Chapter Fourteen
Radiation by Moving Charges

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Contents

We have already calculated the radiation produced by some known charge and current distribution. Now we are going to do it again. This time, however, we shall consider that the source is a single charge moving in some fairly arbitrary, possibly relativistic, fashion. Here the methods of chapter 9, e.g., multipole expansions, are impractical and there are better ways to approach the problem.

1 Liénard-Wiechert potentials

The current and charge densities produced by a charge e in motion are

$$\begin{aligned} \rho(\mathbf{x},t) &= e\delta(\mathbf{x}-\mathbf{x}(t)) \\ \mathbf{J}(\mathbf{x},t) &= e\mathbf{v}(t)\delta(\mathbf{x}-\mathbf{x}(t)) \end{aligned} \tag{1}$$

if $\mathbf{x}(t)$ is the position of the particle at time t and $\mathbf{v}(t) \equiv d\mathbf{x}(t)/dt \equiv \dot{\mathbf{x}}(t)$ is its velocity. In four-vector notation,

$$J^\mu(\mathbf{x},t) = ec\beta^\mu\delta(\mathbf{x}-\mathbf{x}(t)) \tag{2}$$

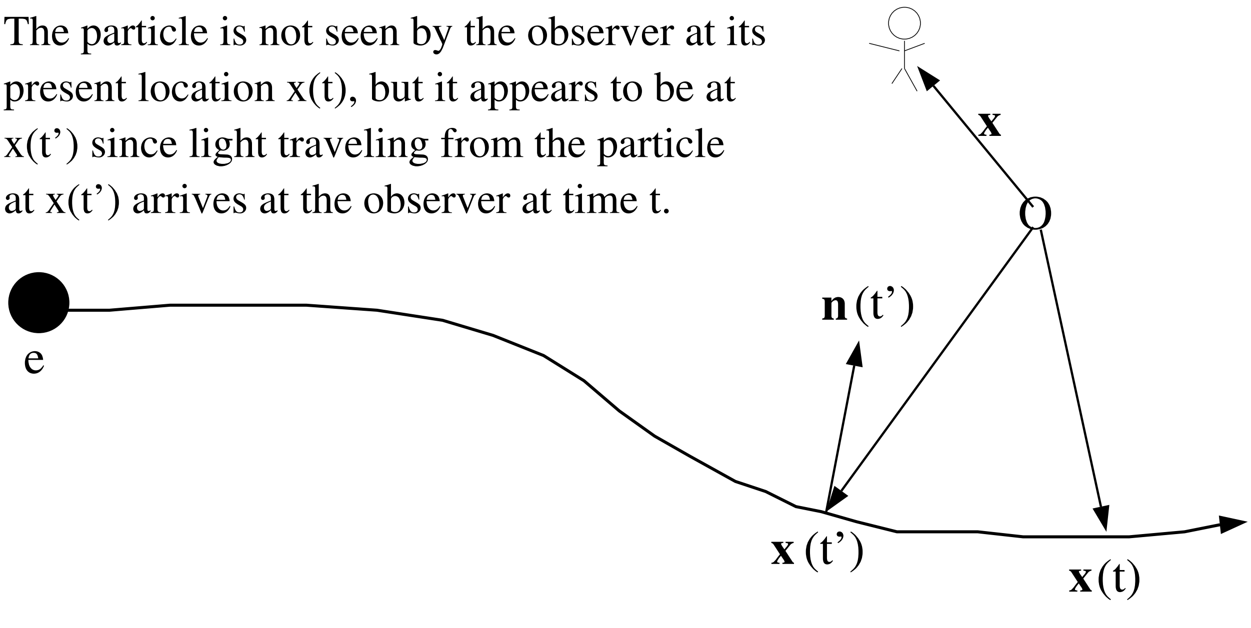
where $\beta^\mu \equiv (1, \boldsymbol{\beta})$ is **not** a four-vector; $\boldsymbol{\beta} = \mathbf{v}/c$.

From $\mathbf{J}$ and ρ, one finds $\mathbf{A}$ and Φ. We can do this in an infinite space by making use of the retarded Green's function $G(\mathbf{x}, t; \mathbf{x}', t') = \delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c)/|\mathbf{x} - \mathbf{x}'|$. From here, we can evaluate the electromagnetic field as appropriate derivatives of the potentials. All of these manipulations are straightforward. Furthermore, the integrations are relatively easy because there are many delta functions. The problem becomes interesting and unfamiliar, however, for highly relativistic particles which produce large retardation effects.

Let us start from the integral expression for the potentials:

$$\begin{aligned}
A^\mu(\mathbf{x}, t) &= \int d^3x' dt'\, G(\mathbf{x}, t; \mathbf{x}', t') J^\mu(\mathbf{x}', t') \\
&= \frac{1}{c} \int d^3x' dt' \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c) J^\mu(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \\
&= e \int d^3x' dt'\, \beta^\mu(t') \frac{\delta(\mathbf{x}' - \mathbf{x}(t'))\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} \\
&= e \int dt'\, \beta^\mu(t') \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}(t')|/c)}{|\mathbf{x} - \mathbf{x}(t')|} \qquad (3)
\end{aligned}$$

This form reflects the retarded nature of the problem.



The evaluation of this integral is not totally simple because the argument of the δ-function is not a simple function of the time t'. In general when one faces an integral of this form, one invokes the rule

$$\int dt'\, f(t')\delta[g(t')] = f(t_0) \Big/ \left|\frac{dg}{dt'}\right|_{t_0} \tag{4}$$

where t_0 is the zero (there may be more than one) of g[1], i.e., $g(t_0) = 0$. Applying this to the present case, we have

$$g(t') = t' + |\mathbf{x} - \mathbf{x}(t')|/c - t = t' + [(\mathbf{x} - \mathbf{x}(t')) \cdot (\mathbf{x} - \mathbf{x}(t'))]^{1/2}/c - t, \tag{5}$$

so

$$\frac{dg}{dt'} = 1 + \frac{1}{2c}\left(-2\frac{d\mathbf{x}(t')}{dt'}\right) \cdot \frac{(\mathbf{x} - \mathbf{x}(t'))}{|\mathbf{x} - \mathbf{x}(t')|} = 1 - \frac{\mathbf{v}(t') \cdot (\mathbf{x} - \mathbf{x}(t'))}{c|\mathbf{x} - \mathbf{x}(t')|} \equiv 1 - \boldsymbol{\beta}(t') \cdot \mathbf{n}(t'). \tag{6}$$

The unit vector $\mathbf{n}$ points from the point $\mathbf{x}(t')$ on the particle's path toward the field point $\mathbf{x}$; it, and $\boldsymbol{\beta}$, must be evaluated at a time t' which is earlier than t, the time at which the field is evaluated, by some amount which is determined by solving the equation

$$t' + |\mathbf{x} - \mathbf{x}(t')|/c = t. \tag{7}$$

Combining Eqs. (3), (4), and (6) we find that the potentials are given simply by

$$A^\mu(\mathbf{x}, t) = e\left[\frac{\beta^\mu}{|\mathbf{x} - \mathbf{x}(t')|(1 - \boldsymbol{\beta} \cdot \mathbf{n})}\right]_{ret} = \left[\frac{e\beta^\mu}{R\kappa}\right]_{ret} \tag{8}$$

where $R \equiv |\mathbf{x} - \mathbf{x}(t')|$, $\kappa = 1 - \boldsymbol{\beta} \cdot \mathbf{n}$, and the subscript ret means that the quantity in brackets [...] must be evaluated at the retarded time t' determined from Eq. (7).

