Boundary Value Problems in Electrostatics II

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Contents

Although this is a new chapter, we continue to do things begun in the previous chapter. In particular, the first topic is the separation of variable method in spherical polar coordinates.

1 Laplace Equation in Spherical Coordinates

The Laplacian operator in spherical coordinates is

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{1}$$

This is also a coordinate system in which it is possible to find a solution in the form of a product of three functions of a single variable each:

$$\Phi(r,\theta,\phi) = R(r)P(\theta)Q(\phi) = U(r)P(\theta)Q(\phi)/r. \tag{2}$$

Operate on Φ with ∇^2, and set the result equal to zero to find

$$\frac{PQ}{r}\frac{d^2U}{dr^2} + \frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{UP}{r^3\sin^2\theta}\frac{d^2Q}{d\phi^2} = 0 \tag{3}$$

Multiply by $r^3\sin^2\theta/UPQ$ to find

$$\frac{r^2\sin^2\theta}{U}\frac{d^2U}{dr^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0. \tag{4}$$

The first two terms are independent of ϕ while the third depends only on this variable. Thus the third must be a constant as must the sum of the first two; the first of these conditions is

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = C \text{ or } \frac{d^2Q}{d\phi^2} = CQ, \tag{5}$$

from which it follows that $Q \sim e^{\sqrt{C}\phi}$.

Now, a change in ϕ by 2π corresponds to no change whatsoever in spatial position; therefore, we must have $Q(\phi + 2\pi) = Q(\phi)$ because a function describing a measurable quantity must be a single-valued function of position. Hence we can conclude that $\sqrt{C} = im$ where m is an integer so that $e^{im2\pi} = 1$. Thus $C = -m^2$, and $Q(\phi) \to Q_m(\phi) = e^{im\phi}$, with $m = 0, \pm 1, \pm 2, \ldots$. We recognize that the functions Q_m can be used to construct a Fourier series and are a complete orthogonal set on the interval $\phi_0 \leq \phi \leq \phi + 2\pi$.

Returning now to Laplace's equation, Eq. (4), and using $-m^2$ for $\frac{1}{Q}\frac{d^2Q}{d\phi^2}$, we find

$$\frac{r^2 \sin^2\theta}{U}\frac{d^2U}{dr^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - m^2 = 0 \tag{6}$$

or

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{1}{\sin^2\theta}m^2 = 0. \tag{7}$$

In this expression we recognize that the first term depends only on r and next two, only on θ, so we as usual conclude that each term must be separately a constant and that the constants must add to zero. The first equation extracted by this device is

$$\frac{d^2U}{dr^2} = \frac{A}{r^2}U \tag{8}$$

where A is the constant. It is a standard convention to write A as $l(l+1)$ which is still quite general if l is allowed to be complex. Thus the preceding equation becomes

$$\frac{d^2U}{dr^2} = \frac{l(l+1)}{r^2}U. \tag{9}$$

The solutions of this ordinary, second-order, linear, differential equation are two in number and are $U \sim r^{l+1}$ and $U \sim 1/r^l$. Before commenting further on that, let us go on to the equation for $P(\theta)$.

1.1 Legendre Equation and Polynomials

Substitution of $l(l+1)$ for the first term in Eq. (7) produces

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0. \tag{10}$$

This is the *generalized Legendre Equation*; it is commonly written in terms of a different variable, namely $u \equiv \cos\theta$. Then one has

$$\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = \frac{du}{d\theta}\frac{d}{du}\left(\sqrt{1-u^2}\frac{du}{d\theta}\frac{dP}{du}\right) = -\sqrt{1-u^2}\frac{d}{du}\left(-(1-u^2)\frac{dP}{du}\right) \tag{11}$$

and

$$l(l+1) - \frac{m^2}{\sin^2\theta} = l(l+1) - \frac{m^2}{1-u^2}; \tag{12}$$

hence,

$$\frac{d}{du}\left((1-u^2)\frac{dP}{du}\right) + \left(l(l+1) - \frac{m^2}{1-u^2}\right)P = 0 \tag{13}$$

is the form of the generalized Legendre equation using u as the variable. The interval of interest to us is $0 \le \theta \le \pi$ which is $-1 \le u \le 1$.

We shall discuss first the special case of $m = 0$ which corresponds to $Q(\phi) = 1$, or a system for which $\Phi(\mathbf{x})$ is independent of ϕ; we shall call such a potential *azimuthally invariant*; there are many interesting systems which are more or less of this type. The equation for P is

$$\frac{d}{du}\left((1-u^2)\frac{dP}{du}\right)+l(l+1)P=0, \tag{14}$$

which is called the *Legendre equation.*

A standard procedure for solving this equation (and other similar second-order differential equations) is to assume that the solution can be written as a power series. Then there must be a smallest power α in the series, so we can write

$$P(u)=u^\alpha\sum_{j=0}^{\infty}a_ju^j=\sum_{j=0}^{\infty}a_ju^{j+\alpha} \tag{15}$$

from which we may evaluate the derivatives as

$$\frac{dP}{du}=\sum_{j=0}^{\infty}(j+\alpha)a_ju^{j+\alpha-1}, \tag{16}$$

$$\frac{d^2P}{du^2}=\sum_{j=0}^{\infty}(j+\alpha)(j+\alpha-1)a_ju^{j+\alpha-2}, \tag{17}$$

and

$$-\frac{d}{du}u^2\frac{dP}{du}=-\sum_{j=0}^{\infty}(j+\alpha)(j+\alpha+1)a_ju^{j+\alpha}. \tag{18}$$

Substitution into the Legendre equation gives

$$\sum_{j=0}^{\infty}\left[(\alpha+j)(\alpha+j-1)u^{\alpha+j-2}a_j-(\alpha+j)(\alpha+j+1)u^{\alpha+j}a_j+l(l+1)u^{\alpha+j}a_j\right]=0, \tag{19}$$

or, if we shift the zero of j in each term so as to isolate individual powers of u,

$$\alpha(\alpha-1)u^{\alpha-2}a_0+\alpha(\alpha+1)a_1u^{\alpha-1}+$$

$$\sum_{j=0}^{\infty}\left[(\alpha+j+2)(\alpha+j+1)a_{j+2}+\left(l(l+1)-(\alpha+j)(\alpha+j+1)\right)a_j\right]u^{\alpha+j}=0 \quad (20)$$

The only way that this power series can vanish for all u on the interval is to have the coefficient of each power of u vanish separately. Thus I will list the coefficients:

j	Coefficient of $u^{\alpha+j}$
-2	$a_0\alpha(\alpha-1)$
-1	$a_1\alpha(\alpha+1)$
$j\geq 0$	$[(\alpha+j+2)(\alpha+j+1)a_{j+2}+(l(l+1)-(\alpha+j)(\alpha+j+1))\,a_j]$

The coefficient of the leading (smallest) power $j=-2$ is zero if $\alpha=0$ or 1; $a_0=0$ is not an option because by definition the first term in the expansion has a nonvanishing coefficient. Thus we find at this juncture two possible allowed values of α.

$$\alpha=0,1 \tag{21}$$

In order that the coefficient of the next power $j=-1$ of u vanish we must have either $\alpha=0$ or $a_1=0$ (we can also have both); we can't have $\alpha=-1$ because of our first condition. Finally, the condition that the coefficient of $u^{\alpha+j}$ vanish for $j\geq 0$ is

$$a_{j+2}=\frac{(\alpha+j)(\alpha+j+1)-l(l+1)}{(\alpha+j+1)(\alpha+j+2)}a_j. \tag{22}$$

This is call the *recurrence relation.*

Consider this relation when j is very large, much larger than 1. In this limit it simplifies to the statement that $a_{j+2} = a_j(1+\mathcal{O}(1/j))$ which will produce a power series (for large powers) in the form of a sum of terms proportional to u^{2j}, all with the same coefficient. At $u \to 1$, this sum will not converge, and so P is singular at $u = 1$. This is not an allowed behavior for a solution to the Laplace equation, so we cannot have such a function representing the potential. What must therefore happen is that the series terminates which means that there must be some j such that $a_j \neq 0$ while $a_{j+2} = 0$. Examining Eq. (22), we see that this j is such that

$$(\alpha + j)(\alpha + j + 1) - l(l + 1) = 0. \tag{23}$$

This condition requires that $\alpha + j = l$ which is a condition on l; since α is 0 or 1, and j is a non-negative integer, we see that l must be an integer equal to or larger than α.

$$l \in \mathcal{Z} \quad l \geq \alpha \tag{24}$$

Now, our recurrence relation gives us a_{j+2} from a_j; hence, starting from a_0, we can get only the even coefficients a_j, j even, and starting from a_1, we get the odd coefficients. Lets consider the odd series and the even series separately. First consider the **even series**. Since at termination of the series $\alpha + j = l$, we see that l is even when $\alpha = 0$ and l is odd when $\alpha = 1$. Thus the even series terminates when

$$\alpha = 0 \text{ and } l \text{ even} \tag{25}$$

$$\alpha = 1 \text{ and } l \text{ odd}$$

By similar arguments applied to the **odd series**, we can see that it terminates when

$$\alpha = 0 \text{ and } l \text{ odd} \tag{26}$$

$$\alpha = 1 \text{ and } l \text{ even}$$

Since l cannot be both odd and even, we can only have an even or an odd series (they are actually equivalent). The other must be zero. Since by convention, we choose $a_0 \neq 0$, it must be that the odd series vanishes, and thus $a_1 = 0$. Remembering that l is odd when $\alpha = 1$ and is even when $\alpha = 0$, we see that the solutions are polynomials of degree l. They are known as *Legendre polynomials*. It is easy to generate a few of them, aside from normalization, starting from $l = 0$ and using the recurrence relation. As for normalization, they are traditionally chosen to be such that $P(1) = 1$. Let us add a subscript l to P to designate the particular Legendre polynomial. The first few are

l	$P_l(u)$
0	$P_0(u) = 1$
1	$P_1(u) = u$
2	$P_2(u) = \frac{3}{2}u^2 - \frac{1}{2}$
3	$P_3(u) = \frac{5}{2}u^3 - \frac{3}{2}u$
4	$P_4(u) = \frac{35}{8}u^4 - \frac{15}{4}u^2 + \frac{3}{8}$

There are other ways to generate the Legendre polynomials. For example, one has *Rodrigues' formula* which is

$$P_l(u) = \frac{1}{2^l l!}\frac{d^l}{du^l}(u^2-1)^l; \tag{27}$$

it is easy to see that this generates a polynomial of degree l; one may show that it is a solution to the Legendre equation by direct substitution into that equation. Thus, it must be the Legendre polynomial (one should also check normalization). If one expands the factor $(u^2-1)^l$ in Rodrigues' formula using the binomial expansion and then takes the derivatives, she/he will find that

$$P_l(u) = \frac{(-1)^l}{2^l}\sum_{n\geq l/2}^{l}\frac{(-1)^n(2n)!}{n!(l-n)!(2n-l)!}u^{2n-l} \tag{28}$$

Another way of generating $P_l(u)$ is via the generating function

$$T(u,x) = (1-2ux+x^2)^{-1/2} = \sum_{l=0}^{\infty} x^l P_l(u). \tag{29}$$

If one takes l derivatives of this function with respect to x and then sets $x=0$, the result is $l!P_l(u)$.

