$$\mathbf{A}(\mathbf{x}) = \frac{4\pi}{3} M_0 r \sin\theta \hat{\phi} \text{ for } r < a. \tag{142}$$

These expressions may be summarized as

$$\mathbf{A}(\mathbf{x}) = \frac{4\pi M_0}{3}a^2\left(\frac{r_<}{r_>^2}\right)\sin\theta\hat{\phi}. \tag{143}$$

The preceding example is a typical one involving a permanently magnetized material. Then one knows $\mathbf{M}(\mathbf{x})$ and so can proceed with a solution of the equations for $\mathbf{B}$ or $\mathbf{H}$ via the route of one's choice. A quite different sort of problem is one involving linear, isotropic magnetic materials with $\mathbf{J}(\mathbf{x}) = 0$. These are entirely equivalent to problems in macroscopic electrostatics (Laplace equation problems) with which we have extensive experience. The point is that one has

$$\nabla \times \mathbf{H}(\mathbf{x}) = 0 \tag{144}$$

which means there is a potential $\Phi_H(\mathbf{x})$ such that

$$\mathbf{H}(\mathbf{x}) = -\nabla\Phi_H(\mathbf{x}). \tag{145}$$

Further, since $\nabla \cdot \mathbf{B}(\mathbf{x}) = 0$, and $\mathbf{B}(\mathbf{x}) = \mu\mathbf{H}(\mathbf{x})$, we have

$$0 = \nabla\cdot\mathbf{B}(\mathbf{x}) = \nabla\cdot(\mu\mathbf{H}(\mathbf{x})) = (\nabla\mu)\mathbf{H}(\mathbf{x}) + \mu\nabla\cdot\mathbf{H}(\mathbf{x}) = -(\nabla\mu)\nabla\Phi_H(\mathbf{x}) - \mu\nabla^2\Phi_H(\mathbf{x}), \tag{146}$$

or

$$\nabla^2\Phi_H(\mathbf{x}) = -\frac{\nabla\mu}{\mu}\nabla\Phi_H(\mathbf{x}). \tag{147}$$

In any region of space where μ is a constant, this is just the Laplace equation,

$$\nabla^2\Phi_H(\mathbf{x}) = 0. \tag{148}$$

Hence we may write down an appropriate solution of the Laplace equation for $\Phi_H(\mathbf{x})$ in each region of space where the magnetic properties are uniform and then make sure the continuity conditions on the tangential components of $\mathbf{H}(\mathbf{x})$ and on the normal components of $\mathbf{B}(\mathbf{x})$ are satisfied on the boundaries between such regions. The techniques used are the same as for boundary value problems in electrostatics.

Notice that in problems of this kind, we can equally well make a scalar potential for $\mathbf{B}(\mathbf{x})$ in each of the regions where μ is constant.

8.2 Shielding by a Paramagnetic Cylinder

A standard example is magnetic shielding in which a shell of magnetic material with a very large value of μ (a strongly paramagnetic material) is placed around some region of space. Suppose, for example, that a long cylindrical shell of inner radius a and outer radius b is placed around the z-axis and that this system is subjected to a transverse applied field $\mathbf{H}_0 = H_0\hat{\mathbf{x}}$. Then we have three regions, $\rho < a$, $a < \rho < b$, and $b < \rho$, in which we may make potentials Φ_H which satisfy the Laplace equation. These potentials will have the forms

(i) $\rho < a$:

$$\Phi_H(\mathbf{x}) = H_0 A \rho \cos\phi \tag{149}$$

(ii) $a < \rho < b$:

$$\Phi_H(x) = H_0 \left[C\rho\cos\phi + Da^2\cos\phi/\rho \right] \tag{150}$$

(iii) $b < \rho$:

$$\Phi_H(\mathbf{x}) = H_0 \left[-\rho\cos\phi + Eb^2\cos\phi/\rho \right]. \tag{151}$$

Here, we have guessed (on the basis of long experience) that the only terms in the potential will be the ones with the same dependence on ϕ as the potential of the applied field. The latter is $-H_0 x = -H_0\rho\cos\phi$.

Continuing, one has the boundary conditions at $\rho = a$ and $\rho = b$ which are that H_ϕ and B_ρ must be continuous. These lead to the four equations

$$\begin{aligned} A &= C + D \\ C + D\frac{a^2}{b^2} &= -1 + E \\ A &= \mu C - \mu D \\ &\text{and} \\ \mu C - \mu D\frac{a^2}{b^2} &= -1 - E. \end{aligned} \tag{152}$$

The solution for A in particular is

$$A = -\left(\frac{4\mu}{(\mu+1)^2 - (a^2/b^2)(\mu-1)^2} \right), \tag{153}$$

from which one finds that the field inside of the shield is

$$\mathbf{H}(\mathbf{x}) = \mathbf{B}(\mathbf{x}) = \frac{4\mu}{(\mu+1)^2 - (a^2/b^2)(\mu-1)^2}\mathbf{H}_0. \tag{154}$$

The field inside is decreased relative to the applied field by a factor of A. If one employs a "high-μ" material with a permeability of order 10^4 or 10^5, then $a \sim 4/\{\mu[1-(a^2/b^2)]\}$ which will be much smaller than unity if a is significantly smaller than b. This is the limit in which the sleeve acts as a good shield against the applied magnetic field.