Boundary-value Problems in Electrostatics I

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(1777 - 1855)

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Contents

In this chapter we shall solve a variety of boundary value problems using techniques which can be described as commonplace.

1 Method of Images

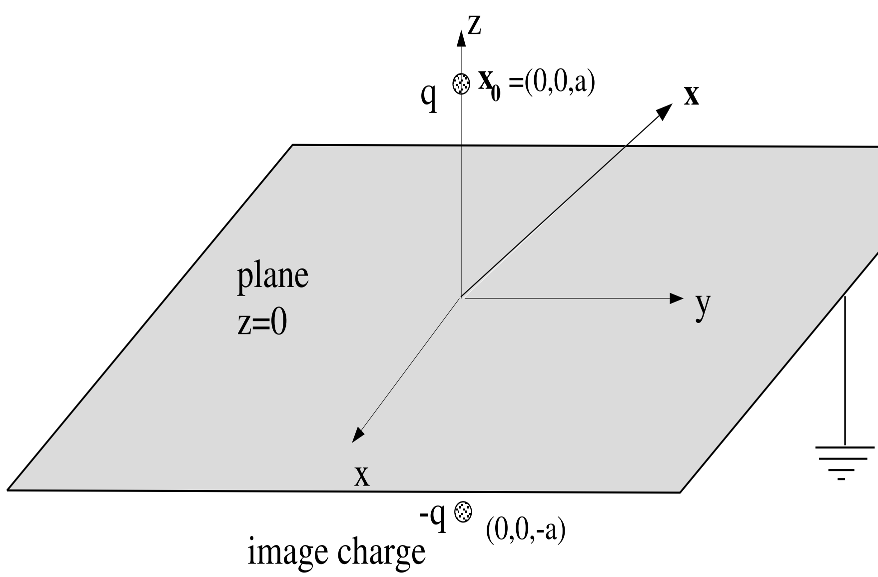
This method is useful given sufficiently simple geometries. It is closely related to the Green's function method and can be used to find Green's functions for these same simple geometries. We shall consider here only conducting (equipotential) bounding surfaces which means the boundary conditions take the form of $\Phi(\mathbf{x})$ = constant on each electrically isolated conducting surface. The idea behind this method is that the solution for the potential in a finite domain V with specified charge density and potentials on its surface S can be the *same* within V as the solution for the potential given the same charge density inside of V but a quite different charge density elsewhere. Thus we consider two distinct electrostatics problems. The first is the "real" problem in which we are given a charge density $\rho(\mathbf{x})$ in V and some boundary conditions on the surface S. The second is a "fictitious problem" in which the charge density inside of V is the same as for the real problem and in which there is some undetermined charge distribution elsewhere; this is to be chosen such that the solution to the second problem satisfies the boundary conditions specified in the first problem. Then the solution to the second problem is also the solution to the first problem inside of V (but not outside of V). If one has found the initially undetermined exterior charge in the second problem, called *image charge*, then the potential is found simply from Coulomb's Law,

$$\Phi(\mathbf{x}) = \int d^3x' \, \frac{\rho_2(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}; \tag{1}$$

ρ_2 is the total charge density of the second problem.

1.1 Point Charge Above a Conducting Plane

This may sound confusing, but it is made quite clear by a simple example. Suppose that we have a point charge q located at a point $\mathbf{x}_0 = (0, 0, a)$ in Cartesian coordinates. Further, a grounded conductor occupies the half-space $z < 0$, which means that we have the Dirichlet boundary condition at $z = 0$ that $\Phi(x, y, 0) = 0$; also, $\Phi(\mathbf{x}) \to 0$ as $r \to \infty$. The first thing that we must do is determine some image charge located in the half-space $z < 0$ such that the potential of the image charge plus the real charge (at $\mathbf{x}_0$) produces zero potential on the $z = 0$ plane. With just a little thought one realizes that a single image charge $-q$ located at the point $\mathbf{x}_0' = (0, 0, -a)$ is what is required.



All points on the $z = 0$ plane are equidistant from the real charge and its image, and so the two charges produce cancelling potentials at each of these points. The solution to the problem is therefore

$$\Phi(\mathbf{x}) = q\left[\frac{1}{|\mathbf{x}-\mathbf{x}_0|} - \frac{1}{|\mathbf{x}-\mathbf{x}_0'|}\right]. \tag{2}$$

This function satisfies the correct Poisson equation in the domain $z > 0$ and also satisfies the correct boundary conditions at $z = 0$; therefore it is the (unique) solution. It is important to realize, however, that it is *not* the correct solution in the space $z < 0$; here, the real potential is zero because this domain in inside of the grounded conductor.

In the real system, there is some surface charge density $\sigma(x, y)$ on the conductor; to determine what this is, we have only to evaluate the normal component of the electric field at the surface of the conductor,

$$\mathbf{E}_n(x, y, 0) = -\left.\frac{\partial\Phi(\mathbf{x})}{\partial z}\right|_{z=0} = -\frac{2qa}{(\rho^2 + a^2)^{3/2}}, \tag{3}$$

where $\rho = \sqrt{x^2 + y^2}$. The surface charge density is just this field, divided by 4π, $E_n = \sigma/4\pi$.

From this example we can also see why this technique has the name 'method of images.' The image charge is precisely the mirror image in the $z = 0$ plane of the real charge.

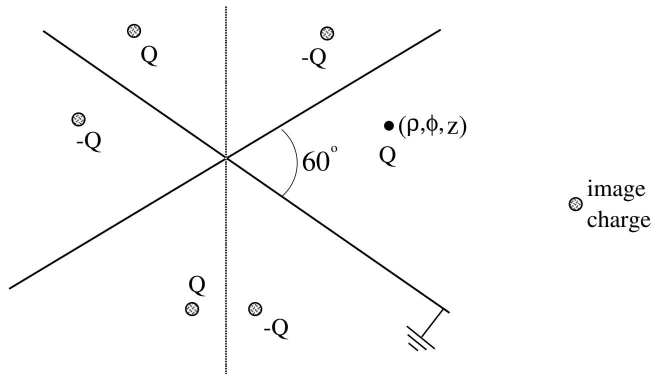
As a by-product of our solution, we have also got the Dirichlet Green's function for the semi-infinite half-space $z > 0$; it is

$$G(\mathbf{x}, \mathbf{x}') = \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x} - \mathbf{x}'_i|} \right) \tag{4}$$

where $\mathbf{x}'_i$ is the mirror image of $\mathbf{x}'$ in the $z = 0$ plane. Hence we can solve, by doing appropriate integrals, any problem in which we are given some $\rho(\mathbf{x})$ in the domain $z > 0$ and an arbitrary potential $\Phi(x, y, 0)$.

1.2 Point Charge Between Multiple Conducting Planes

A simple extension of the problem above is one with a point charge between two intersecting conducting planes. For example, consider two grounded conducting planes that intersect at an angle of 60^o forming a wedge, with point charge Q placed at (ρ, ϕ, z) within the wedge.



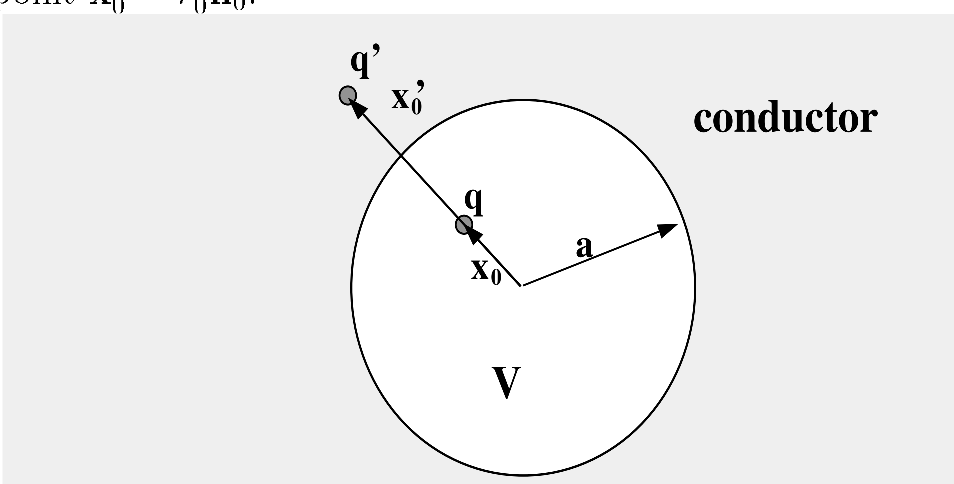
To solve this problem, we again use image charges to satisfy the boundary conditions. There are five image charges, as indicated in the figure above. They all share the same value of ρ and z as the real charge, and their azimuthal angles are given in the table below.

charge	angle
$-Q$	$\frac{2\pi}{3} - \phi$
$+Q$	$\frac{2\pi}{3} + \phi$
$-Q$	$-\frac{2\pi}{3} - \phi$
$+Q$	$-\frac{2\pi}{3} + \phi$
$-Q$	$-\phi$

1.3 Point Charge in a Spherical Cavity

It is also sometimes possible to use the image method when the boundary S involves curved surfaces. However, just as curved mirrors produced distorted images, so do curved surfaces make the image of a point charge more complicated[1]. Let's do the simplest problem of this kind. Suppose that we have a spherical cavity of radius a inside of a conductor; within this cavity is a point charge q located a distance r_0 from the center of the sphere which is also chosen as the origin of coordinates. Thus the charge is at point $\mathbf{x}_0 = r_0\mathbf{n}_0$ where $\mathbf{n}_0$ is a unit vector pointing in the direction from the origin to the charge.