The Legendre functions have many properties that we will need to make use of from time to time. For summaries of these, see *e.g.*, , the section on Legendre functions in Abramowitz and Stegun starting on p. 332 and also the section on orthogonal polynomials starting on p. 771. Here we summarize some of the most significant properties. First, orthogonality and normalization. Consider the integral

$$\int_{-1}^{1} duP_l(u)P_{l'}(u) = \frac{1}{2^{l+l'}l!l'!}\int_{-1}^{1} du\frac{d^l}{du^l}(u^2-1)^l\frac{d^{l'}}{du^{l'}}(u^2-1)^{l'}. \tag{30}$$

Suppose, without loss of generality, that $l' \geq l$ and start by integrating by parts,

$$\int_{-1}^{1} du P_l(u) P_{l'}(u) =$$

$$\frac{1}{2^{l+l'} l! l'!} \left[\frac{d^l}{du^l}(u^2-1)^l \frac{d^{l'-1}}{du^{l'-1}}(u^2-1)^{l'} \Big|_{-1}^{1} - \int_{-1}^{1} du \frac{d^{l+1}}{du^{l+1}}(u^2-1)^l \frac{d^{l'-1}}{du^{l'-1}}(u^2-1)^{l'} \right] \quad (31)$$

The first term in brackets vanishes because (u^2-1) is zero at the end points of the interval. Continuing from the right-hand side, integrate in like fashion $l'-1$ more times. The result is

$$\int_{-1}^{1} du P_l(u) P_{l'}(u) = (-1)^{l'} \frac{1}{2^{l+l'} l! l'!} \int_{-1}^{1} du \frac{d^{l+l'}(u^2-1)^l}{du^{l+l'}} (u^2-1)^{l'}. \quad (32)$$

Now, $d^{l+l'}(u^2-1)^l / du^{l+l'} = 0$ if $l' > l$, and so the integral is zero in this case. If $l' = l$, we have

$$\int_{-1}^{1} du P_l(u) P_l(u) = (-1)^l \frac{(2l)!}{2^{2l}(l!)^2} \int_{-1}^{1} du (u^2-1)^l$$

$$= (-1)^l \frac{(2l)!}{2^{2l}(l!)^2} 2(-1)^l \frac{(2l)!!}{(2l+1)!!} = 2\frac{(2^l l!)^2(2l-1)!!}{2^{2l}(l!)^2(2l+1)!!} = \frac{2}{2l+1}. \quad (33)$$

Thus we have derived the relation

$$\int_{-1}^{1} du P_l(u) P_{l'}(u) = \frac{2}{2l+1} \delta_{ll'} \quad (34)$$

which expresses the orthogonality and normalization of the Legendre polynomials.

Consider next recurrence relations. These provide, among other things, a good way to generate values of Legendre polynomials on computers. A number of recurrence relations can be derived using Rodrigues' formula and the Legendre equation. Consider, for example,

$$\begin{aligned}
\frac{dP_{l+1}}{du} &= \frac{1}{2^{l+1}(l+1)!}\frac{d^{l+2}}{du^{l+2}}(u^2-1)^{l+1} \\
&= \frac{l+1}{2^l(l+1)!}\frac{d^{l+1}}{du^{l+1}}\left((u^2-1)^l u\right) \\
&= \frac{1}{2^l l!}\frac{d^l}{du^l}\left[(u^2-1)^l + 2lu^2(u^2-1)^{l-1}\right] \\
&= \frac{1}{2^l l!}\frac{d^l}{du^l}\left[(u^2-1)^l + 2l(u^2-1)^l + 2l(u^2-1)^{l-1}\right] \\
&= (2l+1)P_l(u) + \frac{dP_{l-1}(u)}{du},
\end{aligned} \tag{35}$$

or

$$\frac{dP_{l+1}(u)}{du} - (2l+1)P_l(u) - \frac{dP_{l-1}(u)}{du} = 0. \tag{36}$$

From this relation and the Legendre equation

$$\frac{d}{du}\left((1-u^2)\frac{dP_l}{du}\right) + l(l+1)P_l = 0 \tag{37}$$

one may derive additional standard recurrence relations for the Legendre polynomials. Several of these are

$$\begin{aligned}
(l+1)P_{l+1} - u(2l+1)P_l + lP_{l-1} &= 0 \\
(1-u^2)\frac{dP_l}{du} + luP_l - lP_{l-1} &= 0 \\
\frac{dP_{l+1}}{du} - u\frac{dP_l}{du} - (l+1)P_l &= 0.
\end{aligned} \tag{38}$$

These may be used to advantage in numerous applications such as doing integrals of products of two Legendre polynomials and a power of u.

1.2 Solution of Boundary Value Problems with Azimuthal Symmetry

Using what we have learned in the previous two sections, we are now in a position to construct a general solution to the Laplace equation in spherical coordinates under conditions of azimuthal invariance, that is, when $\Phi(\mathbf{x})$ is independent of ϕ. The most general form that a solution can have is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta). \tag{39}$$

The Legendre polynomials form a complete set on the interval $-1 \leq u \leq 1$ or $0 \leq \cos\theta \leq \pi$. Thus any specified ϕ-independent potential on a spherical surface can be expressed as a sum of P_l's. If the volume in which a solution is to be found includes the origin, then none of the terms $\sim r^{-(l+1)}$ can be included in the sum as they are singular at the origin, and the potential will not be singular there. Similarly, if the volume extends to $r \rightarrow \infty$, then no terms $\sim r^l$ are allowed. In the former case, the conclusion is that $B_l = 0$ for all l, and in the latter case, all $A_l = 0$.

We consider now some examples.

1.2.1 Example: A Sphere With a Specified Potential

An isolated sphere of radius a is centered at the origin. By unspecified means, the potential on its surface is maintained at

$$\Phi(a, \theta, \phi) = V_o \cos^3(\theta)$$

where θ is the polar angle. Find $\Phi(r, \theta, \phi)$ for all $r > a$.

This problem is azimuthally symmetric. Thus, in general,

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta).$$

Since our volume contains all $r > a$, physics demands that $A_l = 0$ for all l. The constants B_l are then determined by matching the terms in the series the boundary condition on the surface of the sphere. Recall that

$P_0(x)$	1
$P_1(x)$	x
$P_2(x)$	$\frac{1}{2}(3x^2 - 1)$
$P_3(x)$	$\frac{1}{2}(5x^3 - 3x)$
$P_4(x)$	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
$P_5(x)$	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$

So that,

$$\Phi(a, \theta, \phi) = V_o \cos^3(\theta) = V_o \left(\frac{2}{5} P_3(\cos(\theta)) + \frac{3}{5} P_1(\cos(\theta)) \right)$$

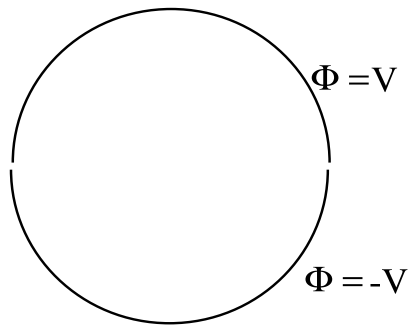
Thus

$$\Phi(r,\theta,\phi) = V_o \left(\frac{2}{5} \left(\frac{a}{r}\right)^4 P_3(\cos(\theta)) + \frac{3}{5} \left(\frac{a}{r}\right)^2 P_1(\cos(\theta)) \right).$$

1.2.2 Example: Hemispheres of Opposite Potential

For the first, suppose that we need to solve the Laplace equation inside of a sphere of radius a given that on the surface, the potential is specified as follows:

$$\Phi(a,\theta) = \begin{cases} V, & 0 \le \theta \le \pi/2 \\ -V, & \pi/2 \le \theta \le \pi. \end{cases} \tag{40}$$



Then the expansion must take the form

$$\Phi(r,\theta) = V \sum_{l=0}^{\infty} A_l \left(\frac{r}{a}\right)^l P_l(\cos\theta). \tag{41}$$

Notice the introduction of the factor V on the right-hand side, along with the use of the powers of a in the sum. These are included for convenience. The scale of the potential and hence the size of a leading term in the sum is set by V which also gives the correct dimensions to the terms in the sum; it is thus natural to put this factor in each term. The powers of a are included for the same reasons; r is of order a and has the same dimensions so that leading coefficients A_l are of order unity and have dimension unity.