Our potentials, Eq. (8), are known as the *Liénard-Wieckert potentials*. Probably their most significant feature is the fact that they vary inversely as $1 - \boldsymbol{\beta} \cdot \mathbf{n}$ or κ; this factor can be very close to zero for $\mathbf{n} \parallel \boldsymbol{\beta}$ if β is close to one, i.e., for highly relativistic

[1] More generally $\delta[g(x)] = \sum_j \delta(x - x_j)/|g'(x_j)|$ where x_j are the simple zeroes of $g(x)$. If $g'(x_j) = 0$ (a complex zero), then $\delta[g(x)]$ makes no sense.

particles, meaning that there is a strong maximum in the potentials produced by a relativistic particle in the direction of the particle's velocity (at some retarded time).

We wish next to find the electromagnetic field. One may do this in a variety of ways. One is simply **carefully** to take derivatives of the Liénard-Wieckert potentials, a procedure followed in, e.g., Landau and Lifshitz' book, *The Classical Theory of Fields*. Another, much more elegant, is to find the fields in the instantaneous rest frame of the particle and to Lorentz-transform them to the frame interest. Another, not very elegant at all, and about to be employed here, is to go back to the integrals, Eq. (3), for the potentials and take derivatives of these expressions. Consider just the vector potential,

$$\mathbf{A}(\mathbf{x},t) = e\int dt'\,\frac{\boldsymbol{\beta}(t')\delta(t'+R(t')/c-t)}{R(t')}. \tag{9}$$

If we wish to find $\mathbf{B}(\mathbf{x},t)$, we have to take derivatives of R with respect to various components of $\mathbf{x}$. Consider, for example,

$$\nabla f(R) = \frac{df}{dR}\nabla R = \frac{df}{dR}\frac{\mathbf{R}}{R} = \mathbf{n}\frac{df}{dR}. \tag{10}$$

Application of this simple rule gives (since $\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$, and $\nabla\times\boldsymbol{\beta}(t')=0$ due to the lack of an $\mathbf{x}$ dependence.)

$$\begin{aligned}\mathbf{B}(\mathbf{x},t) &= \nabla\times\mathbf{A}(\mathbf{x},t) = e\int dt'\,\nabla\left(\frac{\delta(t'+R/c-t)}{R}\right)\times\boldsymbol{\beta}(t')\\ &= e\int dt'\,(\mathbf{n}\times\boldsymbol{\beta})\left[-\frac{1}{R^2}\delta(t'+R/c-t)+\frac{1}{cR}\delta'(t'+R/c-t)\right]\end{aligned} \tag{11}$$

where the prime on the delta function denotes differentiation with respect to the argument. Hence

$$\mathbf{B}(\mathbf{x},t) = e\left\{\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} + \int dt'\,\frac{\kappa}{cR}\left(\frac{\delta'(t'+R/c-t)}{\kappa}\right)(\mathbf{n}\times\boldsymbol{\beta})\right\} \tag{12}$$

Now, from Eqs. (5) and (6)

$$\frac{d(t'+R/c-t)}{dt'} = \kappa \qquad \text{or} \qquad \kappa dt' = d(t'+R/c-t) \tag{13}$$

and so

$$\begin{aligned}
\mathbf{B}(\mathbf{x},t) &= e\left\{\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} + \int d(t'+R/c-t)\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{c\kappa R}\right)\delta'(t'+R/c-t)\right\} \\
&= e\left\{\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} - \int d(t'+R/c-t)\frac{\partial[(\mathbf{n}\times\boldsymbol{\beta})/c\kappa R]}{\partial(t'+R/c-t)}\delta(t'+R/c-t)\right\} \\
&= e\left\{\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} - \left[\frac{1}{\kappa}\frac{\partial}{\partial t'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{cR\kappa}\right)\right]_{ret}\right\} \\
&= e\left\{\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\left[\frac{1}{\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R}\right)\right]_{ret}\right\}. \qquad (14)
\end{aligned}$$

The electric field can be found by similar manipulations:

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= -\nabla\Phi(\mathbf{x},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} \\
&= -e\int dt'\,\mathbf{n}\frac{d}{dR}\left(\frac{\delta(t'+R/c-t)}{R}\right) + \frac{e}{c}\int dt'\,\frac{\boldsymbol{\beta}\delta'(t'+R/c-t)}{R} \\
&= e\int dt'\left\{\mathbf{n}\frac{\delta(t'+R/c-t)}{R^2} + \frac{1}{cR}(\boldsymbol{\beta}-\mathbf{n})\delta'(t'+R/c-t)\right\} \\
&= e\left[\frac{\mathbf{n}}{\kappa R^2}\right]_{ret} - \frac{e}{c}\left[\frac{1}{\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}-\mathbf{n}}{\kappa R}\right)\right]_{ret}. \qquad (15)
\end{aligned}$$

We now have expressions for $\mathbf{E}$ and $\mathbf{B}$, but they involve time derivatives of retarded quantities. We can work out each of these derivatives. First, we consider just the derivative of $\mathbf{n} = \mathbf{R}/R$. One has

$$\frac{\partial\mathbf{R}}{\partial t'} = -\dot{\mathbf{x}}(t') = -\boldsymbol{\beta}c \qquad (16)$$

and

$$\frac{\partial R}{\partial t'} = \frac{\partial[(\mathbf{x}-\mathbf{x}(t'))\cdot(\mathbf{x}-\mathbf{x}(t'))]^{1/2}}{\partial t'} = \frac{1}{2R}[-2(\mathbf{x}-\mathbf{x}(t'))\cdot\dot{\mathbf{x}}(t')] = -\mathbf{n}\cdot\dot{\mathbf{x}}(t') = -\mathbf{n}\cdot\boldsymbol{\beta}c \qquad (17)$$

Hence,

$$\frac{1}{c}\frac{d\mathbf{n}}{dt'} = \frac{1}{cR}\frac{\partial\mathbf{R}}{\partial t'} - \frac{1}{cR^2}\mathbf{R}\frac{\partial R}{\partial t'} = -\frac{1}{R}\left[\boldsymbol{\beta}-(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{n}\right]. \qquad (18)$$

The quantity in brackets is can be written more concisely:

$$(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{n}-\boldsymbol{\beta} = (\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{n}-(\mathbf{n}\cdot\mathbf{n})\boldsymbol{\beta} = \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}); \qquad (19)$$

hence we find

$$\frac{1}{c}\frac{d\mathbf{n}}{dt'} = \frac{1}{R}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}). \tag{20}$$

Employing this useful result we have, for the electric field,

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{\kappa^2 R^2}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) + \frac{\mathbf{n}}{c\kappa}\frac{\partial}{\partial t'}\left(\frac{1}{\kappa R}\right) - \frac{1}{c\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret} \\
&= e\left[\frac{\mathbf{n}(1-\mathbf{n}\cdot\boldsymbol{\beta})}{\kappa^2 R^2} + \frac{1}{\kappa^2 R^2}[(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{n}-\boldsymbol{\beta}]\right]_{ret} \\
&\quad + \left[\frac{\mathbf{n}}{c\kappa}\frac{\partial}{\partial t'}\left(\frac{1}{\kappa R}\right) - \frac{1}{c\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}
\end{aligned} \tag{21}$$

or

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa^2 R^2} + \frac{\mathbf{n}}{c\kappa}\frac{\partial}{\partial t'}\left(\frac{1}{\kappa R}\right) - \frac{1}{c\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}. \tag{22}$$