We need to find the image(s) of the charge in the spherical surface which encloses it. The simplest possible set of images would be a single charge q'; if there is such a solution, symmetry considerations tell us that the image must be located on the line passing through the origin and going in the direction of $\mathbf{n}_0$. Let us therefore put an image charge q' at point $\mathbf{x}'_0 = r'_0\mathbf{n}_0$.



[1]Consider the example of the right-hand side-view mirror of a car. Here the mirror is concave, and images appear to be much farther away than they actually are

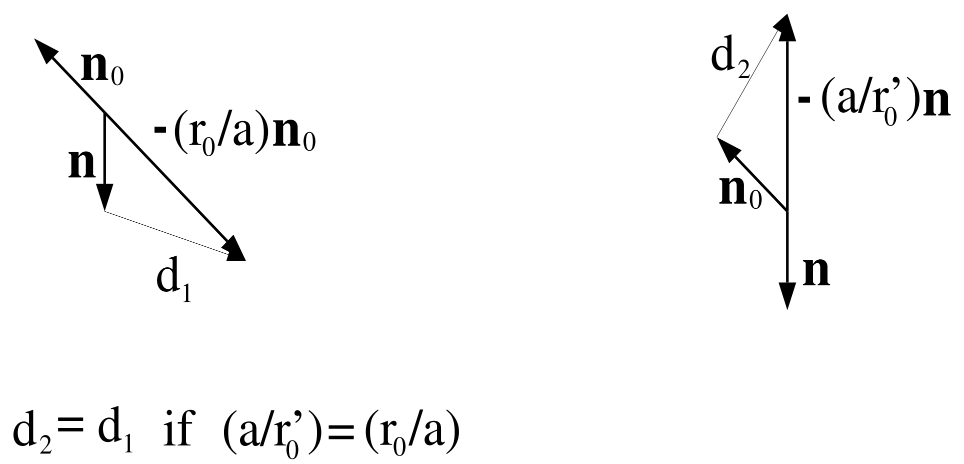
The potential produced by this charge and the real one at $\mathbf{x}_0$ is

$$\Phi(\mathbf{x}) = \frac{q}{|\mathbf{x} - \mathbf{x}_0|} + \frac{q'}{|\mathbf{x} - \mathbf{x}_0'|}. \tag{5}$$

Now we must choose, if possible, q' and r_0' such that $\Phi(\mathbf{x}) = 0$ for $\mathbf{x}$ on the cavity's spherical surface, $\mathbf{x} = a\mathbf{n}$ where the direction of $\mathbf{n}$ is arbitrary. The potential at such a point may be written as

$$\Phi(a\mathbf{n}) = \frac{q/a}{|\mathbf{n} - (r_0/a)\mathbf{n}_0|} + \frac{q'/r_0'}{|\mathbf{n}_0 - (a/r_0')\mathbf{n}|}. \tag{6}$$

The from the figure below, it is clear that denominators are equal if $r_0/a = a/r_0'$, and the numerators are equal and opposite if $q/a = -q'/r_0'$. (The "other" solution, $r_0' = r_0$ and $q' = -q$ is no solution at all since then the image charge would be within the volume V and cancel the real charge. We must have $r_0' > 1$)



Hence, we make Φ zero on S by choosing

$$r_0' = a^2/r_o \text{ and } q' = -q(a/r_0). \tag{7}$$

Thus we have the solution for a point charge in a spherical cavity with an equipotential surface:

$$\Phi(\mathbf{x}) = q\left[\frac{1}{|\mathbf{x}-\mathbf{x}_0|} - \frac{a/r_0}{|\mathbf{x}-(a^2/r_0^2)\mathbf{x}_0|}\right]; \tag{8}$$

we have also found the Dirichlet Green's function for the interior of a sphere of radius a:

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a/r}{|\mathbf{x}'-(a^2/r^2)\mathbf{x}|}. \tag{9}$$

The solution of the "inverse" problem which is a point charge outside of a conducting sphere is the same, with the roles of the real charge and the image charge reversed. The preceding equations for $\Phi(\mathbf{x})$ and $G(\mathbf{x},\mathbf{x}')$ are valid except that r, r_0, and r' are all larger than a.

Let's look at a few more features of the solution for the charge inside of the spherical cavity. First, what is σ, the charge density on the surface of the cavity. From Gauss's Law, we know that the charge density is the normal component of the electric field out of the conductor at its surface divided by 4π. This is the negative of the radial component in spherical polar coordinates, so

$$\sigma = -\frac{E_r}{4\pi} = \frac{1}{4\pi}\frac{\partial\Phi}{\partial r}. \tag{10}$$

If we define the z-direction to be the direction of $\mathbf{n}_0$, then the potential at an arbitrary point within the sphere is

$$\Phi(\mathbf{x}) = q\left[\frac{1}{(r^2+r_0^2-2rr_0\cos\theta)^{1/2}} - \frac{(a/r_0)}{(r^2+(a^4/r_0^2)-2r(a^2/r_0)\cos\theta)^{1/2}}\right] \tag{11}$$

where θ is the usual polar angle between the z-axis, or direction of $\mathbf{x}_0$, and the direction of $\mathbf{x}$, the field point. The radial component of the electric field at $r = a$ is

$$E_r = -\left.\frac{\partial \Phi(\mathbf{x})}{\partial r}\right|_{r=a}$$

$$= -q\left[-\frac{1}{2}\frac{2a - 2r\cos\theta}{(a^2 + r_0^2 - 2ar_0\cos\theta)^{3/2}} + \frac{a}{r_0}\frac{1}{2}\frac{2a - 2(a^2/r_0)\cos\theta}{(a^2 + (a^4/r_0^2) - 2(a^3/r_0)\cos\theta)^{3/2}}\right]$$

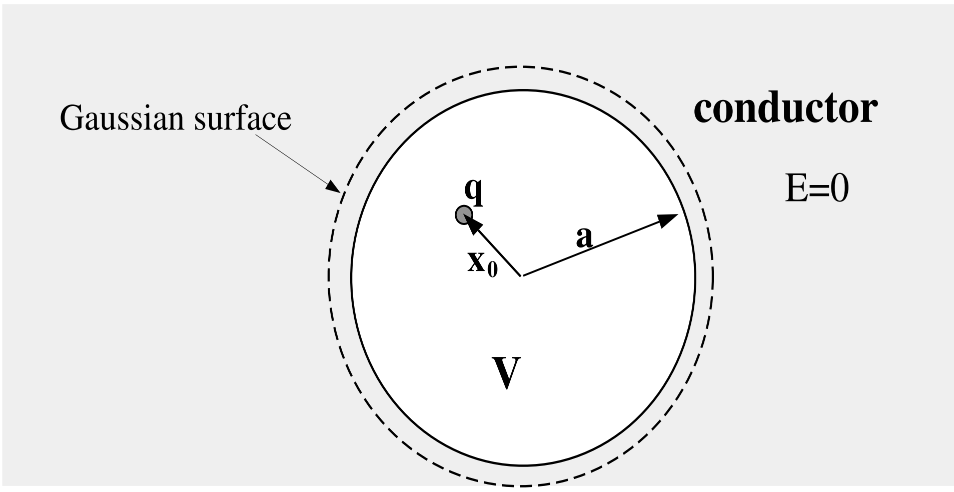
$$= q\left[\frac{a - r_0\cos\theta}{(a^2 + r_0^2 - 2ar_0\cos\theta)^{3/2}} - \frac{r_0^2}{a^2}\frac{a - (a^2/r_0)\cos\theta}{(a^2 + r_0^2 - 2ar_0\cos\theta)^{3/2}}\right]$$

$$= \frac{q}{a^2}\left[\frac{1 - r_0^2/a^2}{(1 + (r_0^2/a^2) - 2(r_0/a)\cos\theta)^{3/2}}\right] \tag{12}$$

If we introduce $\epsilon = r_0/a$, then the surface charge density may be written concisely as

$$\sigma = -\frac{q}{4\pi a^2}\frac{1 - \epsilon^2}{(1 + \epsilon^2 - 2\epsilon\cos\theta)^{3/2}}. \tag{13}$$

The **total charge on the surface** may be found by integrating over σ. But it may be obtained more easily by invoking Gauss's Law; if we integrate the normal component of $\mathbf{E}(\mathbf{x})$ over a closed surface which lies entirely in conducting material and which also encloses the cavity, we know that we will get zero, because the field in the conductor is zero.