On the spherical surface, we have

$$\Phi(a, \theta) = V \sum_{l=0}^{\infty} A_l P_l(\cos\theta). \tag{42}$$

In order to find a given coefficient A_n, we multiply this equation by $P_n(\cos\theta)$ and integrate over $\cos\theta$, recalling that $d\cos\theta = -\sin\theta d\theta$. Making use of the orthogonality and normalization of the Legendre polynomials, we find

$$\int_0^{\pi} d\theta \Phi(a, \theta) P_n(\cos\theta) \sin\theta = V \sum_{l=0}^{\infty} A_l \left(\frac{2}{2l+1}\right) \delta_{ln} = V \left(\frac{2}{2n+1}\right) A_n, \tag{43}$$

or

$$\begin{aligned} A_n = \frac{2n+1}{2} \left[\int_0^{\pi/2} d\theta P_n(\cos\theta) \sin\theta - \int_{\pi/2}^{\pi} d\theta P_n(\cos\theta) \sin\theta\right] \\ = \frac{2n+1}{2} \left[\int_0^1 du P_n(u) - \int_{-1}^{0} du P_n(u)\right] \end{aligned} \tag{44}$$

Now use the inversion property of the Legendre polynomials, $P_n(u) = (-1)^n P_n(-u)$ to conclude that

$$A_n = \frac{2n+1}{2} \left[\int_0^1 du P_n(u) - (-1)^n \int_0^1 du P_n(u)\right] = \begin{cases} 0 & n \text{ even} \\ (2n+1) \int_0^1 du P_n(u) & n \text{ odd.} \end{cases} \tag{45}$$

To complete the integral for the case of odd n we use a recurrence relation

$$\frac{dP_{n+1}}{du} = (2n+1)P_n + \frac{dP_{n-1}}{du} \tag{46}$$

so that:

$$A_n = (2n+1)\int_0^1 duP_n(u) = \int_0^1 du\left[\frac{dP_{n+1}}{du} - \frac{dP_{n-1}}{du}\right] = P_{n-1}(0) - P_{n+1}(0) \tag{47}$$

where we make use of the fact that $P_n(1) = 1$, independent of n. Further, for even l,

$$P_l(0) = (-1)^{l/2}\frac{(l-1)!!}{l!!} \tag{48}$$

where the "double factorial" sign means $l!! = l(l-2)(l-4)...(2 \text{ or } 1)$. Hence

$$P_{n-1}(0) - P_{n+1}(0) = (-1)^{(n-1)/2}\left[\frac{(n-2)!!}{(n-1)!!} + \frac{n!!}{(n+1)!!}\right]$$

$$= (-1)^{(n-1)/2}\frac{(n-2)!!}{(n+1)!!}(n+1-n) = (-1)^{(n-1)/2}\frac{(n-2)!!}{(n+1)!!}(2n+1). \tag{49}$$

Now set $n = 2m+1$, $m = 0,1,2,...$, and have

$$\Phi(r,\theta) = V\sum_{m=0}^{\infty} B_m\left(\frac{r}{a}\right)^{2m+1} P_{2m+1}(\cos\theta) \tag{50}$$

where

$$B_m = (-1)^m\frac{(2m-1)!!}{(2m+2)!!}(4m+3). \tag{51}$$

The first few terms in the expansion are

$$\Phi(r,\theta) = \frac{3}{2}V\left\{\frac{r}{a}P_1(\cos\theta) - \frac{7}{12}\left(\frac{r}{a}\right)^3 P_3 + \frac{11}{24}\left(\frac{r}{a}\right)^5 P_5 - \frac{25}{64}\left(\frac{r}{a}\right)^7 P_7 + ...\right\} \tag{52}$$

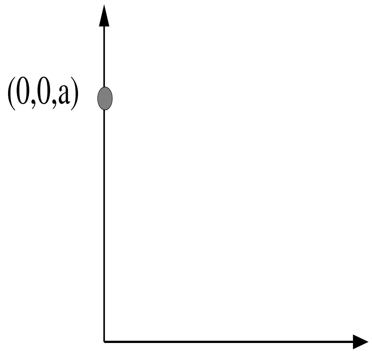
1.2.3 Example: Potential of an Isolated Charge

Another method of finding the coefficients in the expansion makes use of the fact that the expansion is unique. If, for example, we are able to find the potential at fixed $\theta = \theta_0$ for all r,

$$\Phi(\theta_0, r) = g(r) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta_0) \tag{53}$$

then we can infer the form of the expansion by expanding $g(r)$ in powers of r and recognizing that the coefficient of r^l *must* be $P_l(\cos\theta_0)$ times a coefficient A_l while the coefficient of $(1/r)^{-(l+1)}$ must be $B_l P_l(\cos\theta_0)$. The most convenient value of θ_0 to use is certainly 0 or π since we know immediately the value of $P_l(\cos\theta)$ in these instances.

Consider the following specific example: Suppose that there is a charge q at a position $\mathbf{x} = a\hat{\mathbf{z}}$



in which case we know that the potential is

$$\Phi(r, \theta) = \frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}. \tag{54}$$

For $\theta = 0$, we have simply $\phi(\mathbf{x}) = q/|r - a|$. At $r < a$ in particular, this function has a simple power series expansion,

$$\Phi(r,0) = \frac{q}{a-r} = \frac{q}{a}\frac{1}{1-r/a} = \frac{q}{a}\left\{1 + \frac{r}{a} + \frac{r^2}{a^2} + \frac{r^3}{a^3} + ...\right\}. \quad r < a \tag{55}$$

Hence, associating P_l with $(r/a)^l$, we have

$$\Phi(r,\theta) = \frac{q}{a}\sum_{l=0}^{\infty}\left(\frac{r}{a}\right)^l P_l(\cos\theta); \quad r < a \tag{56}$$

the point is, the uniqueness of the expansion in terms of Legendre polynomials tells us that this must be the solution. A similar expansion done for $r > a$ yields

$$\Phi(r,\theta) = \frac{q}{r}\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(\cos\theta) \quad r > a\,. \tag{57}$$

There are two points that are worth making in connection with these expansions. First, as stated earlier, there is a generating function $T(u,x)$ for the Legendre polynomials; see Eq. (29). We have just derived it; that is, it is Φ, Eq. (54), equal to the sum in Eq. (56). Also, we have obtained a convenient and useful expansion for the potential of a point charge; in more general notation, we have derived

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \tag{58}$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$ while $r_<$ ($r_>$) is the smaller (larger) of $|\mathbf{x}|$ and $|\mathbf{x}'|$.

1.3 Behavior of Fields in Conical Holes and Near Sharp Points

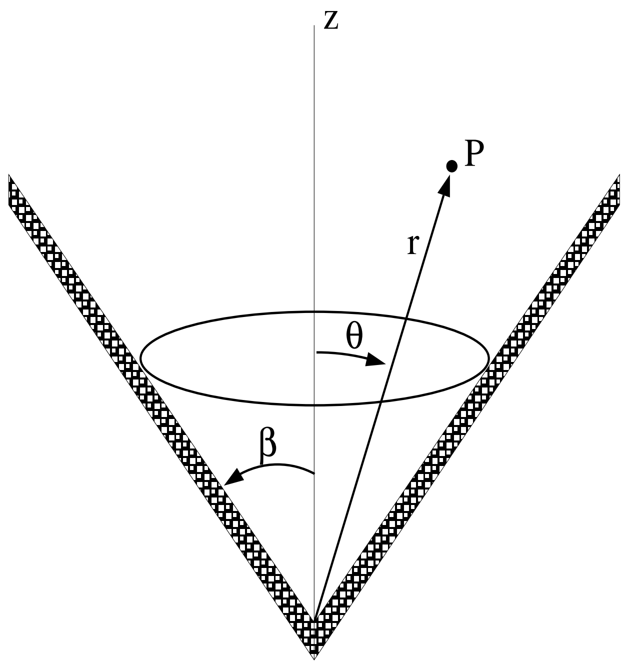
The field in the vicinity of the apex of a cone-shaped tip or depression can also be investigated using the separation of variables method in spherical coordinates. The solution for the potential is of the form, for r small enough,

$$\Phi(r,\theta) \sim r^{\nu} P_{\nu}(\cos\theta) \tag{59}$$

where $P_\nu(u)$ is a solution to the Legendre equation

$$\frac{d}{du}(1-u^2)\frac{dP_\nu}{du} + \nu(\nu+1)P_\nu = 0 \tag{60}$$

with ν to be determined.



For the geometry shown, the solution must be well-behaved as $\theta \to 0$, or $u = \cos\theta \to 1$, but not necessarily as $\theta \to \pi$ or $u = -1$. Introduce the variable $y \equiv \frac{1}{2}(1-u)$ or $u = 1-2y$; then Eq. (60) becomes

$$-\frac{1}{2}\frac{d}{dy}\left(1-(1-2y)^2\right)\left(-\frac{1}{2}\frac{dP_\nu}{dy}\right)+\nu(\nu+1)P_\nu = 0 \tag{61}$$

or

$$\frac{d}{dy}\left(y(1-y)\frac{dP_\nu}{dy}\right)+\nu(\nu+1)P_\nu = 0. \tag{62}$$

Let us look once again for a solution in the form of a power series expansion,

$$P_\nu = y^\alpha \sum_{j=0}^{\infty} a_j y^j, \tag{63}$$

with $0 \leq y \leq y_0 \leq 1$. Then

$$\frac{dP_\nu}{dy} = \sum_{j=0}^{\infty}(\alpha+j)a_j y^{\alpha+j-1}, \tag{64}$$

$$y(1-y)\frac{dP_\nu}{dy} = \sum_{j=0}^{\infty}(\alpha+j)a_j(y^{\alpha+j}-y^{\alpha+j+1}), \tag{65}$$

and

$$\frac{d}{dy}\left(y(1-y)\frac{dP_\nu}{dy}\right) = \sum_{j=0}^{\infty} a_j\left[(\alpha+j)y^{\alpha+j-1}-(\alpha+j+1)y^{\alpha+j}\right](\alpha+j). \tag{66}$$

Now combine these equations to find

$$\sum_{j=0}^{\infty}\left[a_j(\alpha+j)^2 y^{\alpha+j-1}+a_j\left((\alpha+j)(\alpha+j+1)+\nu(\nu+1)\right)y^{\alpha+j}\right]=0 \tag{67}$$

or, isolating individual powers of y,

$$a_0\alpha^2 y^{\alpha-1} = 0 \tag{68}$$

which implies that $\alpha = 0$, and

$$a_{j+1} = a_j \frac{j(j+1) - \nu(\nu+1)}{(j+1)^2}. \tag{69}$$

If one lets $\nu = l$, a non-negative integer, the result is just the Legendre polynomials (no surprise), viewed as functions of y. More generally, for any real $\nu > 0$, one finds that the solutions are *Legendre functions of the first kind of order* ν.

$$P_\nu = \sum_{j=0}^{\infty} a_j(\nu) y^j, \tag{70}$$

These are well-behaved (that is, they are not singular) functions of y for $y < 1$ corresponding to $u > -1$ and are singular at $y = 1$.