This expression may be related to that for the magnetic induction,

$$\begin{aligned}
\mathbf{B}(\mathbf{x},t) &= e\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} + \frac{e}{c}\left[\frac{1}{\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R}\right)\right]_{ret} \\
&= e\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} + \frac{e}{c}\left[\frac{1}{\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\times\mathbf{n}\right]_{ret} \\
&\quad + e\left[\frac{\boldsymbol{\beta}}{\kappa^2 R}\times\left(\frac{1}{R}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right)\right]_{ret} \\
&= e\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa R^2}\right]_{ret} + \frac{e}{c}\left[\frac{1}{\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\times\mathbf{n}\right]_{ret} \\
&\quad + e\left[\frac{1}{\kappa^2 R^2}[\mathbf{n}\left((\mathbf{n}\times\boldsymbol{\beta})\cdot\boldsymbol{\beta}\right) - (\mathbf{n}\times\boldsymbol{\beta})(\mathbf{n}\cdot\boldsymbol{\beta})]\right]_{ret}
\end{aligned} \tag{23}$$

Next, put all terms proportional to $\mathbf{n}\times\boldsymbol{\beta}$ over the same denominator, which is $\kappa^2 R^2$, and also make use of the fact that $(\mathbf{n}\times\boldsymbol{\beta})\cdot\boldsymbol{\beta} = 0$, to find

$$\mathbf{B}(\mathbf{x},t) = e\left[\frac{\boldsymbol{\beta}\times\mathbf{n}}{\kappa^2 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{1}{\kappa}\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{\kappa R}\right)\times\mathbf{n}\right]_{ret} \tag{24}$$

This is our final expression. Comparing it with Eq. (22) for $\mathbf{E}(\mathbf{x},t)$, we can see that

$$\mathbf{B}(\mathbf{x},t) = [\mathbf{n}]_{ret}\times\mathbf{E}(\mathbf{x},t). \tag{25}$$

Thus we can find $\mathbf{B}(\mathbf{x}, t)$ quite easily provided we can find $\mathbf{E}(\mathbf{x}, t)$. Henceforth, we won't spend more time on $\mathbf{B}$ but will examine only $\mathbf{E}$ from which $\mathbf{B}$ then follows trivially.

To have an explicit expression for $\mathbf{E}$ with no time derivatives, we need to evaluate

$$
\begin{aligned}
\frac{\partial}{\partial t'}\left(\frac{1}{\kappa R}\right) &= -\frac{1}{\kappa^2 R^2}\left[\kappa(-\mathbf{n}\cdot\boldsymbol{\beta}c) + R\left(-\mathbf{n}\cdot\frac{\partial\boldsymbol{\beta}}{\partial t'}\right) - R\boldsymbol{\beta}\cdot[(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{n}-\boldsymbol{\beta}]\frac{c}{R}\right] \\
&= -\frac{c}{\kappa^2 R^2}\left[(1-\mathbf{n}\cdot\boldsymbol{\beta})(-\mathbf{n}\cdot\boldsymbol{\beta}) - \frac{R}{c}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) + \boldsymbol{\beta}\cdot[\boldsymbol{\beta}-(\mathbf{n}\cdot\beta)\mathbf{n}]\right] \\
&= -\frac{c}{\kappa^2 R^2}\left[\beta^2 - \mathbf{n}\cdot\boldsymbol{\beta} - \frac{R}{c}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\right] \qquad (26)
\end{aligned}
$$

and so

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}, t) &= e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa^2 R^2} + \frac{\mathbf{n}}{c\kappa}\left(\frac{-c}{\kappa^2 R^2}\right)\left(\beta^2 - \mathbf{n}\cdot\boldsymbol{\beta} - \frac{R}{c}\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\right) - \frac{\dot{\boldsymbol{\beta}}}{c\kappa^2 R}\right]_{ret} \\
&\quad -e\left[\frac{\boldsymbol{\beta}}{c\kappa}\left(\frac{-c}{\kappa^2 R^2}\right)\left(\beta^2 - \mathbf{n}\cdot\boldsymbol{\beta} - \frac{R}{c}\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\right)\right]_{ret} \\
&= e\left[\frac{1}{\kappa^3 R^2}\mathbf{n}\left(1-\beta^2+\frac{R}{c}\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\right) + \frac{\boldsymbol{\beta}}{\kappa^3 R^2}\left(-1+\beta^2-\frac{R}{c}\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\right) - \frac{1}{c\kappa^2 R}\dot{\boldsymbol{\beta}}\right]_{ret} \\
&= e\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{ret} + e\left[\frac{1}{c\kappa^3 R}\left((\mathbf{n}-\boldsymbol{\beta})(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) - (1-\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}}\right)\right]_{ret} \qquad (27)
\end{aligned}
$$

The second bracket in the final expression contains the quantity

$$
(\mathbf{n}-\boldsymbol{\beta})(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) - \mathbf{n}\cdot(\mathbf{n}-\boldsymbol{\beta})\dot{\boldsymbol{\beta}} = \mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}], \qquad (28)
$$

so

$$
\mathbf{E}(\mathbf{x}, t) = e\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\kappa^3 R}\right]_{ret}. \qquad (29)
$$

If there is no acceleration, then $\dot{\boldsymbol{\beta}} = 0$ and only the first term in $\mathbf{E}$ and the corresponding term in $\mathbf{B}$ are finite. These terms fall off with distance as $1/R^2$, and hence cannot give rise to a net flux of radiation to infinity. If there is an acceleration, then the second term of $\mathbf{E}$, and the corresponding term in $\mathbf{B}$ are finite. These fall off as $1/R$, and hence will give rise to radiation, meaning that the charged particle will emit radiation only if it is accelerated. Hence the two terms are interpreted as the non-radiation and radiation parts respectively.

From these results when $\dot{\boldsymbol{\beta}} = 0$, we should be able to recover the results for the fields of a uniformly moving charge which we derived in chapter 11 (Jackson Eq. 11.152). With the particle moving along the x-direction with a constant velocity v, the fields felt by an observer a distance b away are

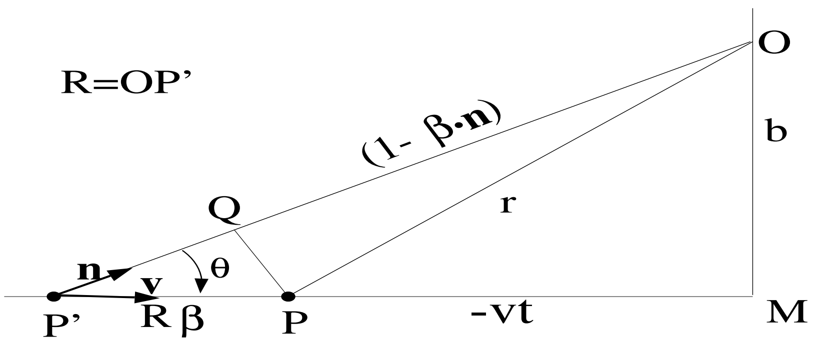
$$\begin{aligned} E_{\|} &= -\gamma q v t/[b^2 + (\gamma v t)^2]^{3/2} \\ E_{\perp} &= \gamma q b/[b^2 + (\gamma v t)^2]^{3/2}. \end{aligned} \tag{30}$$

where the observer and particle are closest at time $t = 0$. It is a reasonably simple task to show that this is the same result as Eq. (29) above when $\dot{\boldsymbol{\beta}} = 0$. Consider the diagram below in which the path of a charged particle is along the abscissa. At time t, the retarded and real location of the particle are P' and P respectively, while O is the observation point. The time required for light to travel from P' to the observer O is $t = R/c$, in which time the particle travels βR =PP'. Thus the distance P'Q is $\beta R \cos\theta = \boldsymbol{\beta} \cdot \mathbf{n} R$. From this it follows that the distance OQ is $R - \boldsymbol{\beta} \cdot \mathbf{n} R$. Then $[(1 - \boldsymbol{\beta} \cdot \mathbf{n})R]^2 = r^2 - (PQ)^2 = r^2 - \beta^2 R^2 \sin^2(\theta)$. Thus, as $R \sin\theta = b$