$$0 = \int_S d^2x\, \mathbf{E}\cdot\mathbf{n} = 4\pi Q = 4\pi\left(q + \int_S d^2x\,\sigma\right)$$

charge within this surface. What is inside is the charge q in the cavity and the surface charge on the conductor. The implication is that the total surface charge is equal to $-q$. It is perhaps useful to actually do the integral over the surface as a check that we have gotten the charge density there right:

$$\begin{aligned}
Q_i &= \int_S d^2x\,\sigma(\mathbf{x}) \\
&= -\frac{q}{2}(1-\epsilon^2)\int_{-1}^{1}\frac{du}{(1+\epsilon^2-2\epsilon u)^{3/2}} \\
&= -\frac{q}{2}\frac{1-\epsilon^2}{2\epsilon}\left[\frac{2}{(1+\epsilon^2-2\epsilon)^{1/2}} - \frac{2}{(1+\epsilon^2+2\epsilon)^{1/2}}\right] \\
&= -\frac{q}{2}\frac{1-\epsilon^2}{\epsilon}\left(\frac{1}{1-\epsilon} - \frac{1}{1+\epsilon}\right) = -q.
\end{aligned} \tag{14}$$

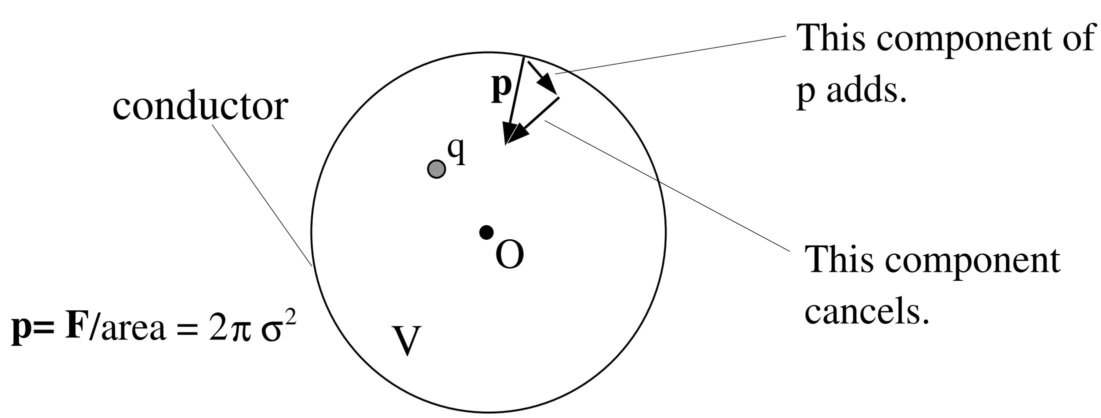
Notice that $|\sigma|$ is largest in the direction of $\mathbf{n}_0$ and is

$$|\sigma_{max}| = -\frac{q}{4\pi a^2}\frac{1+\epsilon}{(1-\epsilon)^2}. \tag{15}$$

In the opposite direction, the magnitude of the charge density is at its minimum which is

$$|\sigma_{min}| = -\frac{q}{4\pi a^2}\frac{1-\epsilon}{(1+\epsilon)^2}. \tag{16}$$

The **total force on the charge** may also be computed. This is the negative of the total force on the conductor. Now, we know that the force per unit area on the surface of the conductor is $2\pi\sigma^2$ and is directed normal to the conductor's surface into the cavity. Because of the rotational invariance of the system around the direction of $\mathbf{n}_0$, only the component of the force along this direction need be computed; the other components will average to zero when integrated over the surface.



Hence we find

$$|\mathbf{F}_n| = 2\pi a^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \sigma^2(\theta)\cos\theta$$

$$= \frac{4\pi^2 a^2 q^2}{16\pi^2 a^4}(1-\epsilon^2)^2 \int_{-1}^{1} \frac{udu}{(1+\epsilon^2-2\epsilon u)^3} = \frac{1}{4}\frac{q^2}{a^2}(1-\epsilon^2)^2 \int_{-1}^{1} \frac{udu}{(1+\epsilon^2-2\epsilon u)^3}$$

$$= \frac{1}{4}\frac{q^2}{a^2}(1-\epsilon^2)^2 \frac{1}{4\epsilon^2}\left[-\frac{1}{1+\epsilon^2-2\epsilon u} + \frac{1+\epsilon^2}{2(1+\epsilon^2-2\epsilon u)^2}\right]_{-1}^{1}$$

$$
\begin{aligned}
&= \frac{1}{4}\frac{q^2}{a^2}\frac{(1-\epsilon^2)^2}{4\epsilon^2}\left[\frac{-1}{(1-\epsilon)^2}+\frac{1}{(1+\epsilon)^2}+\frac{1+\epsilon^2}{2(1-\epsilon)^4}-\frac{1+\epsilon^2}{2(1+\epsilon)^4}\right] \\
&= \frac{q^2}{4a^2}\frac{(1-\epsilon^2)^2}{4\epsilon^2}\left[\frac{-4\epsilon}{(1-\epsilon^2)^2}+\frac{(1+\epsilon^2)(8\epsilon+8\epsilon^3)}{2(1-\epsilon^2)^4}\right] \\
&= \frac{q^2}{4a^2}\frac{-4\epsilon(1-\epsilon^2)^2+4\epsilon(1+\epsilon^2)^2}{4\epsilon^2(1-\epsilon^2)^2} = \frac{q^2}{a^2}\frac{\epsilon}{(1-\epsilon^2)^2}. \qquad (17)
\end{aligned}
$$

The direction of this force is such that the charge is attracted toward the point on the cavity wall that is closest to it.

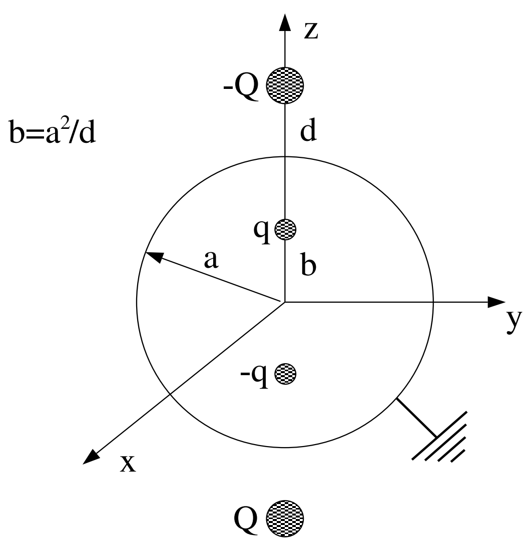
We may also ask what is the "force" between the charge and its image. The distance between them is $r_0'-r_0 = a\epsilon(1/\epsilon^2-1) = a(1-\epsilon^2)/\epsilon$, and the product of the charges is $qq' = -q^2/\epsilon$, so

$$
|\mathbf{F}| = \frac{q^2}{a^2}\frac{\epsilon}{(1-\epsilon^2)^2} \qquad (18)
$$

which is the same as the real force between the charge and the surface. One is led to ask whether the real force on the charge is always the same as that between the charge and its images. The answer is yes. The electric field produced by the real surface charge at the position of the real charge is the same as that produced by the image charge at the real charge, and so the same force will arise in both systems. It is generally much easier to calculate the force between the real charge and its images than the force between the real charge and the surface charges.

1.4 Conducting Sphere in a Uniform Applied Field

Consider next the example of a grounded conducting sphere, which means that $\Phi(\mathbf{x}) = 0$ on the sphere, placed in a region of space where there was initially a uniform electric field $\mathbf{E}_0 = E_0\hat{\mathbf{z}}$ produced by some far away fixed charges. Here, $\hat{\mathbf{z}}$ is a unit vector pointing in the z-direction. We approach this problem by replacing it with another one which will become equivalent to the first one in some limit. Let the sphere be centered at the origin and let there be not a uniform applied field but rather a charge Q placed at the point $(0, 0, -d)$ and another charge $-Q$ placed at the point $(0, 0, d)$ in Cartesian coordinates.



The resulting potential configuration is easily solved by the image method; there are images of the charges $\pm Q$ in the sphere at $(0, 0, -a^2/d)$ and at $(0, 0, a^2/d)$; they have size $-Qa/d$ and Qa/d, respectively. The potential produced by these four charges is zero on the surface of the sphere. Thus we have solved the problem of a grounded sphere in the presence of two symmetrically located equal and opposite charges. We could equally well think of the sphere as isolated (not electrically connected to anything) and neutral, because the total image charge is zero.