For $1 > \nu > 0$, $P_\nu(y)$ has a single zero; for $2 > \nu > 1$, $P_\nu(y)$ has two zeroes, etc. This is important because if we have a cone of half-angle β with equipotential surfaces, we need ν to be such that $P_\nu\left((1-\cos\beta)/2\right) = 0$. There will thus be a sequence of allowed values of ν, which we designate by ν_k, $k = 1, 2, 3, \ldots$, which are such that $y_\beta \equiv \frac{1}{2}(1-\cos\beta) \equiv k^{th}$ zero of P_ν.

The general solution at finite values of r, and including the point $r = 0$, is

$$\Phi(r,\theta) = \sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta). \tag{71}$$

For small r, the leading term is the one with the smallest power of r, that is the $k = 1$ term. Hence we may approximate the sum sufficiently close to the origin by its leading term

$$\Phi(r,\theta) \approx A r^{\nu_1} P_{\nu_1}(\cos\theta). \tag{72}$$

The dominant contribution to the electric field in this region comes from this term; we have, by the usual $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$,

$$E_r = \frac{d\Phi}{dr} = -\nu_1 A r^{\nu_1 - 1} P_{\nu_1}(\cos\theta) \tag{73}$$

and

$$E_\theta = -\frac{1}{r}\frac{d\Phi}{d\theta} = A \sin\theta \, r^{\nu_1 - 1} \left.\frac{dP_{\nu_1}(u)}{du}\right|_{\cos\theta} \tag{74}$$

The behavior of ν_1 as a function of β is shown below. For β less than about 0.8π, one has[1] $\nu_1 \approx \frac{2.405}{\beta} - \frac{1}{2}$, while for β larger than about the same number, $\nu_1 \approx \left[2\ln\left(\frac{2}{\pi-\beta}\right)\right]^{-1}$. As $\beta \to \pi$, $\nu_1 \to 0$ and so $E_r \sim E_\theta \sim 1/r$ in this limit. The enhancement of a field near *e.g.*, a lightning rod is thus $\sim (R/\delta)$ if R is the size of the system and δ is the radius of curvature of the tip of the rod. Recall that in two dimensions

[1] This relation comes from study of the properties of the Legendre functions.

we found an enhancement of order $(R/\delta)^{1/2}$ The enhancement is much more pronounced in three dimensions; a three dimensional tip is a much sharper thing than an edge.

1.4 Associated Legendre Polynomials; Spherical Harmonics

Let us now return to the more general case of a solution to Laplace's equation (i.e. a potential) which depends on the azimuthal angle ϕ. Then we must have the functions of ϕ $e^{im\phi}$, or, equivalently, $\sin m\phi$ and $\cos m\phi$, and the differential equation we have to face on the space of θ is

$$\frac{d}{du}(1-u^2)\frac{dP}{du} + \left(l(l+1) - \frac{m^2}{1-u^2}\right) P = 0. \tag{75}$$

The solutions are not finite polynomials in u in general but can be expressed as infinite power series. They are only "well-behaved" on the interval $-1 \leq u \leq 1$ when $l \geq |m|$, with l an integer. Then there is just one well-behaved solution which is known as the *associated Legendre function of degree l and order m*. For $m \geq 0$, the associated Legendre function can be written in terms of the Legendre polynomial of the same degree as

$$P_l^m(u) = (-1)^m (1-u^2)^{m/2} \frac{d^m}{dx^m}\left(P_l(u)\right); \tag{76}$$

one can read all about this in Abramowitz and Stegun on pages 332 to 353. Making use of Rodrigues' formula for the Legendre polynomials, we see that

$$P_l^m(u) = (-1)^m \frac{(1-u^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{du^{l+m}}[(u^2-1)^l]. \tag{77}$$

This last formula is also valid for negative m [2]; comparing the two cases, one may see that

$$P_l^{-m}(u) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(u). \tag{78}$$

As for the Legendre polynomials, there is a generating function for the associated Legendre functions as well as a variety of recurrence relations. For example, a recurrence relation in degree is given by

$$(2l+1)uP_l^m(u) = (l-m+1)P_{l+1}^m(u) + (l+m)P_{l-1}^m(u) \tag{79}$$

and one in order is

$$P_l^{m+1} + \frac{2mu}{\sqrt{1-u^2}} P_l^m(u) + (l-m+1)(l+m)P_l^{m-1}(u) = 0 \tag{80}$$

Out of all of this, what is of importance to us is that the product

$$(Ar^l + Br^{-l-1})P_l^m(\cos\theta)e^{im\phi} \tag{81}$$

is a solution of the Laplace equation and that the set of functions $e^{im\phi}P_l^m(\cos\theta)$ with $l = 0, 1, 2, ...,$ and $m = -l, -l+1, ... l-1, l$ form a complete orthogonal set on the two-dimensional domain $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. As usual, completeness is difficult to demonstrate

[2] That's in part a matter of definition.

but orthogonality is quite straightforward using the formulae we have already written down. Consider the integral

$$I = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta e^{-im\phi} P_l^m(\cos\theta) e^{im'\phi} P_{l'}^{m'}(\cos\theta) = 2\pi\delta_{mm'} \int_{-1}^{1} du P_l^m(u) P_{l'}^m(u) \tag{82}$$

Assume $l' \geq l$, $m \geq 0$, and write $P_{l'}^m$ in terms of $P_{l'}^{-m}$:

$$I = 2\pi\delta_{mm'}(-1)^m \frac{(l'+m)!}{(l'-m)!} \int_{-1}^{1} du P_{l'}^{-m}(u) P_l^m(u)$$

$$= 2\pi\delta_{mm'}(-1)^m \frac{(l'+m)!}{(l-m)!} \frac{1}{2^{l+l'} l! l'!} \int_{-1}^{1} du \frac{d^{l'-m}}{du^{l'-m}}(u^2-1)^{l'} \frac{d^{l+m}}{du^{l+m}}(u^2-1)^l$$

$$= 2\pi\delta_{mm'}(-1)^{l'} \frac{(l'+m)!}{(l'-m)!} \frac{1}{2^{l+l'} l! l'!} \int_{-1}^{1} du (u^2-1)^{l'} \frac{d^{l+l'}}{du^{l+l'}}(u^2-1)^l$$

$$= 2\pi\delta_{ll'}\delta_{mm'} \frac{(l+m)!(2l)!(2l)!!}{(l-m)! 2^{2l} (l!)^2 (2l+1)!!} 2 = \frac{4\pi}{2l+1} \delta_{ll'}\delta_{mm'} \frac{(l+m)!}{(l-m)!} \tag{83}$$

Thus we may construct an orthonormal set of functions on the surface of the unit sphere; these are called *spherical harmonics* and are defined as

$$Y_{l,m}(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}, \tag{84}$$

with $m = -l, -l+1, ..., l-1, m$ and $l = 0, 1, 2, ...$ These functions have the property that

$$Y_{l,m}^*(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi). \tag{85}$$

The condition of orthonormality is

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{l,m}^*(\theta,\phi) Y_{l'm',}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{86}$$

The completeness relation (not derived as usual) is

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{m} Y_{l,m}^{*}(\theta',\phi')Y_{l,m}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta'). \tag{87}$$

A general function $g(\theta,\phi)$ is expanded in terms of the spherical harmonics as

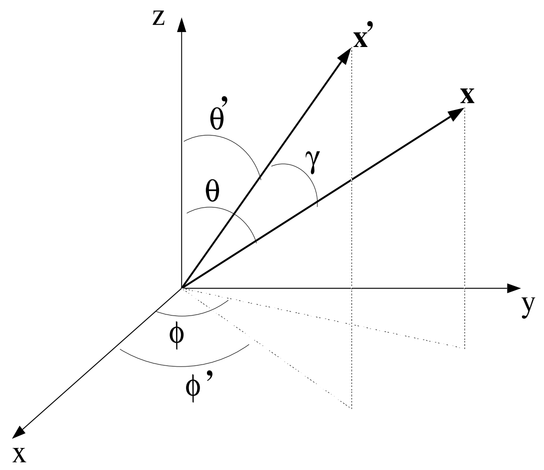
$$g(\theta,\phi) = \sum_{l,m} A_{lm}Y_{l,m}(\theta,\phi) \tag{88}$$

with

$$A_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta g(\theta,\phi)Y_{l,m}^{*}(\theta,\phi). \tag{89}$$

1.5 The Addition Theorem

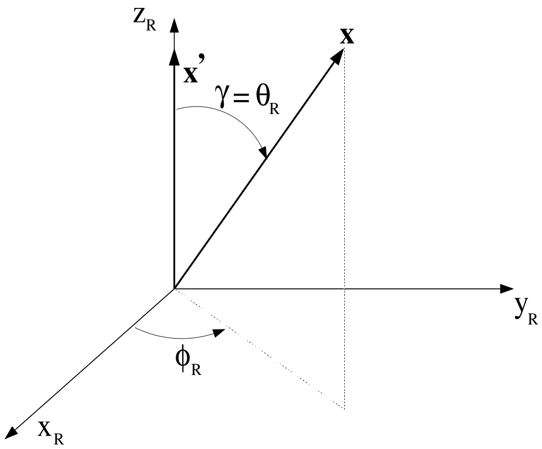
In applications we will occasionally have need to know the function $P_l(\cos\gamma)$ where γ is the angle between two vectors $\mathbf{x}$ and $\mathbf{x}'$; it will prove to be useful to be able to write this function in terms of the variables θ, ϕ, θ', and ϕ'.



It must be possible to do so in terms of any complete sets of functions of these variables such as the spherical harmonics. In fact, the expansion is

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{l,m}(\theta',\phi')Y_{l,m}(\theta,\phi). \tag{90}$$

We shall derive this expression as an example of the use of spherical harmonics and their properties. First, let us set up a second coordinate system rotated relative to the original one in such a way that its polar axis lies along the direction of $\mathbf{x}'$. In this system, the vector $\mathbf{x}$ has components (r_R, θ_R, ϕ_R).