$$[(1 - \boldsymbol{\beta} \cdot \mathbf{n})R]^2 = b^2 + v^2 t^2 - \beta^2 b^2 = \frac{1}{\gamma^2}\left(b^2 + \gamma^2 v^2 t^2\right) , \tag{31}$$

and transverse component of Eq. (29) when $\dot{\boldsymbol{\beta}} = 0$ is

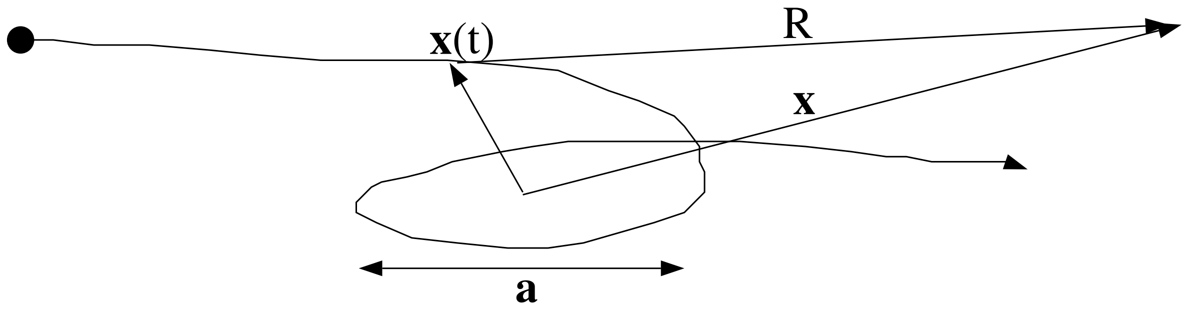
$$e\left[\frac{(R\sin\theta)}{\gamma^2 (1 - \boldsymbol{\beta} \cdot \mathbf{n})^3 R^3}\right]_{ret} = \frac{eb\gamma}{[b^2 + (\gamma v t)^2]^{3/2}} \tag{32}$$



2 Radiation from an Accelerated Charge; the Larmor Formula

Even given Eqs. (25) and (29), the problem of calculating the fields emitted from a charge moving along an arbitrary trajectory is non-trivial. This is largely due to the effect of retardation. The problem is greatly simplified if the particle is not moving too fast.

In order to consider this limit, let's consider the trajectory of a particle shown below.



We will assume that the origin is located in the center of the region of interest of the particles trajectory, and that this region of interest is of linear dimension a (an example would be an electron bound to a classical atom, where a would be the Bohr radius).

There are two ways in which the problem simplifies in the nonrelativistic limit, $\beta << 1$. First, we may approximate

$$\kappa \approx 1 \qquad \mathbf{n} - \boldsymbol{\beta} \approx \mathbf{n} \qquad 1 - \beta^2 \approx 1 \,. \tag{33}$$

Second, and much more importantly, we can approximate functions of the retarded time

$$f(t - |\mathbf{x} - \mathbf{x}(t')|/c) \tag{34}$$

by making a Taylor series expansion around the origin

$$f(t - |\mathbf{x} - \mathbf{x}(t')|/c) \approx f(t - r/c) + \mathbf{x}(t') \cdot \nabla f(t - r/c) + \cdots \tag{35}$$

where $r = |\mathbf{x}|$. The ratio of the second term to the first is roughly $f' x(t')/fc$, where the prime means differentiation of f with respect to $t - r/c$. If this is small, as it must be for $\beta \ll 1$, then the second term in the series may be neglected, then

$$f(t - |\mathbf{x} - \mathbf{x}(t')|/c) \approx f(t - r/c)\,. \tag{36}$$

This approximation is sometimes called the dipole approximation (we will see why directly). Here the effects of retardation are simple but not insignificant. They are simply due to the separation of the charge from the observer, not due to the motion of the charge. Thus, the only way they are significant is if the acceleration of the charge changes abruptly and it takes a while for the emitted fields to reach the observer. [2]

With the approximations described above, the electric field becomes

$$\mathbf{E}_{nr}(\mathbf{x}, t) = e\left[\frac{\mathbf{n}}{R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}})}{R}\right]_{ret}. \tag{37}$$

The Poynting vector is given by

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B}) = \frac{c}{4\pi}\mathbf{E} \times ([\mathbf{n}]_{ret} \times \mathbf{E}) \\ &= \frac{c}{4\pi}\left\{[\mathbf{n}]_{ret}\mathbf{E} \cdot \mathbf{E} - \mathbf{E}([\mathbf{n}]_{ret} \cdot \mathbf{E})\right\}. \end{aligned} \tag{38}$$

If we take the limit of large R and keep just the radiation terms, then we find that

$$\mathbf{E}(\mathbf{x}, t) = \frac{e}{c}\left[\frac{\mathbf{n}(\mathbf{n} \cdot \dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}}{R}\right]_{ret} \;;\; \mathbf{B}(\mathbf{x}, t) = [\mathbf{n}]_{ret} \times \mathbf{E} = \left[-\frac{e}{cR}\mathbf{n} \times \dot{\boldsymbol{\beta}}\right]_{ret} \tag{39}$$

Note that $\mathbf{n} \cdot \mathbf{E} = \mathbf{n} \cdot \mathbf{B} = 0$ for these radiation fields, thus (as usual) the radiation is transverse. The Poynting vector due to the radiation is then given by

$$\mathbf{S} = \frac{ce^2}{4\pi R^2 c^2}\left[\mathbf{n}\left|\mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}})\right|^2\right]_{ret} \tag{40}$$

[2]For example, suppose a particle starts to oscillate back and forth at time $t = 0$, there will be no signal felt at the observer's location $\mathbf{x}$ for what may be a very long time $t = r/c$.

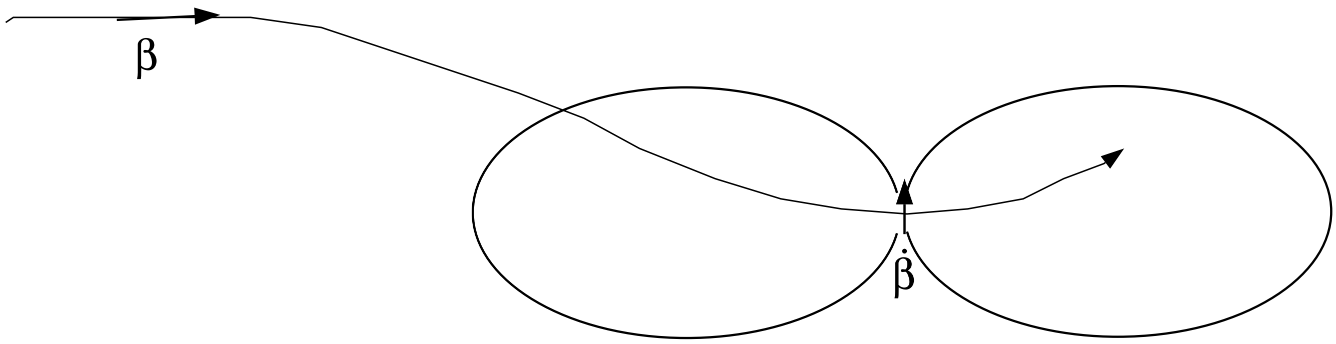
The angular distribution of radiated power is

$$\frac{d\mathcal{P}}{d\Omega} = R^2(\mathbf{S}\cdot\mathbf{n}) \tag{41}$$

which is easily evaluated by expanding the cross products. The result is

$$\frac{d\mathcal{P}}{d\Omega} = \frac{e^2}{4\pi c}(\dot{\beta}^2\sin^2\theta) = \frac{e^2}{4\pi c^3}(\dot{\mathbf{v}}^2\sin^2\theta) \tag{42}$$

where θ is the angle between $\mathbf{n}$ and $\dot{\boldsymbol{\beta}}$ (at the retarded time).