Now we want to think about what happens if we let Q become increasingly large and at the same time move the real charges farther and farther away from the sphere in such a way that the field they produce at the origin is constant. This field is $\mathbf{E}(\mathbf{x}) = (2Q/d^2)\hat{\mathbf{z}}$, so if Q is increased at a rate proportional to d^2, the field at the origin is unaffected. As d becomes very large in comparison with the radius a of the sphere, not only will the applied field at the origin have this value, but it will have very nearly this value everywhere in the vicinity of the sphere. The difference becomes negligible in the limit $d/a \to \infty$. Hence we recover the configuration presented in the original problem of a sphere placed in a uniform applied field. If we pick $E_0 = 2Q/d^2$, or, more appropriately, $Q = E_0 d^2/2$, we have the solution in the limit of $d \to \infty$:

$$\Phi(\mathbf{x}) = \lim_{d\to\infty}\left[\frac{E_0 d^2/2}{(d^2+r^2+2rd\cos\theta)^{1/2}} - \frac{E_0 d^2 a/2d}{(a^4/d^2+r^2+2r(a^2/d)\cos\theta)^{1/2}}\right]$$

$$+\lim_{d\to\infty}\left[-\frac{E_0 d^2/2}{(d^2+r^2-2rd\cos\theta)^{1/2}} + \frac{E_0 d^2 a/2d}{(a^4/d^2+r^2-2r(a^2/d)\cos\theta)^{1/2}}\right]$$

$$= \lim_{d\to\infty}\left[\pm\frac{E_0 d/2}{(1\pm 2(r/d)\cos\theta + r^2/d^2)^{1/2}} \mp \frac{E_0 da/2r}{(1\pm(a^2/rd)\cos\theta + a^4/d^2r^2)^{1/2}}\right]$$

$$= -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta \qquad (19)$$

The first term, $-E_0 r\cos\theta$, is the potential of the applied constant field, $\mathbf{E}_0$. The second is the potential produced by the induced surface charge density on the sphere. This has the characteristic form of an *electric dipole* field, of which we shall hear more presently. The *dipole moment* $\mathbf{p}$ associated with any charge distribution is defined by the equation

$$\mathbf{p} = \int d^3x\, \mathbf{x}\rho(\mathbf{x}); \qquad (20)$$

in the present case the dipole moment of the sphere may be found either from the surface charge distribution or from the image charge distribution. Taking the latter tack, we find

$$\mathbf{p} = \int d^3x\, \mathbf{x}\frac{E_0 da}{2}\left[-\delta(z+a^2/d)\delta(y)\delta(x) + \delta(z-a^2/d)\delta(y)\delta(x)\right]$$
$$= \frac{E_0 da}{2}\left[(a^2/d)\hat{\mathbf{z}} + (a^2/d)\hat{\mathbf{z}}\right] = E_0 a^3\hat{\mathbf{z}}. \qquad (21)$$

Comparison with the expression for the potential shows that the dipolar part of the potential may be written as

$$\Phi(\mathbf{x}) = \mathbf{p}\cdot\mathbf{x}/r^3 \qquad (22)$$

The charge density on the surface of the sphere may be found in the usual way:

$$4\pi\sigma = E_r = -\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = E_0\cos\theta + \frac{2E_0}{a^3}a^3\cos\theta = 3E_0\cos\theta. \qquad (23)$$

Hence,

$$\sigma(\theta) = \frac{3}{4\pi} E_0 \cos\theta. \tag{24}$$

2 Green's Function Method for the Sphere

Next, let us do an example of the use of the Green's function method by considering a Dirichlet potential problem inside of a sphere. The task is to calculate the potential distribution inside of an empty ($\rho(\mathbf{x}) = 0$, $x \in V$) spherical cavity of radius a, given some specified potential distribution $V(\theta, \phi)$ on the surface of the sphere. We can immediately invoke the Green's function expression

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \int_S d^2x' \, \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'}, \tag{25}$$

and we already know that,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{a/r'}{|\mathbf{x} - (a^2/r'^2)\mathbf{x}'|} \tag{26}$$

since $G(\mathbf{x}, \mathbf{x}')$ is the potential at $\mathbf{x}$ due to a unit point charge at $\mathbf{x}'$ ($\mathbf{x}\,, \mathbf{x}' \in V$), and we have just solved this problem. If we let γ be the angle between $\mathbf{x}$ and $\mathbf{x}'$,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} - \frac{a/r'}{(r^2 + (a^4/r'^2) - 2r(a^2/r')\cos\gamma)^{1/2}}. \tag{27}$$

Then

$$\left.\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'}\right|_S = \left.\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial r'}\right|_{r'=a}$$

$$= -\frac{1}{2}\left[\frac{2a - 2r\cos\gamma}{(r^2 + a^2 - 2ra\cos\gamma)^{3/2}} - a\frac{2ar^2 - 2ra^2\cos\gamma}{(r^2a^2 + a^4 - 2ra^3\cos\gamma)^{3/2}}\right]$$

$$= -\frac{a(1 - r^2/a^2)}{(r^2 + a^2 - 2ra\cos\gamma)^{3/2}} = -\frac{1}{a^2}\frac{(1 - \epsilon^2)}{(1 + \epsilon^2 - 2\epsilon\cos\gamma)^{3/2}} \quad (28)$$

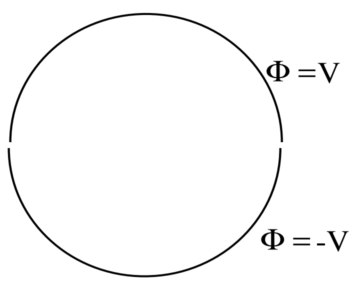
where $\epsilon = r/a$. For simplicity, let us suppose that $\rho(\mathbf{x}) = 0$ inside of the sphere. Then

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} \frac{\sin\theta' d\theta' V(\theta', \phi')(1 - \epsilon^2)}{(1 + \epsilon^2 - 2\epsilon\cos\gamma)^{3/2}}. \quad (29)$$

In terms of θ, ϕ and θ' and ϕ',

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \quad (30)$$

This integral can rarely be done in closed form in terms of simple functions; however, it is generally a simple matter to carry out the integrals numerically. As an example, consider



$$V(\theta, \phi) = \begin{cases} V, & 0 \le \theta \le \pi/2 \\ -V, & \pi/2 \le \theta \le \pi. \end{cases} \quad (31)$$

Then the answer will not depend on ϕ, so we may arbitrarily set ϕ equal to zero and proceed. Using $\epsilon \equiv r/a$, we have

$$\Phi(\epsilon, \theta) = \frac{V}{4\pi}(1-\epsilon^2)\int_0^{2\pi} d\phi' \left[\int_0^{\pi/2} \frac{\sin\theta' d\theta'}{(1 + \epsilon^2 - 2\epsilon\cos\gamma)^{3/2}} - \int_{\pi/2}^{\pi} \frac{\sin\theta' d\theta'}{(1 + \epsilon^2 - 2\epsilon\cos\gamma)^{3/2}}\right] \quad (32)$$

The integral is still difficult in the general case. For $\theta = 0$, it is easier:

$$\Phi(\epsilon, 0) = \frac{V}{4\pi}(1-\epsilon^2)2\pi \left[\int_0^1 \frac{du}{(1+\epsilon^2-2\epsilon u)^{3/2}} - \int_{-1}^0 \frac{du}{(1+\epsilon^2-2\epsilon u)^{3/2}}\right] \tag{33}$$

These integrals are easily completed with the result that

$$\Phi(\epsilon, 0) = \frac{V}{\epsilon}\left(1 - \frac{1-\epsilon^2}{\sqrt{1+\epsilon^2}}\right). \tag{34}$$

An alternative approach, valid for $r/a << 1$, is to expand the integrand in powers of ϵ and then to complete the integration term by term. This is straightforward with a symbolic manipulator but tedious by hand. Either way, a solution in powers of ϵ is generated.

$$\Phi(\epsilon, \theta) = \frac{3V}{2}\left[\epsilon\cos\theta - \frac{7}{12}\epsilon^3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + O(\epsilon^5)\right]. \tag{35}$$

The alert student will recognize that the functions of $\cos\theta$ that are being generated are *Legendre polynomials*;

$$\begin{aligned} P_1(\cos\theta) &= \cos\theta, \\ P_3(\cos\theta) &= \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta, \end{aligned} \tag{36}$$

etc. Note that only terms which are odd in $\cos\theta$ enter into the sum, due to the symmetry of the boundary conditions.

3 Orthogonal Functions and Expansions; Separation of Variables

We turn now to a quite different, much more systematic approach to the solution Laplace's equation

$$\nabla^2\Phi(\mathbf{x}) = 0 \tag{37}$$

as a boundary value problem. It is implemented by expanding the solution in some domain V using complete sets of orthogonal functions

$$\Phi(\eta, \xi, \nu) = \sum_{nlm} A_n(\eta) B_l(\xi) C_m(\nu) \tag{38}$$

and determining the coefficients in the expansion by requiring that the solution take on the proper values on the boundaries. For simple geometries for which Laplace's equation separates (spheres, cylinders, rectangular parallelepipeds) this method can always be utilized[2]. Before launching into a description of how one proceeds in specific cases (or geometries), let us take a few minutes to review the terminology of orthogonal function expansions and some basic facts.