Further, $\theta_R = \gamma$. Next, we may regard $P_l(\cos\gamma)$ as a function of θ and ϕ for fixed θ' and ϕ' and so can certainly expand it as

$$P_l(\cos\gamma) = \sum_{l'=0}^{\infty}\sum_{m'=-l}^{l} A_{l'm'}(\theta',\phi')Y_{l'm',}(\theta,\phi). \tag{91}$$

Similarly, in terms of spherical harmonics whose arguments are coordinates in the rotated system, it is easy to see that

$$P_l(\cos\gamma) = \sqrt{\frac{4\pi}{2l+1}}Y_{l,0}(\theta_R,\phi_R). \tag{92}$$

Now, the spherical harmonics satisfy the differential equation

$$\nabla^2 Y_{l,m}(\theta,\phi) + \frac{l(l+1)}{r^2} Y_{l,m}(\theta,\phi) = 0 \tag{93}$$

and they also satisfy this equation with variables θ_R, ϕ_R. But the Laplacian operator $\nabla^2 = \nabla \cdot \nabla$ is a scalar object which is invariant under coordinate rotations which means we can write it in the unrotated frame while writing the spherical harmonic in the rotated frame:

$$\nabla^2 Y_{l,m}(\theta_R,\phi_R) + \frac{l(l+1)}{r^2} Y_{l,m}(\theta_R,\phi_R) = 0. \tag{94}$$

Now recall that

$$Y_{l,0}(\theta_R,\phi_R) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\gamma) = \sqrt{\frac{2l+1}{4\pi}} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} A_{l'm'}(\theta',\phi') Y_{l'm',}(\theta,\phi). \tag{95}$$

If we plug this into the differential equation above, we obtain:

$$\sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} A_{l'm'}(\theta',\phi') [\nabla^2 Y_{l'm',}(\theta,\phi) + \frac{l(l+1)}{r^2} Y_{l'm',}(\theta,\phi)] = 0 \tag{96}$$

or

$$\sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} A_{l'm'}(\theta',\phi') \left[-\frac{l'(l'+1)}{r^2} + \frac{l(l+1)}{r^2} \right] Y_{l'm',}(\theta,\phi) = 0. \tag{97}$$

This equation can be true only if $l = l'$ **or** if $A_{l'm'} = 0$. Thus we have demonstrated that $A_{l'm'} = 0$ for $l' \neq l$, and the expansion of P_l reduces to

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_{lm}(\theta',\phi') Y_{l,m}(\theta,\phi). \tag{98}$$

The coefficients in this expansion are found in the usual way for an orthogonal function expansion,

$$A_{lm} = \int d\Omega P_l(\cos\gamma) Y^*_{l,m}(\theta,\phi) = \int d\Omega_R P_l(\cos\theta_R) Y^*_{l,m}(\theta,\phi). \tag{99}$$

Following the same line of reasoning, we may express $\sqrt{4\pi/(2l+1)}Y^*_{l,m}(\theta,\phi)$ as a sum of the form

$$\sqrt{\frac{4\pi}{2l+1}} Y^*_{l,m}(\theta,\phi) = \sum_{m'=-l}^{l} B_{lm'}(m) Y_{lm'}(\theta_R,\phi_R) \tag{100}$$

where B_{l0} in particular is

$$\begin{aligned} B_{l0}(m) &= \int d\Omega_R \sqrt{\frac{4\pi}{2l+1}} Y^*_{l,m}(\theta,\phi) Y^*_{l,0}(\theta_R,\phi_R) \\ &= \int d\Omega_R Y^*_{l,m}(\theta,\phi) P_l(\cos\theta_R) \equiv A_{lm}. \end{aligned} \tag{101}$$

However, from Eq. (76), it is clear that when $u = 1$ $P_l^m(u) = 0$ when $m \neq 0$,and $P_l^m(u) = P_l(u)$ when $m = 0$, thus

$$\sqrt{\frac{4\pi}{2l+1}} \left. Y^*_{l,m}(\theta,\phi)\right|_{\theta_R=0} = B_{l0}(m)\sqrt{\frac{2l+1}{4\pi}} P_l(1) = B_{l0}(m)\sqrt{\frac{2l+1}{4\pi}}, \tag{102}$$

so

$$A_{lm} = B_{l0}(m) = \frac{4\pi}{2l+1} \left. Y^*_{l,m}(\theta,\phi)\right|_{\theta_R=0} = Y^*_{l,m}(\theta',\phi')\frac{4\pi}{2l+1}, \tag{103}$$

where the last step follow since when $\theta_R = 0$, $\mathbf{x} = \mathbf{x}'$. Thus,

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{l,m}(\theta',\phi') Y_{l,m}(\theta,\phi). \tag{104}$$

Thus ends our demonstration of the spherical harmonic addition theorem.

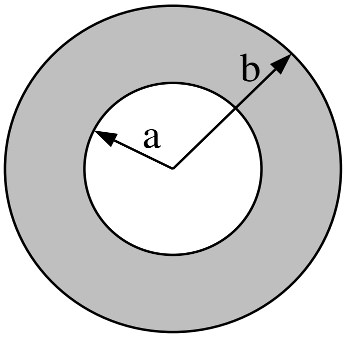
An application if this theorem is that we can write $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ as an expansion in spherical harmonics:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) = \sum_{l=0}^{\infty} \left\{ \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{l} \left[Y_{l,m}^*(\theta', \phi') Y_{l,m}(\theta, \phi) \right] \right\}. \tag{105}$$

This expansion is often useful when faced with common integrals in electrostatics such as as $\int d^3x' \rho(\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|$.

1.6 Expansion of the Green's Function in Spherical Harmonics

More as an example of the use of the addition theorem than anything else, let us devise an expansion for the Dirichlet Green's function for the region V bounded by $r = a$ and $r = b$, $a < b$.



This function can be written as

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}') \tag{106}$$

where $\nabla^2 F = 0$ in V. Thus it must be possible to write

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{l} Y_{l,m}^*(\theta', \phi') Y_{l,m}(\theta, \phi)$$

$$+\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\frac{r^l}{b^{l+1}}+B_{lm}\frac{a^l}{r^{l+1}}\right)Y_{l,m}(\theta,\phi) \tag{107}$$

where A_{lm} and B_{lm} can be functions of $\mathbf{x}'$. The first term on the right-hand side is $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ and the second is a general solution of the Laplace equation. The coefficients are determined by requiring that the boundary conditions on $G(\mathbf{x},\mathbf{x}')$ are satisfied ($G(\mathbf{x},\mathbf{x}')=0$ for $\mathbf{x}$ on one of the two bounding spherical surfaces). At $r=a$ ($r_<=r=a$ $r_>=r'$), we have

$$0=\sum_{l,m}\left\{\frac{4\pi}{2l+1}\frac{a^l}{r'^{l+1}}Y^*_{l,m}(\theta',\phi')+A_{lm}\frac{a^l}{b^{l+1}}+B_{lm}\frac{1}{a}\right\}Y_{l,m}(\theta,\phi)=0, \tag{108}$$

from which it follows, using the orthogonality of the spherical harmonics, that

$$\frac{4\pi}{2l+1}\frac{a^l}{r'^{l+1}}Y^*_{l,m}(\theta',\phi')+A_{lm}\frac{a^l}{b^{l+1}}+B_{lm}\frac{1}{a}=0. \tag{109}$$

By similar means applied at $r=b$ ($r_<=r'$ $r_>=r=b$), one may show that

$$\frac{4\pi}{2l+1}\frac{r'^l}{b^{l+1}}Y^*_{l,m}(\theta',\phi')+A_{lm}\frac{1}{b}+B_{lm}\frac{a^l}{b^{l+1}}=0. \tag{110}$$

These present us with two linear equations that may be solved for A_{lm} and B_{lm}; the solutions are

$$A_{lm}=\frac{4\pi}{(2l+1)}Y^*_{l,m}(\theta',\phi')\left(\frac{a^{2l+1}}{b^l r'^{l+1}}-\frac{r'^l}{b^l}\right)\Big/\left[1-\left(\frac{a}{b}\right)^{2l+1}\right] \tag{111}$$

and

$$B_{lm}=\frac{4\pi}{2l+1}Y^*_{l,m}(\theta',\phi')\left(\frac{a^{l+1}r'^l}{b^{2l+1}}-\frac{a^{l+1}}{r'^{l+1}}\right)\Big/\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]. \tag{112}$$

From these and the expansion Eq. (107), we find that the Green's function is

$$G(\mathbf{x},\mathbf{x}') = \sum_{l,m} \frac{4\pi}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} Y^*_{l,m}(\theta',\phi')Y_{l,m}(\theta,\phi)$$

$$\left\{\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]\frac{r_<^l}{r_>^{l+1}} + \left(\frac{a}{b}\right)^{2l+1}\frac{r^l}{r'^{l+1}} - \frac{r'^l r^l}{b^{2l+1}} + \left(\frac{a}{b}\right)^{2l+1}\frac{r'^l}{r^{l+1}} - \frac{a^{2l+1}}{r^{l+1}r'^{l+1}}\right\} \quad (113)$$

This result, if we can call it that, can be written in a somewhat more compact form by factoring the quantity $\{...\}$. Suppose that $r_> = r'$ and $r_< = r$; then

$$\begin{aligned}\{...\} &= \left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \\ &\equiv \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right).\end{aligned} \quad (114)$$

If, on the other hand, $r_> = r$ and $r_< = r'$, then

$$\{...\} = \left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \equiv \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right). \quad (115)$$

Comparing these results, we see that we may in general write

$$G(\mathbf{x},\mathbf{x}') = \sum_{l,m} \frac{4\pi/(2l+1)}{1-\left(\frac{a}{b}\right)^{2l+1}} Y^*_{l,m}(\theta',\phi')Y_{l,m}(\theta,\phi)\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right). \quad (116)$$

Notice that the Green's functions for the interior of a sphere of radius b and for the exterior of a sphere of radius a are easily obtained by taking the limits $a \to 0$ and $b \to \infty$, respectively. In the former case, for example, one finds

$$G(\mathbf{x},\mathbf{x}') = \sum_{l,m} \frac{4\pi}{2l+1} Y^*_{l,m}(\theta',\phi')Y_{l,m}(\theta,\phi)\left(\frac{r^l_<}{r^{l+1}_>} - \frac{r^l_< r^l_>}{b^{2l+1}}\right)$$
$$= \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \sum_{l,m} \frac{4\pi}{2l+1} Y^*_{l,m}(\theta',\phi')Y_{l,m}(\theta,\phi)\frac{b}{r'}\frac{r^l}{(b^2/r')^{l+1}}$$
$$= \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{b/r'}{|\mathbf{x}-\mathbf{x}'_R|} \tag{117}$$

where $\mathbf{x}'_R = (b^2/r', \theta', \phi')$ in spherical coordinates.