In these last expressions we have dropped the reminder that $\mathbf{n}$ and R must be evaluated at the retarded time.

The radiation pattern is characteristic of dipole radiation with the reference z axis parallel to the direction of the particle's acceleration. The shape and magnitude of the distribution are independent of the particle's velocity and proportional to the square of the acceleration. The total power radiated is the integral over all directions of the distribution,

$$\mathcal{P} = \int d\Omega \frac{d\mathcal{P}}{d\Omega} = \frac{2}{3}\frac{e^2\dot{\mathbf{v}}^2}{c^3} \tag{43}$$

which is known as the *Larmor formula* for the power radiated by an accelerated particle.

2.1 Relativistic Larmor Formula

We turn now to the relativistic generalization of the Larmor formula. One can determine this generalization by consideration of how power transforms under Lorentz transformations. There are actually two ways (that I know of) to do this, of which Jackson does one. One can also find it from direct computation and that is what we shall do. The first step is to consider just how we shall define the power radiated by a particle. The point is that the rate at which energy crosses a closed surface surrounding the particle depends on that surface because of retardation. We are going to calculate the power as a function not of the time t at which the fields are measured on the surface but rather as a function of the retarded time t'. Consider a surface S which encloses the particle at all times during which it is radiating. The power crossing unit area at $\mathbf{x}$ on S at time t is $\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}$, where $\mathbf{n}$ is a unit outward normal, and so the total energy crossing this unit area is

$$W = \int_{-\infty}^{\infty} dt\, \mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}. \tag{44}$$

Now let us transform to the retarded time t':

$$W = \int_{-\infty}^{\infty} dt' \,\frac{dt}{dt'}\mathbf{S}(\mathbf{x},t(t'))\cdot\mathbf{n} = \int_{-\infty}^{\infty} dt'\, \kappa[\mathbf{S}(\mathbf{x},t(t'))\cdot\mathbf{n}]. \tag{45}$$

The integrand of this expression we identify as dW/dt', the rate at which the particle radiates what eventually passes through the unit area on S at $\mathbf{x}$. This is the *instantaneous* radiated power, and if we multiply it by R^2 we get $d\mathcal{P}(t')/d\Omega$:

$$\frac{d\mathcal{P}(t')}{d\Omega} = \kappa R^2 \mathbf{S}\cdot\mathbf{n}. \tag{46}$$

If we suppose that R is large enough that only the radiation fields need be retained, then we find, from our results for the fields and the definition of the Poynting vector, that

$$\frac{d\mathcal{P}(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}})]^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}\bigg|_{t'} \tag{47}$$

Of course, if one wants to know the radiated power per unit area at the time that it actually gets there, then he will have to address the calculation of the retardation. In other words, the radiated intensity at time t depends upon the behavior of the particle at t', and the differential time elements are different as well: $dt = dt'(1 - \mathbf{n} \cdot \boldsymbol{\beta})_{ret}$. For example, if a particle of velocity $\boldsymbol{\beta}$ is impulsively accelerated for a time τ, and then brought to rest, a pulse of radiation will appear at the observer at time $t = r/c$, of duration $\tau(1 - \mathbf{n} \cdot \boldsymbol{\beta})_{ret}$. The total energy lost of the particle, of course, must eventually equal the energy radiated, but the energy lost by the charge per unit time will differ from the energy radiated per unit time by the factor $(1 - \mathbf{n} \cdot \boldsymbol{\beta})_{ret}$

However, by calculating the instantaneous power radiated, we have avoided having to calculate the retardation. Thus we can proceed with a calculation of the energy lost. Our result, Eq. (47), depends in a complicated way on both $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ making it possible for rather remarkable angular distributions to occur. The total power radiated can also be computed by integrating the distribution over directions:

$$\begin{aligned}
\mathcal{P}(t') &= \frac{e^2}{4\pi c}\int \frac{d\Omega}{\kappa^5}[\mathbf{n} \times ((\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})]^2 \\
&= \frac{e^2}{4\pi c}\int \frac{d\Omega}{\kappa^5}\left\{\left[(\mathbf{n} - \boldsymbol{\beta})(\mathbf{n} \cdot \dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}(1 - \boldsymbol{\beta} \cdot \mathbf{n})\right]^2\right\} \\
&= \frac{e^2}{4\pi c}\int \frac{d\Omega}{\kappa^5}\left\{(1 - 2\mathbf{n} \cdot \boldsymbol{\beta} + \beta^2)(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})^2\right. \\
&- \left. 2(\mathbf{n} \cdot \dot{\boldsymbol{\beta}} - \boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})(\dot{\boldsymbol{\beta}} \cdot \mathbf{n})(1 - \mathbf{n} \cdot \boldsymbol{\beta}) + \dot{\beta}^2(1 - 2\boldsymbol{\beta} \cdot \mathbf{n} + (\boldsymbol{\beta} \cdot \mathbf{n})^2)\right\} \qquad (48)
\end{aligned}$$

Let the angle between $\boldsymbol{\beta}$ and $\mathbf{n}$ be θ; further, let $\cos\theta = u$; also, let the angle between $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ be θ_0. Then

$$\begin{aligned}
\mathcal{P}(t') &= \frac{e^2\dot{\beta}^2}{2c}\int_{-1}^{1}\frac{du}{\kappa^5}\left\{(\beta^2 - 1)\left[u^2\cos^2\theta_0 + \frac{1}{2}(1 - u^2)\sin^2\theta_0\right]\right. \\
&\quad \left. +2\cos^2\theta_0\, u\beta(1 - \beta u) + (1 - 2\beta u + \beta^2 u^2)\right\} \\
&= \frac{\dot{\beta}^2 e^2}{2c}\int_{-1}^{1}\frac{du}{(1 - \beta u)^5}\left\{-u^2(1 - \beta^2) + 2u\beta(1 - \beta u) + 1 - 2\beta u + \beta^2 u^2\right. \\
&\quad \left. + \sin^2\theta_0\left[(1 - \beta^2)u^2 - (1 - u^2)(1 - \beta^2)/2 - 2\beta u + 2\beta^2 u^2\right]\right\} \qquad (49) \\
&= \frac{\dot{\beta}^2 e^2}{2c}\int_{-1}^{1}\frac{du}{(1 - \beta u)^5}\left\{(1 - u^2) + \sin^2\theta_0\left[\left(\frac{3}{2} + \frac{1}{2}\beta^2\right)u^2 - 2\beta u - \frac{1}{2}(1 - \beta^2)\right]\right\}.
\end{aligned}$$