Suppose that we have a set of functions $U_n(\eta), n = 1, 2, \ldots$ which are orthogonal on the interval $a \le \eta \le b$, by which we mean that

$$\int_a^b d\eta\, U_n^*(\eta) U_m(\eta) = 0, \text{ if } m \neq n; \tag{39}$$

[2] It can also be very tedious.

the superscript * denotes complex conjugation. Further, the functions $U_n(\eta)$ are normalized on the interval,

$$\int_a^b d\eta U_n^*(\eta) U_n(\eta) = \int_a^b d\eta |U_n(\eta)|^2 = 1. \tag{40}$$

Combining these equations we have

$$\int_a^b d\eta U_n^*(\eta) U_m(\eta) = \left\{ \begin{array}{ll} 0, & n \neq m \\ 1, & n = m \end{array} \right\} = \delta_{nm}. \tag{41}$$

The functions $U_n(\eta)$ are said to be *orthonormal*; δ_{nm} is called a *Kronecker delta function.*

Next, we attempt to expand, on the interval $a \leq \eta \leq b$, an arbitrary function $f(\eta)$ as a linear combination of the functions $U_n(\eta)$, which are referred to as *basis* functions. Keeping just N terms in the expansion, one has

$$f(\eta) \approx \sum_{n=1}^{N} a_n U_n(\eta). \tag{42}$$

We need a criterion for choosing the coefficients in the expansion; a standard criterion is to minimize the mean square error E which may be defined as follows:

$$\begin{aligned} E &= \int_a^b d\eta |f(\eta) - \sum_{n=1}^{N} a_n U_n(\eta)|^2 \\ &= \int_a^b d\eta \left(f^*(\eta) - \sum_{n=1}^{N} a_n^* U_n^*(\eta) \right) \left(f(\eta) - \sum_{m=1}^{N} a_m U_m(\eta) \right). \end{aligned} \tag{43}$$

The conditions for an extremum are

$$\left(\frac{\partial E}{\partial a_k} \right)_{a_k^*} = 0 = \left(\frac{\partial E}{\partial a_k^*} \right)_{a_k}. \tag{44}$$

where a_k and a_k^* have been treated as independent variables [3] Application of these conditions leads to

$$0 = \int_a^b d\eta \left(f^*(\eta) - \sum_{n=1}^{N} a_n^* U_n^*(\eta) \right) U_k(\eta)$$
$$= \int_a^b d\eta \left(f(\eta) - \sum_{n=1}^{N} a_n U_n(\eta) \right) U_k^*(\eta) \qquad (45)$$

or, making use of the orthonormality of the basis functions,

$$a_k = \int_a^b d\eta f(\eta) U_k^*(\eta). \qquad (46)$$

with a_n^* given by the complex conjugate of this relation. If the basis functions are orthogonal but not normalized, then one finds

$$a_k = \frac{\int_a^b d\eta f(\eta) U_k^*(\eta)}{\int_a^b d\eta |U_k(\eta)|^2}. \qquad (47)$$

The set of basis functions $U_n(\eta)$ is said to be complete if the mean square error can be made arbitrarily small by keeping a sufficiently large number of terms in the sum. Then one says that the sum converges in the mean to the given function. If we are a bit careless, we can then write

$$f(\eta) = \sum_n a_n U_n(\eta) = \sum_n \int_a^b d\eta' f(\eta') U_n^*(\eta') U_n(\eta)$$
$$= \int_a^b d\eta' \left(\sum_n U_n(\eta) U_n^*(\eta') \right) f(\eta'), \qquad (48)$$

from which it is evident that

$$\sum_n U_n(\eta) U_n^*(\eta') = \delta(\eta - \eta') \qquad (49)$$

[3] This is always possible, since a_k and a_k^* are linearly related to $\mathrm{Re}(a_k)$ and $\mathrm{Im}(a_k)$.

for a complete set of functions. This equation is called the *completeness* or *closure relation*.

We may easily generalize to a space of arbitrary dimension. For example, in two dimensions we may have the space of η and ζ with $a \leq \eta \leq b$, and $c \leq \zeta \leq d$ and complete sets of orthonormal functions $U_n(\eta)$ and $V_m(\zeta)$ on the respective intervals. Then the arbitrary function $f(\eta, \zeta)$ has the expansion

$$f(\eta, \zeta) = \sum_{n,m} A_{nm} U_n(\eta) V_m(\zeta), \tag{50}$$

where

$$A_{nm} = \int_a^b d\eta \int_a^b d\zeta f(\eta, \zeta) U_n^*(\eta) V_m^*(\zeta). \tag{51}$$

3.1 Fourier Series

Returning to the one-dimensional case, suppose that the interval is infinite, $-\infty < \eta < \infty$. Then the index n of the functions $U_n(\eta)$ may become a continuous index, $U_n(\eta) \rightarrow U(\eta; \rho)$. A familiar example of this is the Fourier integral which is the limit of a Fourier series when the interval on which functions are expanded becomes infinite. Consider that we have the interval $-a/2 < \eta < a/2$. Then the Fourier series may be built from the basis functions

$$U_m(\eta) = \frac{1}{\sqrt{a}} e^{i2\pi m\eta/a}, \quad \text{with } m = 0, \pm 1, \pm 2, ...; \tag{52}$$

these functions form a complete orthonormal set. The expansion of $f(\eta)$ is

$$f(\eta) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{\infty} A_m e^{i2\pi m\eta/a} \tag{53}$$

with

$$A_m = \frac{1}{\sqrt{a}} \int_{-a/2}^{a/2} d\eta f(\eta) e^{-i2\pi m\eta/a}. \tag{54}$$

The closure relation is

$$\frac{1}{a} \sum_m e^{i2\pi m(\eta-\eta')/a} = \delta(\eta - \eta'). \tag{55}$$

Now define $k \equiv 2\pi m/a$ or $m = ka/2\pi$. Also, define $A_m = \sqrt{2\pi/a}\, A(k)$. Note that for $a \to \infty$, k takes on a set of values that approach a continuum. Thus

$$f(\eta) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \frac{a}{2\pi} dk e^{ik\eta} \sqrt{\frac{2\pi}{a}} A(k) = \frac{1}{\sqrt{2\pi}} \int dk e^{ik\eta} A(k), \tag{56}$$

while

$$\sqrt{\frac{2\pi}{a}} A(k) = \frac{1}{\sqrt{a}} \int d\eta f(\eta) e^{-ik\eta} \tag{57}$$

or

$$A(k) = \frac{1}{\sqrt{2\pi}} \int d\eta f(\eta) e^{-ik\eta} \tag{58}$$

while the closure relation now reads

$$\frac{1}{2\pi} \int dk e^{ik(\eta-\eta')} = \delta(\eta - \eta')\,, \tag{59}$$

thus $e^{ik\eta}$ form a complete set (this is also a useful representation of the Dirac delta function).

Note that we can also write this equation as

$$\frac{1}{2\pi}\int d\eta e^{i\eta(k-k')} = \delta(k-k') \tag{60}$$

which is the orthonormalization expression of the complete set of functions $U(\eta;k)$ on the infinite η interval. These functions are

$$U(\eta;k) = \frac{1}{\sqrt{2\pi}}e^{i\eta k}. \tag{61}$$

3.2 Separation of Variables

We are going to attempt to find solutions to boundary value problems in three dimensions by writing the solution as a sum of products of three one-dimensional functions,

$$\Phi(\eta,\zeta,\nu) = \sum_{n,l,m} A_{nlm}E_n(\eta)Z_l(\zeta)N_m(\nu). \tag{62}$$

We will do this for the particular cases of rectangular, cylindrical, and spherical polar coordinates. Now, if the functions E, Z, and N are members of complete sets on appropriate intervals, we can certainly write any three-dimensional function as a linear combination of such products. Because we are looking for special three-dimensional functions, however, that is, solutions to the Laplace equation, we do not actually have to employ complete sets of functions of all three variables. To determine just what we do have to use, we will try to demand that each term in the sum is itself a solution to the Laplace equation, which is more restrictive than just requiring the sum to be a solution. It turns out that this is possible in the Cartesian, cylindrical, and spherical coordinate systems and also in eight more (see Landau and Lifshitz for more information)! The simplification that takes place when one makes this *separation of variables* is that each of the functions of a single variables has to be a solution of a relatively simple second order ordinary differential equation rather than a partial differential equation.