2 Laplace Equation in Cylindrical Coordinates; Bessel Functions

In cylindrical coordinates the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}. \tag{118}$$

We once again look for solutions of the Laplace equation in the form of products of functions of a single variable,

$$\Phi(\mathbf{x}) = R(\rho)Q(\phi)Z(z); \tag{119}$$

Following the usual procedure (substitute into the Laplace equation; divide by appropriate functions to obtain terms which appear to depend on a single variable; argue that such terms must be constants; etc.), we wind up with the following three ordinary differential equations:

$$\frac{d^2Q}{d\phi^2} + \nu^2 Q = 0, \;\; \nu = 0, \pm1, \pm2, ..., \tag{120}$$

$$\frac{d^2Z}{dz^2} - k^2 Z = 0, \tag{121}$$

and

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0 \tag{122}$$

where the value of k is yet to be determined, and ν is determined as indicated by the same argument as in the case of spherical coordinates; the functions $Q(\phi)$ are the same as in spherical coordinates also, $Q(\phi) \sim e^{i\nu\phi}$.

The choice of k is specified by the sort of boundary conditions one has. One could imagine having to satisfy quite arbitrary conditions on an end face $z = c$ where c is constant; alternatively, one may have to fit some function on a side wall $\rho = c$. In the former case, one wants to have functions of ρ which form a complete set on an appropriate interval of ρ; and in the latter case, one wants functions of z to form a complete set on some interval of z; in both cases we will need a complete set of functions of ϕ, which we have. Now, looking at the equations for R and Z, we can see that the latter function in particular is going to be simple exponentials of kz; for k real, these do not form a complete set; for k imaginary, they are sines and cosines and can form a complete set. We may not recognize it yet, but a similar thing happens to R; for k imaginary, it is roughly exponential in character and we cannot get a complete set of functions in this way. But for k real, the functions R are oscillatory (although not sines and cosines) and can form a complete set.

k	$Z(z)$	$R(\rho)$
real	incomplete ($e^{\pm kz}$)	complete (oscillatory)
imaginary	complete ($e^{i\pm\|k\|z}$)	incomplete

The functions of z in either case (k real or imaginary) are familiar to us and do not require further discussion. The functions of R, although probably known to all of us at least vaguely, are much less familiar so we will spend some time presenting their most important, to us, properties. Let's start by defining a dimension-free variable $x = k\rho$. Then Eq. (122) becomes

$$k^2\frac{d^2R}{dx^2} + k^2\frac{1}{x}\frac{dR}{dx} + k^2\left(1 - \frac{\nu^2}{x^2}\right)R = 0 \tag{123}$$

or

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \tag{124}$$

which is *Bessel's Equation.* Its solutions are *Bessel functions of order* ν. In our particular case, ν is an integer, although this need not be true in general. If k is imaginary, then x is imaginary, so we must deal with Bessel functions of imaginary argument; viewed as functions of a real variable $|x|$, these are known as *modified Bessel functions.*

For a given ν, there are two linearly independent solutions of Bessel's equation. Their choice is somewhat arbitrary since any linear combination of them is also a solution. One possible and common way to choose them is as follows:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j}}{j!\Gamma(j+\nu+1)} \tag{125}$$

and

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j}}{j!\Gamma(j-\nu+1)} \tag{126}$$

where

$$\Gamma(z) \equiv \int_0^\infty dt\, t^{z-1} e^{-t} \tag{127}$$

is the *gamma function* which, for z a real, positive integer n, is $\Gamma(n) = (n-1)!$. For $z = 0$ or a negative integer, it is singular.

The two Bessel functions introduced above are linearly independent solutions of Bessel's equation so long as ν is not an integer. It is easy to verify that they are solutions by direct substitution into the differential equation. If, however, ν is an integer, they become identical (aside from a possible sign difference) and so do not provide us with everything we need in this case, which is the important one for us. Another function, which is a solution and which is linearly independent of either of the two solutions introduced above (taken one at a time) is given by

$$N_\nu(x) \equiv \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{128}$$

This is the *Neumann function*; it is also called the Bessel function of the second kind of order ν. [3]

[3]Why does this work? Consider the limit $\nu \to m$ by La Hopital's rule. To formally show that N_m and J_m are independent, one must calculate the Wronskian and show that $W[N_m, J_m] \neq 0$.Note that in Abramowitz and Stegun's book, this function is written as $Y_\nu(x)$; see p. 358.

For $\nu = n$, a non-negative integer, the Neumann function has a series representation which is

$$N_n(x) \equiv -\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j} + \frac{2}{\pi}\ln(x/2)J_n(x)$$
$$-\frac{1}{\pi}\left(\frac{x}{2}\right)^{n}\sum_{j=0}^{\infty}\left\{[\psi(j+1)+\psi(n+j+1)]\frac{(x/2)^{2j}(-1)^j}{j!(n+j)!}\right\} \quad (129)$$

where $\psi(y) = d(\ln\Gamma(y))/dy$ is known as the *digamma or psi function.*

Finally, for some purposes it is more useful to use *Bessel functions of the third kind*, also called *Hankel functions*; these are given by

$$H_\nu^{(1)} = J_\nu(x) + iN_\nu(x) \quad (130)$$

and

$$H_\nu^{(2)} = J_\nu(x) - iN_\nu(x). \quad (131)$$

It is not easy to see what are the properties of the various kinds of Bessel functions from the expansions we have written down so far. As it turns out, their behavior is really quite simple; many of the important features are laid bare by their behavior at small and large arguments. For $x << 1$ and real, non-negative ν, one finds

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu [1 - O(x^2)] \quad (132)$$

and

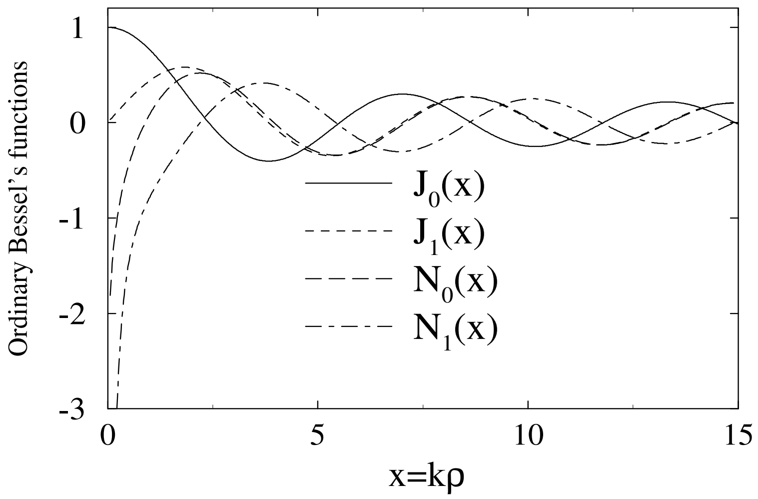
$$N_\nu(x) = \begin{cases} \frac{2}{\pi}[\ln(x/2) + 0.5772 + ...] & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases} \quad (133)$$

For $x >>> 1,\ \nu$,

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{134}$$

and

$$N_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) . \tag{135}$$



Notice in particular that as $x \to 0$, the Bessel functions of the first kind are well-behaved (finite) whereas the Neumann functions are singular; $N_0(x)$ has only a logarithmic singularity while the higher-order functions are progressively more singular. At large (real) argument, on the other hand, both $J_\nu(x)$ and $N_\nu(x)$ are finite and oscillatory. Hence, the Bessel functions (by which we mean the $J_\nu(x)$) of non-negative order are allowable as solutions of the Laplace equation at all values of ρ; the Neumann functions, on the other hand, are not allowable on a domain which includes the point $\rho = 0$.

Bessel functions of all kinds satisfy certain recurrence relations. It is a straightforward if tedious matter to show by direct substitution of the series expansions that they obey the following:

$$\Omega_{\nu+1} - \frac{2\nu}{x}\Omega_\nu + \Omega_{\nu-1} = 0 \tag{136}$$

and

$$\Omega_{\nu+1} + 2\frac{d\Omega_\nu}{dx} - \Omega_{\nu-1} = 0. \tag{137}$$

By taking the sum and difference of these relations, we find also

$$\Omega_{\nu\pm1} = \frac{\nu}{x}\Omega_\nu \mp \frac{d\Omega_\nu}{dx}. \tag{138}$$

These are valid for all three kinds of Bessel functions.

The Bessel function $J_\nu(x)$ can form a complete orthogonal set on an interval $0 \leq x \leq x_0$ in much the same way as the sine function $\sin(n\pi x/x_0)$ does (Note that x_0 is a zero of the sine function.) Similarly, let us denote the n^{th} zero of $J_\nu(x)$ by $x_{\nu n}$ and then form the functions $J_\nu(x_{\nu n}y)$, with n = 1,2,... . Then it turns out that for fixed ν, these functions provide a complete orthogonal set on the interval $0 \leq y \leq 1$. As usual, we shall not demonstrate completeness. Orthogonality can be demonstrated by making use of the Bessel equation and recurrence relations. It is a useful exercise to do so. Start from the Bessel equation for $J_\nu(xy)$,

$$\frac{1}{y}\frac{d}{dy}\left(y\frac{dJ_\nu(xy)}{dy}\right) + \left(x^2 - \frac{\nu^2}{y^2}\right)J_\nu(xy) = 0 \tag{139}$$

Multiply this equation by $yJ_\nu(x'y)$ and integrate from 0 to 1:

$$\int_0^1 dy \left\{ J_\nu(x'y) \frac{d}{dy}\left(y \frac{dJ_\nu(xy)}{dy}\right) + yJ_\nu(x'y)\left(\mathbf{x}^2 - \frac{\nu^2}{y^2}\right) J_\nu(xy) \right\} = 0 \tag{140}$$

or if we integrate the first term by parts:

$$J_\nu(x'y)y \left.\frac{dJ_\nu(xy)}{dy}\right|_0^1 - \int_0^1 dy \frac{dJ_\nu(x'y)}{dy} y \frac{dJ_\nu(xy)}{dy} + \int_0^1 dy y J_\nu(x'y) J_\nu(xy)\left(\mathbf{x}^2 - \frac{\nu^2}{y^2}\right) = 0 \tag{141}$$

Similarly, if we start from the differential equation for $J_\nu(x'y)$ and perform the same manipulations, we find the same equation with x and x' interchanged. Subtract the second equation from the first to find

$$J_\nu(x')x \left.\frac{dJ_\nu(u)}{du}\right|_x - J_\nu(x)x' \left.\frac{dJ_\nu(u)}{du}\right|_{x'} + (x^2 - x'^2)\int_0^1 y dy J_\nu(xy)J_\nu(x'y) = 0 \tag{142}$$

If we let $x = x_{\nu n}$ and $x' = x_{\nu n'}$, two distinct zeros of the Bessel function, then the integrated terms vanish and we may conclude that the Bessel functions $J_\nu(x_{\nu n}y)$ and $J_\nu(x_{\nu n'}y)$ are orthogonal when integrated over y from 0 to one, provided a factor of y is included in the integrand.