Now introduce $x \equiv 1 - \beta u$; then

$$\begin{aligned}
\mathcal{P}(t') &= \frac{\dot\beta^2 e^2}{2c\beta^3}\int_{1-\beta}^{1+\beta}\frac{dx}{x^5}\Big\{(\beta^2 - 1 + 2x - x^2) \\
&\quad + \sin^2\theta_0\Big[\left(3+\beta^2\right)\left(1-2x+x^2\right)/2 - 2\beta^2(1-x) - \beta^2\left(1-\beta^2\right)/2\Big]\Big\} \\
&= \frac{\dot\beta^2 e^2}{2c\beta^3}\int_{1-\beta}^{1+\beta}\frac{dx}{x^5}\Big\{\left(\beta^2 - 1 + 2x - x^2\right) \\
&\quad + \sin^2\theta_0\Big[\left(3 - 4\beta^2 + \beta^4\right)/2 + (-2+\beta^2)x + \left(3+\beta^2\right)x^2/2\Big]\Big\} \\
&= \frac{\dot\beta^2 e^2}{2c\beta^3}\left\{\left(\frac{\beta^2-1}{4x^4} + \frac{2}{3x^3} - \frac{1}{2x^2}\right)\right. \\
&\quad \left. + \sin^2\theta_0\left(\frac{(3/2-\beta^2/2)(1-\beta^2)}{4x^4} + \frac{\beta^2-3}{3x^3} + \frac{3+\beta^2}{4x^2}\right)\right\} \\
&= \frac{\dot\beta^2 e^2\gamma^6}{c\beta^3}\Big\{\Big(-(\beta+\beta^3) + 2(3\beta+\beta^3)/3 - \beta(1-\beta^2)\Big) \\
&\quad + \sin^2\theta_0\Big[(3-\beta^2)(\beta+3\beta^3)/2 + (-3+\beta^2)(\beta+\beta^3/3) + (3+\beta^2)(\beta-\beta^3)/2\Big]\Big\} \\
&= \frac{\dot\beta^2 e^2\gamma^6}{c\beta^3}\left\{\frac{2}{3}\beta^3 - \frac{2}{3}\beta^5\sin^2\theta_0\right\} = \frac{2\dot\beta^2 e^2\gamma^6}{3c}(1 - \beta^2\sin^2\theta_0). \qquad (50)
\end{aligned}$$

Putting the result back in terms of vectors and their products, we have

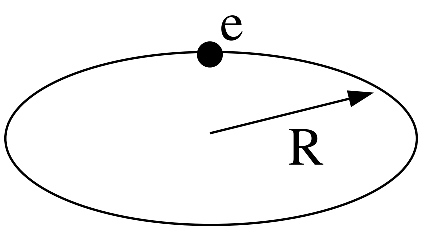
$$\mathcal{P} = \frac{2e^2\gamma^6}{3c}[\dot\beta^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2]. \qquad (51)$$

This is the relativistic generalization of the Larmor formula.

It is instructive to look at some simple examples.

2.1.1 Example: Synchrotron

An electron moves in a circle of radius R at constant speed.



The acceleration is then entirely centripetal and has magnitude $\dot{\beta} = c\beta^2/R$. Then

$$\dot{\beta}^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2 = \frac{c^2\beta^4}{R^2}(1-\beta^2) = \frac{c^2\beta^4}{R^2\gamma^2}, \tag{52}$$

and so

$$\mathcal{P} = \frac{2e^2c\gamma^4\beta^4}{3R^2} \tag{53}$$

which can be rewritten in terms of the particle's kinetic energy as

$$\mathcal{P} = \frac{2e^2c}{3R^2}\left(\frac{E}{mc^2}\right)^4\beta^4. \tag{54}$$

The energy emitted per cycle of the motion is $\Delta E = \mathcal{P}\tau$ where τ is the period of the motion, $\tau = 2\pi R/\beta c$. Hence,

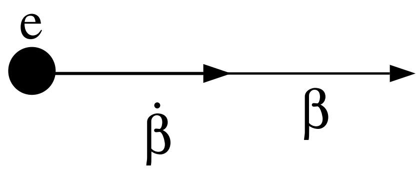
$$\Delta E = \frac{4\pi e^2\beta^3}{3R}\left(\frac{E}{mc^2}\right)^4. \tag{55}$$

An electron of energy $E = 500\,Mev$ in a synchrotron of radius $R = 10^2\,cm$ will radiate in each cycle and energy $\Delta E \approx 10^{-8}\,erg \sim 10^4\,ev$ which is a non-trivial amount if one wants to increase the electron's energy or even to maintain it. Basically, one must apply an accelerating voltage of at least ten thousand volts during each cycle to break even, i.e., to maintain the electron's energy. This radiation is the reason why circular very-high-energy electron accelerators don't exist; however, the synchrotron is a great Xray source.

But if one wants to produce high-energy protons in a circular accelerator, that is a lot easier, especially if one is willing to make R rather large. For example, a $10\,Tev$ proton in a machine of radius $R = 3 \times 10^6\,cm$, which is about nineteen miles, radiates away considerably less than $10^4\,ev$ in one cycle.

2.1.2 Example: Linear Acceleration

An electron is accelerated in the direction of its velocity, $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$.



In this instance one trivially finds from Eq. (51) that

$$\mathcal{P} = \frac{2e^2\gamma^6}{3c}\dot{\beta}^2. \tag{56}$$

This doesn't look very encouraging (for an accelerator design) because of the factor of γ^6, but against this one has the fact that the only acceleration the particle feels is the one produced by the fields acting to increase the particle's kinetic energy; in the case of the round accelerator, there is a large acceleration even for a particle of constant energy. Thus, in the case of a linear accelerator, there is much less acceleration to produce radiation. To make this point more clearly, let's write the acceleration in terms of the time rate of change of the particle's momentum,

$$\dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{1}{m\gamma^3}\frac{d\mathbf{p}}{dt} \tag{57}$$

and so

$$\mathcal{P} = \frac{2e^2}{3m^2c^3}\left(\frac{d\mathbf{p}}{dt}\right)^2. \tag{58}$$

Now relate the rate of change of momentum to the rate of change of energy of the particle,

$$\frac{dp}{dt} = F = \frac{dE}{dx}, \tag{59}$$

and so

$$\mathcal{P} = \frac{2e^2}{3m^2c^3}\left(\frac{dE}{dx}\right)^2 \tag{60}$$

or

$$\frac{\mathcal{P}}{dE/dt} = \frac{2e^2}{3m^2c^3}\frac{dE/dx}{dx/dt} = \frac{2}{3\beta}\frac{(e^2/mc^2)}{mc^2}\frac{dE}{dx}, \tag{61}$$

which says that the power radiated away is quite negligible in comparison with the rate at which energy is being pumped into the particle unless one pumps the energy in at a rate (in space) dE/dx comparable to the rest energy of the particle, mc^2, in a distance $r_c = e^2/mc^2$. This distance is rather small, being about $3 \times 10^{-13}\,cm$ for an electron, and it is difficult to put so much energy into the particle in such a small distance ($500,000\,volts$ in a distance of $10^{-13}\,cm$!). Hence particles in linear accelerators lose an insignificant amount of energy to radiation. The difficulty with such accelerators is that, like all things excepting round ones, they must sooner or later end.

3 Angular distribution of radiation

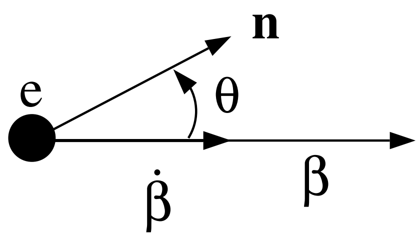
In this section we consider the angular distribution of the radiation emitted by an accelerated charge. On the basis of what we said about the potentials, we expect to find that it is strongly focussed in the forward direction, or parallel to the velocity, in a frame where the particle has a speed close to c. As at the end of the preceding section, we shall consider some particular examples. The basic equation is

$$\frac{d\mathcal{P}}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left\{\mathbf{n} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right\}^2}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^5}; \tag{62}$$

the right-hand side of this expression is to be evaluated at the retarded time t'.