3.3 Rectangular Coordinates

Let us look for a solution of Laplace's equation in the form of a product of functions of x, y, and z,

$$\Phi(\mathbf{x}) = X(x)Y(y)Z(z). \tag{63}$$

Substitution into Laplace's equation $\nabla^2\phi(\mathbf{x}) = 0$ yields

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0. \tag{64}$$

Dividing by Φ we find

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{65}$$

Each term on the LHS of this equation depends on a single variable; consequently, since the equation must remain true when any one variable is varied with the others held fixed, it must be the case that each term is a constant, independent of the variable. Since the three terms add to zero, at least one must be a positive constant, and at least one must be a negative constant. Let us suppose that two are negative, and one, positive. Thus we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2;\ \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2;\ \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 = \alpha^2 + \beta^2. \tag{66}$$

For negative constants, the solutions are oscillatory; when they are positive, solutions are exponential:

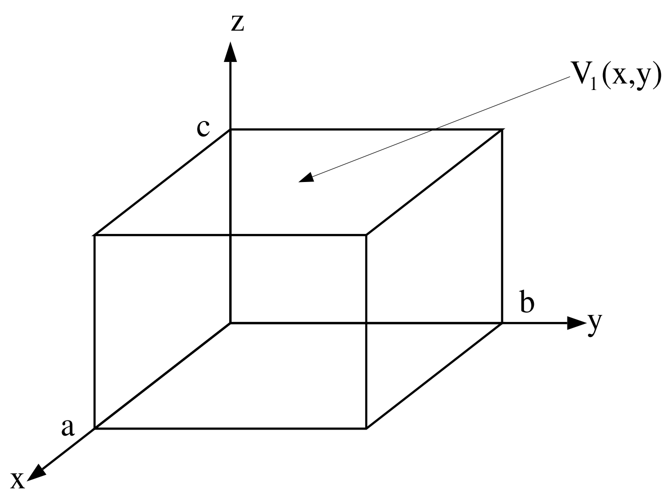
$$Z(z) \sim e^{\pm\gamma z};\ X(x) \sim e^{\pm i\alpha x};\ Y(y) \sim e^{\pm i\beta y}, \tag{67}$$

or, equivalently,

$$\begin{aligned} X(x) &\sim \sin(\alpha x) \text{ or } \cos(\alpha x) \\ Y(y) &\sim \sin(\beta y) \text{ or } \cos(\beta y) \\ Z(z) &\sim \sinh(\gamma z) \text{ or } \cosh(\gamma z). \end{aligned} \tag{68}$$

Now, α and β can be any real constants whatsoever, which means that by taking linear combinations of solutions of the kind outlined above, we can construct **any** function of x and y at some particular value of z.

This is just what we need to solve boundary value problems with planar surfaces. For example, suppose that we need to solve the Laplace equation inside of a rectangular parallelepiped of edge lengths a, b, and c with the potential given on the surface. We can find a solution by considering six distinct problems and superposing the six solutions to them. Each of these six problems has on one face (a different one in each problem) of the box the same potential as that given in the original problem while on the other five faces the potential is zero. Summing the six solutions gives a potential which has the same values on each face of the box as given in the original problem. Let's see how to solve one of these six problems; the others follow trivially. For this problem we may suppose that the faces of the box are given by the planes $z = 0, c$, $x = 0, a$, and $y = 0, b$. Let the potential on the face $z = c$ be $\Phi(x, y, c) = V(x, y)$ while $\Phi(\mathbf{x}) \equiv 0$ on the other five faces.



$$\Phi(x, y, c) = V(x, y) \quad \Phi(\mathbf{x}) \equiv 0 \text{ on the other faces} \tag{69}$$

In order to satisfy the B.C., we must choose the constants α, β and γ so that we have a complete set of functions [4] on the face with the non-trivial boundary condition. Our expansion for the potential now

[4] As we have seen, the sinusoidal function form a complete set, the hyperbolic functions do not

takes the form

$$\Phi(x,y,z) = \sum_{\alpha,\beta} A_{\alpha\beta} \sinh(\gamma_{\alpha\beta} z)(\sin\alpha x \sin\beta y) \tag{70}$$

where α and β are such that $\alpha a = n\pi$ and$\beta b = m\pi$ which makes the basis functions of x and y orthogonal and complete on the domain of the constant-z face of the box. Thus,

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sinh(\gamma_{nm} z) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \tag{71}$$

where

$$\gamma_{nm} = \pi \left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]^{1/2}. \tag{72}$$

Notice that only the sine functions are used and also only the hyperbolic sine. The reason for the latter is that the potential must vanish at $z = 0$; this condition rules out the use of the hyperbolic cosine which is not zero at zero argument. The cosine could be used but is not needed as the sine functions with arguments introduced above are complete on the appropriate intervals.

The coefficients in the sum for the solution are determined by looking at the potential on the top face of the box:

$$V(x,y) = \sum_{nm} A_{nm} \sinh(\gamma_{nm} c) \sin(n\pi x/a) \sin(m\pi y/b). \tag{73}$$

Multiply by $\sin(l\pi x/a)\sin(p\pi y/b)$ and integrate[5] over the face of the box:

$$\int_0^a dx \int_0^b dy V(x,y) \sin(l\pi x/a) \sin(p\pi y/b) = A_{lp} \sinh(\gamma_{lp} c) \frac{1}{4} ab, \tag{74}$$

[5]Here we use the relation $\int_0^a \sin^2(n\pi x/a) = a/2$

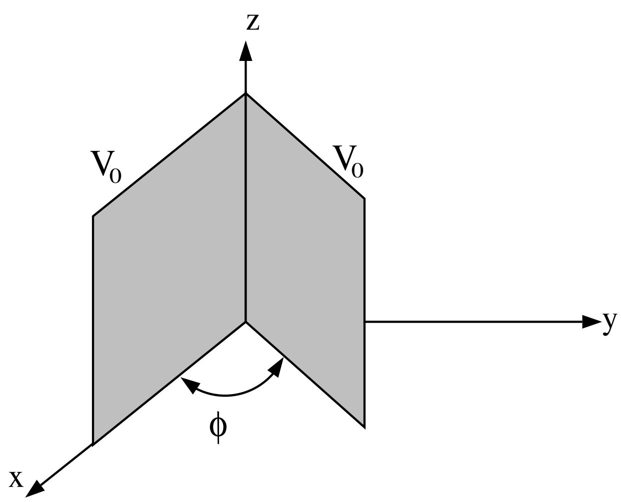
or

$$A_{lp} = \frac{4}{ab \sinh(\gamma_{lp} c)} \int_0^a dx \int_0^b dy V(x,y) \sin(l\pi x/a) \sin(p\pi y/b) \qquad (75)$$

In this manner one can do any Dirichlet problem on a rectangular parallelepiped in the form of an infinite series.

3.4 Fields and Potentials on Edges

What we will never find very accurately from the expansion devised in the preceding section is the behavior of the potential and field close to an edge of the box where many terms must be kept to have decent convergence of the series. However, in these regions we may devise a quite different approximation which converges well very close to the edge. Suppose then that one is very close to such an edge where the boundary may be considered to consist of two infinite intersecting planes. Let the edge be coincident with the z-axis with the planes lying at constant values of th azimuthal angle ϕ.



The solution will then depend only on ϕ and ρ where $\rho = \sqrt{x^2 + y^2}$. In these variables, the Laplace equation is

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0. \tag{76}$$

Once again we use separation of variables, writing

$$\Phi(\rho, \phi) = R(\rho)\Psi(\phi). \tag{77}$$

Substitution into the Laplace equation and division by Φ itself yields the equation

$$\frac{1}{\rho R(\rho)}\frac{d}{d\rho}\left(\rho \frac{dR(\rho)}{d\rho}\right) + \frac{1}{\rho^2 \Psi(\phi)}\frac{d^2\Psi(\phi)}{d\phi^2} = 0. \tag{78}$$

If we multiply by ρ^2, we find that the first term on the LHS depends only on ρ and the second one depends only on ϕ; consequently they must be equal and opposite constants,

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = C, \; C = \text{constant}, \tag{79}$$

and

$$\frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = -C. \tag{80}$$

What must be the sign of C? If $C > 0$, the $\Psi(\phi)$ is an oscillatory function while $R(\rho)$ is not. But if $C < 0$, then the converse is true. If our boundary value problem has Φ equal to some constant on the edges of a wedge with the surfaces of the wedge at $\phi = 0$ and $\phi = \beta$, then we will need to have an oscillatory $\Psi(\phi)$. Hence choose $C \geq 0$. Write $C \equiv \nu^2$, where ν is real. There is the special case when $\nu = 0$, for which $\Psi(\phi) = a + b\phi$ and $R(\nu) = c + d\ln\rho$. When $C > 0$, then

$$\Psi(\phi) = A\sin(\nu\phi) + B\cos(\nu\phi) = A'\sin(\nu\phi + \phi_0), \tag{81}$$

and $R(\rho)$ is the general solution of Eq. (105). Try $R = a\rho^p$; substitution into the differential equation gives

$$ap^2\rho^{p-1} - a\nu^2\rho^{p-1} = 0 \tag{82}$$

from which we find $p = \pm\nu$. The most general solution is

$$R(\rho) = a\rho^{\nu} + b\rho^{-\nu}, \tag{83}$$

and so a single term in the expansion for Φ is

$$\Phi(\rho,\phi) = (A\rho^{\nu} + B\rho^{-\nu})\sin(\nu\phi + \phi_0) \tag{84}$$

where $A,\ B$, and ϕ_0 are constants to be determined by some boundary conditions.