We still have to determine normalization in the case $n = n'$. In the preceding equation, let $x' = x_{\nu n}$ and rearrange the terms to have

$$-J_\nu(x)x_{\nu n} \left.\frac{dJ_\nu(u)}{du}\right|_{x_{\nu n}} = -\int_0^1 dy\, y(x^2 - x_{\nu n}^2) J_\nu(x_{\nu n}y) J_\nu(xy), \tag{143}$$

or

$$\int_0^1 dy\, y J_\nu(x_{\nu n}y) J_\nu(xy) = \frac{J_\nu(x)x_{\nu n}(dJ_\nu(u)/du)|_{x_{\nu n}}}{x^2 - x_{\nu n}^2}. \tag{144}$$

Use L'Hôpital's Rule to evaluate the limit of this expression as $x \to x_{\nu n}$:

$$\int_0^1 dy\, y[J_\nu(x_{\nu n}y)]^2 = \frac{1}{2}\left.\frac{dJ_\nu(x)}{dx}\right|_{x_{\nu n}}\left.\frac{dJ_\nu(u)}{du}\right|_{x_{\nu n}} = \frac{1}{2}[J'_\nu(x_{\nu n})]^2, \tag{145}$$

where the prime ' denotes a derivative with respect to argument. Now employ the recurrence relation

$$xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x) \tag{146}$$

to find $J'_\nu(x_{\nu n}) = -J_{\nu+1}(x_{\nu n})$, from which the normalization integral becomes

$$\int_0^1 dy\, y[J_\nu(x_{\nu n}y)]^2 = \frac{1}{2}[J_{\nu+1}(x_{\nu n})]^2 \tag{147}$$

The expansion of an arbitrary function of ρ on the interval $0 \le \rho \le a$ may be written as

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_\nu(\rho x_{\nu n}/a) \tag{148}$$

with coefficients which may be determined from the orthonormalization properties of the basis functions as

$$A_n = \frac{\int_0^a \rho d\rho f(\rho) J_\nu(\rho x_{\nu n}/a)}{\frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2}. \tag{149}$$

This type of expansion is termed a *Fourier-Bessel series.* The completeness relation for the basis functions is

$$\sum_{n=1}^{\infty} \frac{J_\nu(\rho x_{\nu n}/a) J_\nu(\rho' x_{\nu n}/a)}{(a^2/2)[J_{\nu+1}(x_{\nu n})]^2} = \delta(\rho^2/2 - \rho'^2/2) \equiv \frac{1}{\rho}\delta(\rho - \rho'). \tag{150}$$

It is also of importance to consider the case of imaginary k, $k = i\kappa$ with real κ. Then the functions of z are oscillatory, being of the form $Z(z) \sim e^{\pm i\kappa z}$, and the functions of ρ will be Bessel functions of imaginary argument, *e.g.*, $J_\nu(i\kappa\rho)$. For given ν^2, there are two linearly independent solutions which are conventionally chosen to be J_ν and $H_\nu^{(1)}$, the reason being that they have particularly simple behaviors at large and small arguments. Let us introduce the *modified Bessel functions* $I_\nu(x)$ and $K_\nu(x)$,

$$I_\nu(x) \equiv i^{-\nu} J_\nu(ix) \tag{151}$$

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix). \tag{152}$$

These have the forms at small argument, $x << 1$,

$$I_\nu(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \tag{153}$$

and

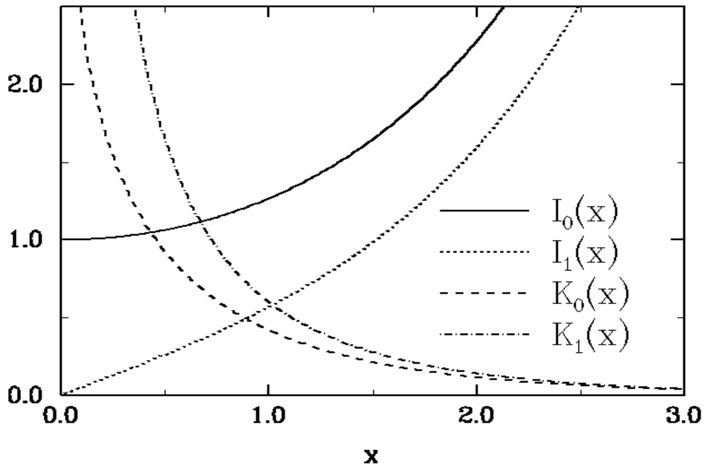
$$K_\nu(x) = \begin{cases} -\ln(x/2) + 0.5772 + \ldots & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases} \tag{154}$$

while at large argument, $x >>> 1, \nu$

$$I_\nu(x) = \frac{1}{\sqrt{2\pi x}} e^x \left[1 + O\left(\frac{1}{x}\right)\right] \tag{155}$$

and

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + O\left(\frac{1}{x}\right)\right]. \tag{156}$$

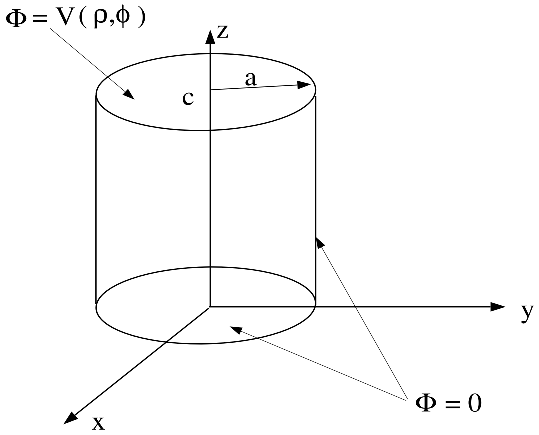


From these equations we see that $I_\nu(x)$ is well-behaved for $x < \infty$, corresponding to $\rho < \infty$, while $K_\nu(x)$ is well-behaved for $x > 0$ or $\rho > 0$, from which we can decide which function(s) to use in expanding any given potential problem.

Let's look at some examples of expansions in cylindrical coordinates.

2.1 Example I

Consider a charge-free right-circular cylinder bounded by S given by $\rho = a$, $z = 0$, and $z = c$. Let $\Phi(\mathbf{x})$ be zero on S except for the top face $z = c$ where $\Phi(\rho, \phi, c) = V(\rho, \phi)$ with V given.



For this distribution of boundary potential, we need a complete set of functions of the space $0 \leq \phi \leq 2\pi$ and $0 \leq \rho \leq a$. Thus we take (k real) $Z(z)$ to be damped exponentials (sinh and cosh) and $R(\rho)$ to be ordinary Bessels functions.

$$\Phi \sim e^{im\phi} \left(AJ_m(k\rho) + BN_m(k\rho) \right) \left(\cos(kz) \pm \sin(kz) \right) \tag{157}$$

It will be convenient if each of these functions is equal to zero when $\rho = a$ and also if each one is zero when $z = 0$. With just a little thought,

1. No Neumann functions N_m since they diverge at $\rho = 0$
2. No cosh since it is finite at $z = 0$, and hence would not satisfy the B.C.
3. Since J_m and J_{-m} are not independent, use $J_{|m|}$.

we realize that we want to use the hyperbolic sine function of z and the Bessel function of the first kind for R. Our expansion is thus of the form

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{im\phi} J_{|m|}(x_{mn}\rho/a) \sinh(x_{mn}z/a) \qquad (158)$$

where x_{mn} is the n^{th} zero of $J_{|m|}(x)$. Each term in the sum is itself a solution of the Laplace equation; each one satisfies the boundary conditions on $z = 0$ and $\rho = a$, and, for given n, we have a complete set of functions of ϕ while for given m, we have a complete set of functions of ρ.

The coefficients in the expansion are determined from the condition that Φ reduce to the given potential V on the top face of the cylinder. Making use of the orthogonality of the basis functions of both ϕ and ρ, we have

$$\begin{aligned} &\int_0^a \rho d\rho \int_0^{2\pi} d\phi V(\rho, \phi) J_{|m|}(x_{mn}\rho/a) e^{-im\phi} \qquad (159) \\ &= A_{mn} 2\pi (a^2/2) [J_{|m|+1}(x_{mn})]^2 \sinh(x_{mn}c/a) \end{aligned}$$

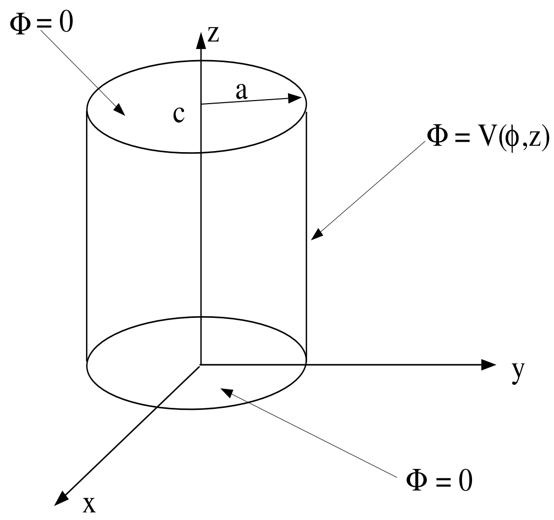
or

$$A_{mn} = \frac{\int_0^{2\pi} d\phi \int_0^a \rho d\rho e^{-im\phi} J_{|m|}(x_{mn}\rho/a) V(\rho, \phi)}{\pi a^2 \sinh(x_{mn}c/a) [J_{|m|+1}(x_{mn})]^2}. \qquad (160)$$

For any given function V, one may now attempt to complete the integrals.