3.1 Example: Parallel acceleration and velocity

Let a particle have parallel velocity and acceleration. If these define the z direction, and $\mathbf{n}$ points at an angle θ to the z axis,

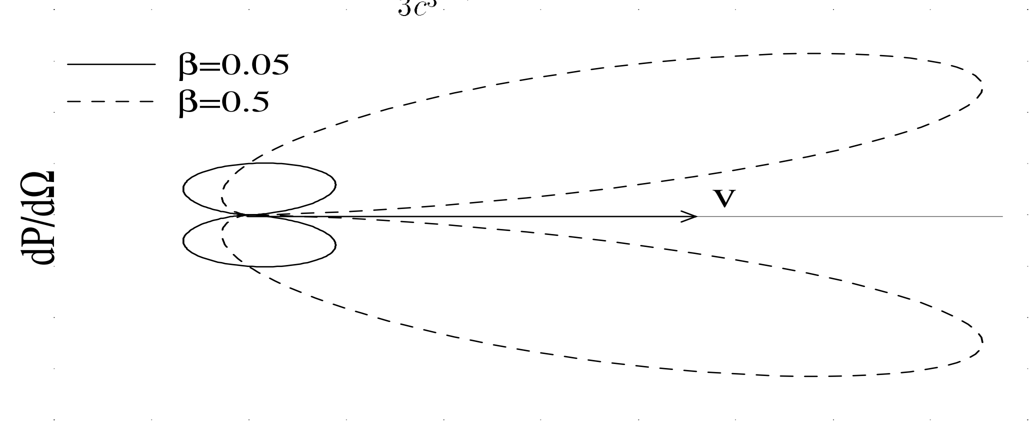


then one finds

$$\frac{d\mathcal{P}}{d\Omega} = \frac{e^2\dot{\beta}^2}{4\pi c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = \frac{e^2\dot{\mathbf{v}}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}, \tag{63}$$

and the total power is

$$\mathcal{P} = \frac{2e^2\dot{\mathbf{v}}^2}{3c^3}\gamma^6$$



For β close to one, this expression provides dramatic confirmation of the inadequacy of the multipole expansion for describing radiation from accelerated charges. This radiation pattern is sharply peaked close to the z axis and is actually zero precisely on the axis. One would have to keep many multipole terms to produce such a distribution.

We can find various basic properties which characterize the distribution. It has a maximum at a value of θ or of $u \equiv \cos\theta$ which we can find by differentiating:

$$0 = \frac{d}{du}\left(\frac{1-u^2}{(1-\beta u)^5}\right) = -\frac{2u}{(1-\beta u)^5} + \frac{5\beta(1-u^2)}{(1-\beta u)^6}, \tag{64}$$

or

$$-2u(1-\beta u) + 5\beta(1-u^2) = 0. \tag{65}$$

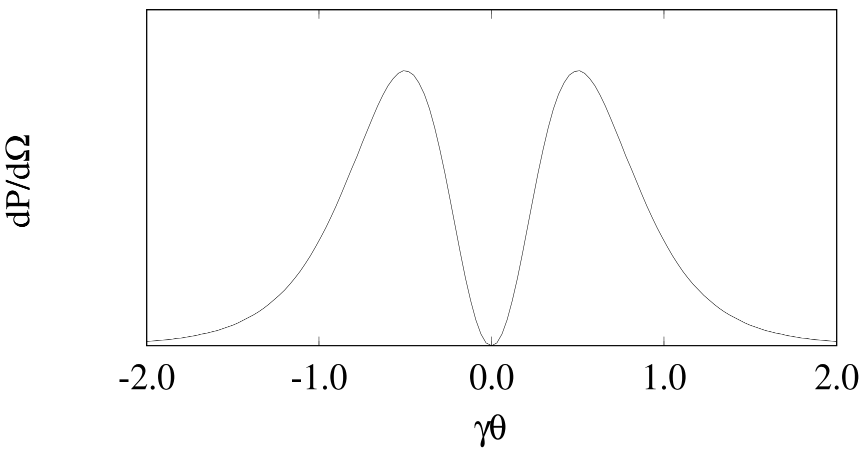
This quadratic equation has the solution

$$u = \frac{1}{3\beta}[\sqrt{1+15\beta^2} - 1] \qquad \text{or} \qquad \theta_m = \arccos\left[\frac{1}{3\beta}(\sqrt{1+15\beta^2} - 1)\right]. \tag{66}$$

Further, $\beta^2 = 1 - 1/\gamma^2$, so for β close to one, $\beta \approx 1 - 1/2\gamma^2$ and

$$\begin{aligned} \theta_m &\approx \arccos\left\{\frac{1}{3}\left(1+\frac{1}{2\gamma^2}\right)\left[4\left(1-\frac{15}{32\gamma^2}\right)-1\right]\right\} \\ &\approx \arccos\left(1-\frac{1}{8\gamma^2}\right) \approx \frac{1}{2\gamma}. \end{aligned} \tag{67}$$

Also, the power distribution becomes, in the relativistic limit,

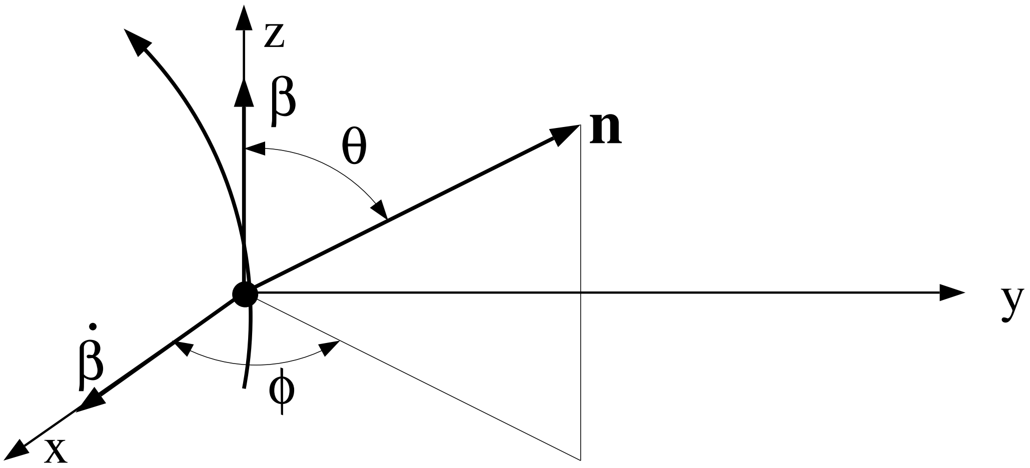


$$\begin{aligned} \frac{d\mathcal{P}}{d\Omega} &= \frac{e^2\dot{v}^2}{4\pi c^3}\frac{\theta^2}{(1-\beta+\beta\theta^2/2)^5} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{32\theta^2}{[2(1-\beta)+\beta\theta^2]^5} \\ &= \frac{e^2\dot{v}^2}{4\pi c^3}\frac{32\gamma^5\theta^2}{(1+\gamma^2\theta^2)^5} = \frac{8e^2\dot{v}^2}{\pi c^3}\gamma^3\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^5} \end{aligned} \tag{68}$$

This may be integrated to find the rms angle

$$< \theta^2 >= \frac{1}{\gamma} = \frac{mc^2}{E} \tag{69}$$

3.2 Example: Acceleration Perpendicular to Velocity



A particle in instantaneously circular motion has its acceleration perpendicular to its velocity. Let $\boldsymbol{\beta} = \beta \boldsymbol{\epsilon_3}$ and $\dot{\boldsymbol{\beta}} = \dot{\beta}\boldsymbol{\epsilon_1}$. Then, letting θ and ϕ specify the direction from the particle (at time t') to the observation point $\mathbf{x}$, we have

$$\boldsymbol{\epsilon_3} = \cos\theta\,\mathbf{n} - \sin\theta\,\boldsymbol{\theta} \qquad \text{and} \qquad \boldsymbol{\epsilon_1} = \sin\theta\ \cos\phi\,\mathbf{n} + \cos\theta\ \cos\phi\,\boldsymbol{\theta} - \sin\phi\,\boldsymbol{\phi}. \tag{70}$$