There is also still the question of allowed values of ν. Let us specify that on the sides of the wedge, $\Phi(\rho, 0) = \Phi(\rho, \beta) = V_0$. To match this, we use $\nu = 0$ with $b = d = 0$ and $ac = V_0$. Then the boundary condition is matched on the edges of the wedge. Further, we must pick $\nu \neq 0$ (and ϕ_0) so that

$$\sin(0 + \phi_0) = 0 \tag{85}$$

and

$$\sin(\nu\beta + \phi_0) = 0; \tag{86}$$

we can easily see that $\phi_0 = 0$ and $\nu\beta = n\pi$, $n = 1, 2, ...$ will work. Now add up solutions of the kind generated, that is, solutions with different values of n and undetermined coefficients, to find the most general solution (of this kind),

$$\Phi(\rho, \phi) = V_0 + \sum_{n=1}^{\infty} \left(A_n \rho^{n\pi/\beta} + B_n \rho^{-n\pi/\beta}\right) \sin\left(\frac{n\pi\phi}{\beta}\right). \tag{87}$$

where the constant term V_0 corresponds to $\nu = 0$.

If the physical region includes the origin ($\rho \to 0$), then we cannot have any negative powers of ρ because they will lead to singularities in Φ at the origin; physically, we know that that won't happen. Hence all B_n are zero (And that is also why we didn't keep the $\ln \rho$ part of the $\nu = 0$ solution). Thus we have

$$\Phi(\rho, \phi) = V_0 + \sum_{n=1}^{\infty} A_n \rho^{n\pi/\beta} \sin(n\pi\phi/\beta). \tag{88}$$

The remaining coefficients are determined by boundary conditions on a surface that closes the system; for example a surface specified by $\rho = \rho_0$ for $0 \leq \phi \leq \beta$.

Without concerning ourselves with the details of fitting the expansion to such a function, we can still see what are the interesting qualitative features of the potential and fields for ρ very small, which means $\rho << \rho_0$. There the potential will be dominated by the term proportional to the smallest power of ρ, which is the $n = 1$ term,

$$\Phi(\rho, \phi) \approx V_0 + A_1 \rho^{\pi/\beta} \sin(\pi\phi/\beta), \text{ at small } \rho, \tag{89}$$

assuming that $A_1 \neq 0$. Taking appropriate derivatives of the potential, we may find the components of the electric field,

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = -\frac{\pi A_1}{\beta} \rho^{\pi/\beta - 1} \sin(\pi\phi/\beta) \tag{90}$$

and

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = -\frac{\pi A_1}{\beta} \rho^{\pi/\beta - 1} \cos(\pi\phi/\beta). \tag{91}$$

Also, the charge density on the conductor close to the origin is found from

$$\sigma(\rho, 0) = \frac{E_\phi(\rho, 0)}{4\pi} = -\frac{A_1}{4\beta} \rho^{\pi/\beta - 1} \tag{92}$$

and

$$\sigma(\rho, \beta) = -\frac{E_\phi(\rho, \beta}{4\pi} = -\frac{A_1}{4\beta} \rho^{\pi/\beta - 1}. \tag{93}$$

Depending on whether $\beta < \pi$ or $\beta > \pi$, one gets dramatically different fields and charge densities as $\rho \to 0$. For $\beta < \pi$, $\pi/\beta - 1 > 0$ and fields and σ vanish as ρ goes to 0. But for $\beta > \pi$, $\pi/\beta - 1 < 0$, and consequently they become very large.

Of course, no real conductor has a perfectly sharp point; there is some rounding on a scale of length δ, leading to a maximum field of order

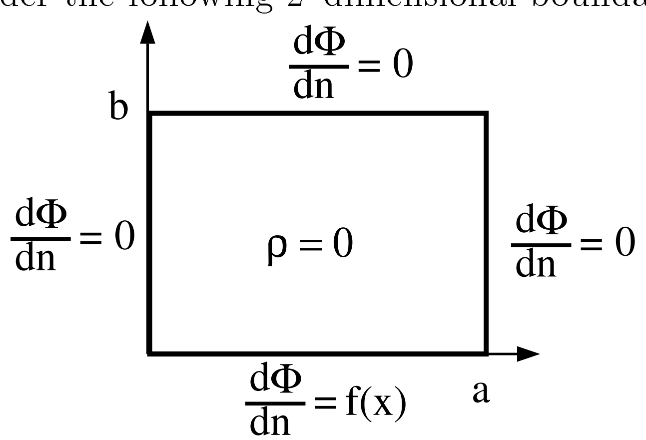
$$E_{max} \approx \frac{A_1}{4\beta}\delta^{\pi/\beta-1} \sim \frac{V_0}{R}\left(\frac{R}{\delta}\right)^{1-\pi/\beta} \sim E_0\left(\frac{R}{\delta}\right)^{1-\pi/\beta}, \qquad (94)$$

where R is the overall size of the system, that is, the distance from the point or wedge to ground. For a potential difference of, say 10^4 $statv$, $R = 1$ km, $\delta = 1$ mm, and $\beta = 2\pi$, we have $E_{max} \sim 30$ $statv/cm$ or 9000 v/cm.

4 Examples

4.1 Two-dimensional box with Neumann boundaries

Consider the following 2–dimensional boundary value problem.



Find $\Phi(\mathbf{x})$ inside the rectangle (Note that due to the Neumann B.C. $\Phi(\mathbf{x})$ can only be determined up to an arbitrary additive constant). Show that we must have $\int_0^a dx f(x) = 0$.

This problem is very similar to that discussed in Sec. III.C. The difference is that this is in 2–d instead of 3–d, and has Neumann rather than Dirichlet B.C. Thus, we search for solutions of

$$\nabla^2 \Phi(x, y) = 0$$

in the form

$$\Phi(\mathbf{x}) = X(x)Y(y)$$

subject to the boundary conditions indicated above. Combining these equations yields

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0.$$

As in class, for arbitrary x and y, the only way to satisfy this equation is for both parts to be constant. Thus

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2; \ \frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha^2.$$

We choose this sign convention for the constant α so that we can easily satisfy the B.C.. Thus the solutions take the form

$$X(x) = A\sin(\alpha x) + B\cos(\alpha x); \ Y(y) = C\sinh\{\alpha(b-y)\} + D\cosh\{\alpha(b-y)\}$$

Here we choose the form $\sinh\{\alpha(b-y)\}$ rather than $\sinh\{\alpha y\}$ with an eye toward satisfying the B.C. easily.

We can eliminate some of these coefficients by imposing the simple B.C.

$$\left.\frac{dX}{dx}\right|_{x=0} = \left.\frac{dX}{dx}\right|_{x=a} = 0$$

Clearly, to satisfy the first of these $A = 0$, and to satisfy the second $\alpha = n\pi/a$. Thus

$$X_n(x) = \cos(n\pi x/a)$$

Similarly, $\left.\frac{dY}{dy}\right|_{y=b} = 0$ indicates that $C = 0$. Thus

$$\Phi(x,y) = \sum_{n=0}^{\infty} a_n \cos(n\pi x/a) \cosh\left(\frac{n\pi}{a}(b-y)\right)$$

The set $\{a_n\}$ are determined by the remaining B.C.

$$\frac{\partial\Phi}{\partial n} = -\left.\frac{\partial\Phi}{\partial y}\right|_{y=0} = f(x)$$

$$f(x) = \sum_n \{\frac{n\pi}{a} \sinh)n\pi b/a)a_n\} cos(n\pi x/a)$$

or if we identify $b_n = \frac{n\pi}{a} \sinh(n\pi b/a)a_n$

$$f(x) = \sum_n b_n cos(n\pi x/a)$$

Now if $f(x)$ is a regular function defined on the interval $0 < x < a$, then it may be represented as a cosine sum over all terms. However, the sum above is incomplete since it does not include the b_0 term. Thus we can only solve the problem if $b_0 = 0$, or equivalently, $\int dx f(x) \propto b_0 = 0$. Physically, what does this mean? (Hint, consider Gauss' law in 2–d, and the fact that the rectangle encloses no charge since $\nabla^2\Phi = 0$).

The remaining b_n may be determined in the usual way.

$$\int_0^a dx f(x) cos(l\pi x/a) = \sum_{n=1}^{\infty} b_n \int_0^a dx \cos(n\pi x/a)\cos(l\pi x/a)$$

then as

$$\int_0^a cos(n\pi x/a) cos(l\pi x/a) = \delta_{l,n}$$

and using the relation between a_l and b_l above we get

$$a_l = \frac{2}{l\pi \sinh(l\pi b/a)} \int_0^a dx f(x)\cos(l\pi x/a)$$

which define Φ through

$$\Phi(x,y) = \Phi_o + \sum_{n=1}^{\infty} a_n \cos(n\pi x/a)\cosh\left(\frac{n\pi}{a}(b-y)\right).$$

Now suppose that $f(x)$ is defined as $f(x) = E_o(1 - 2x/a)$.