2.2 Example II

Consider the same geometry as in the first example but now with boundary condition $\Phi = 0$ on the constant-z faces and some given value $V(\phi, z)$ on the surface at $\rho = a$.



For this system we need a complete set of functions on the domain $0 \leq z \leq c$ and $0 \leq \phi \leq 2\pi$ which means picking k imaginary, $k = i\kappa$. The appropriate functions of z are sin and cos, and the appropriate functions of ρ are the modified Bessels Functions.

$$\Phi \sim e^{im\phi} \left(\sin(kz) \pm \cos(kz)\right) \left(AI(k\rho) + BK(k\rho)\right) \tag{161}$$

1. We may eliminate the K modified Bessels functions since they diverge when $\rho \to 0$.

2. Since I_m is not independent of I_{-m}, we use $I_{|m|}$.

3. The cos function of z cannot be zero at both $z = 0$ and $z = c$, and so may be eliminated.

4. Take $k = n\pi/c$ so that $\sin(n\pi z/c)|_{z=c} = 0$

Thus the expansion is

$$\Phi(\rho,\phi,z)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{im\phi}\sin(n\pi z/c)I_{|m|}(n\pi\rho/c) \tag{162}$$

with coefficients given by

$$A_{mn}=\frac{\int_0^c dz\int_0^{2\pi}d\phi e^{-im\phi}\sin(n\pi z/c)V(\phi,z)}{\pi c I_{|m|}(n\pi a/c)} \tag{163}$$

2.3 B.V.P. on Large Cylinders

By applying the same considerations, one may solve other boundary-value problems on cylinders. A case of special interest, and requiring special treatment, is one in which $a\to\infty$; then $k_{\nu n}\equiv x_{\nu n}/a$ becomes a continuous variable and instead of a Fourier-Bessel series, we come up with an integral. The orthogonality condition is

$$\int_0^\infty xdxJ_m(kx)J_m(k'x)=\frac{1}{k}\delta(k-k') \tag{164}$$

and the completeness relation is the same, with different names for the variables,

$$\int_0^\infty kdkJ_m(kx)J_m(kx')=\frac{1}{x}\delta(x-x'). \tag{165}$$

To see how this comes to be, consider the completeness relation on a finite interval,

$$\sum_{n=1}^{\infty}\frac{J_m(x_{mn}\rho/a)J_m(x_{mn}\rho'/a)}{(a^2/2)[J_{m+1}(x_{mn})]^2}=\frac{1}{\rho}\delta(\rho-\rho') \tag{166}$$

and then let $a\to\infty$, defining x_{mn}/a as k while noting that the interval between roots of the Bessel function at large argument is π. Also, the asymptotic form of the Bessel function, valid at large argument, is

$$J_{m+1}(x_{mn}) \sim \sqrt{\frac{2}{\pi x_{mn}}}\cos[n\pi + (m-1/2)\pi/2 - (m+1)\pi/2 - \pi/4]$$
$$= \sqrt{\frac{2}{\pi x_{mn}}}\cos[(n-1)\pi] = -(-1)^n\sqrt{\frac{2}{\pi x_{mn}}} \quad (167)$$

Using this substitution in the closure relation for the finite interval and taking the limit of large a, one finds that the sum becomes the integral, Eq. (165). Of course, this is not a rigorous derivation because the asymptotic expression is not arbitrarily accurate for all roots.

2.4 Green's Function Expansion in Cylindrical Coordinates

We can expand the Dirichlet Green's function in cylindrical coordinates in much the same manner as we did in spherical coordinates. We shall go through the derivation to expose a somewhat different approach from what we employed in the latter case. We consider a domain between two infinitely long right-circular cylindrical surfaces. Then G must vanish on these surfaces and it must also satisfy a Poisson equation,

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\rho^2/2 - \rho'^2/2)\delta(\phi-\phi')\delta(z-z')$$
$$= -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z'). \quad (168)$$

Let us write the delta functions of ϕ and z using closure relations,

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')} = \frac{1}{\pi}\int_0^{\infty} dk \cos[k(z-z')] \quad (169)$$

and

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}. \tag{170}$$

Similarly, expand the ϕ-dependence and z-dependence of G using the same basis functions,

$$G(\mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{dk}{2\pi^2} g_m(k, \rho, \rho') e^{im(\phi-\phi')} \cos[k(z - z')]. \tag{171}$$

Now operate on this expansion with the Laplacian using cylindrical coordinates:

$$\begin{aligned} \nabla^2 G(\mathbf{x}, \mathbf{x}') &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{dk}{2\pi^2} \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left[\frac{m^2}{\rho^2} + k^2\right]\right) g_m e^{im(\phi-\phi')} \cos[k(z - z')] \\ &= -\frac{4\pi}{\rho}\delta(\rho - \rho') \sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{dk}{2\pi^2} e^{im(\phi-\phi')} \cos[k(z - z')] \end{aligned} \tag{172}$$

Multiply by members of the basis sets, *i.e.*, $e^{-im'(\phi-\phi')}$ and $\cos[k'(z-z')]$ and integrate over the appropriate intervals of $\phi - \phi'$ and $z - z'$ to find a differential equation for g_m,

$$\frac{d^2 g_m}{d\rho^2} + \frac{1}{\rho}\frac{dg_m}{d\rho} - \left(k^2 + \frac{m^2}{\rho^2}\right) g_m = -\frac{4\pi}{\rho}\delta(\rho - \rho'). \tag{173}$$

For $\rho \neq \rho'$, this is Bessel's equation with solutions (viewed as functions or ρ) which are Bessel functions of imaginary argument, or, as we have described, modified Bessel functions of argument $k\rho$. Because of the delta function inhomogeneous term, the solution for $\rho < \rho'$ is different from the solution for $\rho > \rho'$. Hence we may write that, for $\rho < \rho'$,

$$g_m(k, \rho, \rho') = A_<(\rho') K_m(k\rho) + B_<(\rho') I_m(k\rho), \tag{174}$$

and, for $\rho > \rho'$,

$$g_m(k,\rho,\rho') = A_>(\rho')K_m(k\rho) + B_>(\rho')I_m(k\rho). \tag{175}$$

The various coefficients are functions of ρ' and must in fact be linear combinations of $K_m(\rho')$ and $I_m(\rho')$ because $G(\mathbf{x},\mathbf{x}')$ is also a solution of $\nabla'^2 G(\mathbf{x},\mathbf{x}') = 0$ when $\rho \neq \rho'$; another way to see this same point is to recall that $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$. Finally, the coefficients are further constrained by the condition that the Green's function must vanish when ρ becomes equal to the radius of either the inner or outer cylinder.

Let us at this point restrict our attention to a special (and simple) limiting case which is the infinite space. The radius of the inner cylinder is 0 and that of the outer one becomes infinite in this limit. Then we have to have a function g_m which remains finite as $\rho \to 0$ which can only be $I_m(k\rho)$; also, we must have g_m vanish as $\rho \to \infty$, which can only be $K_m(k\rho)$. Thus we have

$$g_m(k,\rho,\rho') = \begin{cases} A_<(\rho')I_m(k\rho) & \rho < \rho' \\ A_>(\rho')K_m(k\rho) & \rho > \rho'. \end{cases} \tag{176}$$

The symmetry condition on G tells us that $A_<(\rho') = AK_m(k\rho')$ while $A_>(\rho') = AI_m(k\rho')$. All of these conditions are included in the statement

$$g_m(k,\rho,\rho') = AI_m(k\rho_<)K_m(k\rho_>) \tag{177}$$

where $\rho_<$ ($\rho_>$) is the smaller (larger) of ρ and ρ'.

The remaining constant in the determination of g_m can be found from the required normalization of G. Let us integrate Eq. (173) across the point $\rho = \rho'$:

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \left[k^2 + \frac{m^2}{\rho^2} \right] \right) g_m = -\frac{4\pi}{\rho'} \tag{178}$$

If we take the limit of $\epsilon \to 0$ and realize that g_m is continuous while its first derivative is not, we find that this equation gives

$$\lim_{\epsilon \to 0} \frac{dg_m}{d\rho}\bigg|_{\rho'-\epsilon}^{\rho'+\epsilon} = A \left[I_m(k\rho')k \left.\frac{dK_m(x)}{dx}\right|_{k\rho'} - K_m(k\rho')k \left.\frac{dI_m(x)}{dx}\right|_{k\rho'} \right] = -\frac{4\pi}{\rho'}, \tag{179}$$

or

$$A[I_m(x)K_m'(x) - K_m(x)I_m'(x)] = -\frac{4\pi}{x}; \tag{180}$$

the primes denote derivatives with respect to the argument x. The quantity in [...] here is the *Wronskian* of I_m and K_m. One may learn by consulting, *e.g.,* the section on Bessel functions in Abramowitz and Stegun, that Bessel functions have simple Wronskians:

$$I_m(x)K_m'(x) - K_m(x)I_m'(x) \equiv W[I_m(x), K_m(x)] = -\frac{1}{x}. \tag{181}$$

Comparison of the two preceding equations leads one to conclude that $A = 4\pi$. Hence our expansion of $G(\mathbf{x}, \mathbf{x}')$, which is just $1/|\mathbf{x} - \mathbf{x}'|$, is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk e^{im(\phi-\phi')} \cos[k(z-z')] I_m(k\rho_<) K_m(k\rho_>) \tag{182}$$

which may also be written entirely in terms of real functions as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{4}{\pi}\int_0^\infty dk \cos[k(z-z')]$$
$$\times\left\{\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty}\cos[m(\phi-\phi')]I_m(k\rho_<)K_m(k\rho_>)\right\}. \quad (183)$$

This turns out to be a useful expansion of $1/|\mathbf{x}-\mathbf{x}'|$; it also provides a starting point for the derivation of some other equally useful expansions. For example, if we let $\mathbf{x}' = 0$, then $\rho_< = 0$ and all I_m vanish except for $m = 0$, while $1/|\mathbf{x}-\mathbf{x}'| = 1/|\mathbf{x}| = 1/\sqrt{\rho^2+z^2}$, so we find, using also $I_0(0) = 1$,

$$\frac{1}{\sqrt{\rho^2+z^2}} = \frac{2}{\pi}\int_0^\infty dk \cos(kz)K_0(k\rho). \quad (184)$$

Other useful identities may be obtained.