$$a_l = \frac{2}{l\pi \sinh(l\pi b/a)} \int_0^a dx E_o(1-2x/a)\cos(l\pi x/a)$$

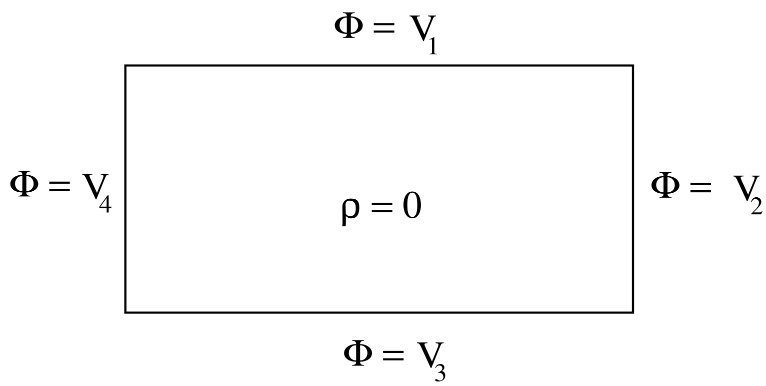
or, after integrating,

$$a_l = \frac{2}{l\pi \sinh(l\pi b/a)} \frac{2a}{l^2\pi^2}\left(1-(-1)^l\right)$$

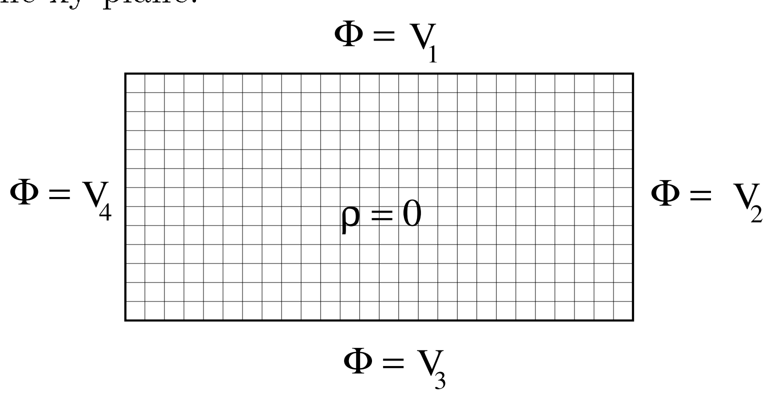
4.2 Numerical Solution of Laplace's Equation

As we discussed earlier, it is possible to solve Laplace's equation through separation of variables and special functions only for a restricted set of problems with separable geometries. When this is not the case, i.e. when the bounding surfaces, or charge distribution (when solving Poisson's equation) involve complex geometries, we generally solve for the potential numerically.

To illustrate how this is done, consider the following (exactly solvable) problem of a two-dimensional box with Dirichlet boundary conditions.



In order to solve Laplace's equation numerically with these boundary conditions, we will introduce a regular grid of width δ and dimensions $N \times M$ in the xy plane.



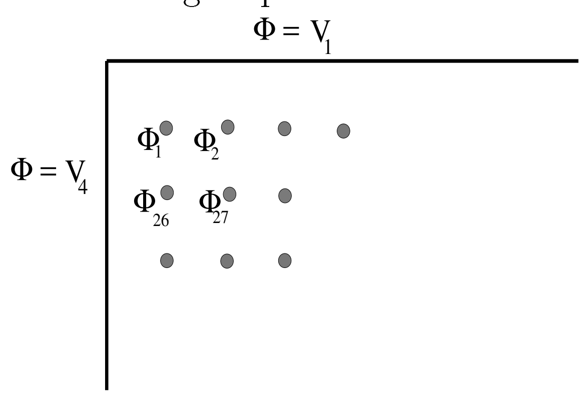
It is then a simple matter to approximately solve

$$\begin{aligned} \nabla^2 \Phi(\mathbf{x}) &= \frac{\partial^2 \Phi(\mathbf{x})}{\partial x^2} + \frac{\partial^2 \Phi(\mathbf{x})}{\partial y^2} = 0 \qquad (95) \\ &\approx \frac{\Phi(x+\delta, y) - 2\Phi(x,y) + \Phi(x-\delta, y)}{\delta^2} + \frac{\Phi(x, y+\delta) - 2\Phi(x,y) + \Phi(x, y-\delta)}{\delta^2} \end{aligned}$$

or, equivalently,

$$\Phi(x+\delta, y) + \Phi(x-\delta, y) + \Phi(x, y+\delta) + \Phi(x, y-\delta) - 4\Phi(x, y) = 0 \quad (96)$$

Since Laplace's equation is now written as a finite difference equation of the potentials, it is a simple manner to enforce the boundary conditions by replacing each occurrence of $\Phi(\mathbf{x})$ above, when $\mathbf{x}$ is the location of a boundary, by the appropriate boundary value. For example, consider a point in the upper left-hand quarter of the rectangle. This is illustrated below, where the potential at each internal grid point is denoted by Φ_n where integers n index each grid point.



At the point $n = 1$, the finite-difference form of Laplace's equation is

$$\Phi_2 + V_4 + V_1 + \Phi_{26} - 4\Phi_1 = 0$$

or

$$\Phi_2 + \Phi_{26} - 4\Phi_1 = -V_4 - V_1\,.$$

In fact, we obtain one such linear equation for each grid point. The equations are coupled of course in that each Φ_n occurs in at least two other equations. Thus, we have a set of $N \times M$ linear equations to solve for the potential. This is done numerically by a variety of methods (see the homework). This method is easily extended to handle irregular boundaries, a mixture of Dirichlet and Neumann boundary conditions, as well as finite (non-singular) charge densities in the solution of Poisson's equation.

4.3 Derivation of Eq. 35: A Mathematica Session

In section II, we explore the problem of the potential between two hemispheres maintained at opposite potentials. Using the Dirichlet Greens function method, we had reduced the problem to quadruture.

$$\Phi(\epsilon,\theta) = \frac{V}{4\pi}(1-\epsilon^2)\int_0^{2\pi} d\phi' \left[\int_0^{\pi/2} \frac{\sin\theta' d\theta'}{(1+\epsilon^2-2\epsilon\cos\gamma)^{3/2}} - \int_{\pi/2}^{\pi} \frac{\sin\theta' d\theta'}{(1+\epsilon^2-2\epsilon\cos\gamma)^{3/2}}\right] \tag{97}$$

The integral is still difficult in the general case; however, alternative approach, valid for $\epsilon = r/a << 1$, is to expand the integrand in powers of ϵ and then to complete the integration term by term. This is straightforward but tedious and generates a solution in powers of ϵ. This is quite tedious to perform by hand, but is straightforward with a symbolic manipulator like Mathematica.

```
In[1]:= integrand= Sin[thetap]/(1+ep^2-2 ep Cos[gamma])^(3/2)


                    Sin[thetap]
```

```
Out[1]= ------------------------------
                2                   3/2
        (1 + ep  - 2 ep Cos[gamma])

In[2]:= integrand=integrand/.Cos[gamma]->Cos[theta] Cos[thetap] +
Sin[theta] Sin[thetap] Cos[phip]


Out[2]= Sin[thetap] /

                  2
>    Power[1 + ep  - 2 ep (Cos[theta] Cos[thetap] +

>         Cos[phip] Sin[theta] Sin[thetap]), 3/2]

In[3]:= In[3]:= integrand1=Series[integrand,{ep,0,6}];

In[4]:= integrand1=Simplify[integrand1];

In[5]:= integrand1=Expand[integrand1];

In[6]:= answer1=Integrate[integrand1,{thetap,0,Pi/2}]-
Integrate[integrand1,{thetap,Pi/2,Pi}];
```

```
In[7]:= answer1=Simplify[answer1];

In[8]:= answer2=Simplify[Integrate[answer1,{phip,0,2 Pi}]]

                                                                   3
                         5 Pi (15 Cos[theta] - 7 Cos[3 theta]) ep
Out[8]= 6 Pi Cos[theta] ep + ------------------------------------------ +
                                            16

                                                                5
     21 Pi (130 Cos[theta] - 35 Cos[3 theta] + 33 Cos[5 theta]) ep        7
>    --------------------------------------------------------------- + O[ep]
                              512

In[9]:= Legrules={Cos[5x_]-> 16 (Cos[x])^5 - 20 (Cos[x])^3 + 5 Cos[x],
        Cos[3x_]-> 4 (Cos[x])^3 - Cos[x]  }

                                                    3             5
Out[9]= {Cos[5 (x_)] -> 5 Cos[x] - 20 Cos[x]  + 16 Cos[x] ,

                                             3
>    Cos[3 (x_)] -> -Cos[x] + 4 Cos[x] }
```

```
In[10]:= answer2= Simplify[answer2/.Legrules]

                                                                3
                             5 Pi (Cos[theta] - 7 Cos[3 theta]) ep
Out[10]= 6 Pi Cos[theta] ep + -------------------------------------- +
                                               16

                                                              5
     21 Pi (60 Cos[theta] - 35 Cos[3 theta] + 33 Cos[5 theta]) ep         7
>    ------------------------------------------------------------- + O[ep]
                                   512

In[11]:= answer2= Simplify[answer2 V (1-ep^2)/(4 Pi)]

                                                                 3
         3 V Cos[theta] ep   7 V (13 Cos[theta] + 5 Cos[3 theta]) ep
Out[11]= ----------------- - --------------------------------------- +
                 2                             64

                                                              5
     11 V (100 Cos[theta] + 35 Cos[3 theta] + 63 Cos[5 theta]) ep         7
>    ------------------------------------------------------------- + O[ep]
```

which is the answer we found, Eq. (35).