Using these conventions we can work out the relevant vector products:

$$(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} = \dot{\beta}[\cos\theta\ \cos\phi\,\boldsymbol{\phi} + \sin\phi\,\boldsymbol{\theta} - \boldsymbol{\beta}(\sin\theta\sin\phi\mathbf{n} + \cos\theta\sin\phi\boldsymbol{\theta} + \cos\phi\boldsymbol{\phi})] \tag{71}$$

and so

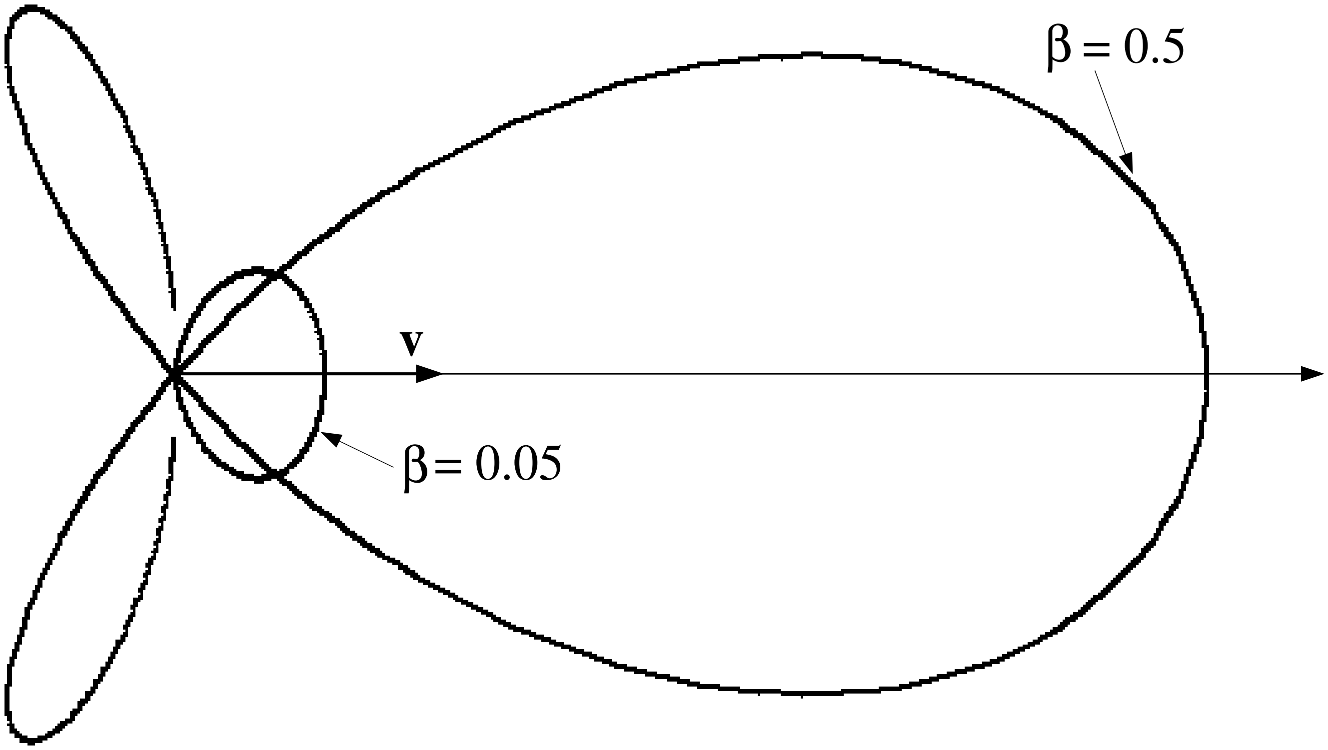
$$\begin{aligned}
[\mathbf{n} \times ((\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})]^2 &= \dot{\beta}^2[\cos^2\phi(\beta - \cos\theta)^2 + \sin^2\phi(1 - \beta\cos\theta)^2] \\
&= \dot{\beta}^2[\beta^2\cos^2\phi - 2\beta\cos\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi \\
&\quad -2\beta\cos\theta\sin^2\phi + \beta^2\cos^2\theta\sin^2\phi] \\
&= \dot{\beta}^2[\cos^2\phi(\beta^2 + \cos^2\theta - 1 - \beta^2\cos^2\theta) + (1 - \beta\cos\theta)^2] \qquad (72)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d\mathcal{P}}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}&\left\{1 - \frac{\cos^2\phi(1-\beta^2)\sin^2\theta}{(1-\beta\cos\theta)^2}\right\}\frac{1}{(1-\beta\cos\theta)^3} \\
&= \frac{e^2\dot{v}^2}{4\pi c^3(1-\beta\cos\theta)^3}\left\{1 - \frac{\cos^2\phi\sin^2\theta}{\gamma^2(1-\beta\cos\theta)^2}\right\}. \qquad (73)
\end{aligned}$$

In this case we find that there is radiation in the forward direction, or $\theta = 0$. In fact the main peak in the radiation is in this direction. If one makes a small angle expansion of the power distribution in the case of highly relativistic particles, he will find the result

$$\frac{d\mathcal{P}}{d\Omega} = \frac{2e^2\dot{v}^2}{\pi c^3}\frac{\gamma^6}{(1+\gamma^2\theta^2)^3}\left\{1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right\}. \tag{74}$$



Further, the total radiated power is

$$\mathcal{P} = \frac{2e^2\dot{v}^2}{4\pi c^3}\gamma^4. \tag{75}$$

3.3 Comparison of Examples

From the expressions for the total power produced in each of our two examples, we see that for a given magnitude of acceleration, there is a factor of γ^2 more radiation produced when the acceleration is parallel to the velocity than when it is perpendicular. This is a misleading statement in some sense because what is actually applied to a particle is not an acceleration but a force, and a given force will produce quite different accelerations when applied perpendicular and parallel to the velocity. For a force applied parallel to the velocity,

$$F = \frac{dp}{dt} = m\gamma^3 \frac{dv}{dt} \tag{76}$$

and for a force applied perpendicular to $\mathbf{v}$,

$$F = \frac{dp}{dt} = m\gamma \frac{dv}{dt}. \tag{77}$$

Hence the powers produced in the two cases, expressed in terms of the forces or dp/dt, are

$$\mathcal{P}_\perp = \frac{2e^2}{3m^2c^3}\gamma^2 \left(\frac{dp}{dt}\right)^2 \qquad \text{and} \qquad \mathcal{P}_\parallel = \frac{2e^2}{3m^2c^3} \left(\frac{dp}{dt}\right)^2. \tag{78}$$

Thus, for a given applied force, γ^2 more radiation is produced if it is perpendicular to $\mathbf{v}$ *than if it is parallel.* Of course, if the force comes from a magnetic field, then it has to be perpendicular to the velocity.

3.4 Radiation of an Ultrarelativistic Charged Particle

The radiation emitted at any instant from an accelerating charged particle may be decomposed into components coming from the parallel and perpendicular accelerations of the particle. From the discussion above, it is clear that the radiation of an ultrarelativistic $\gamma \gg 1$ particle is dominated by the perpendicular acceleration component. Thus the radiation is approximately the same as that emitted by a particle moving instantaneously in a circle of radius

$$\rho = \frac{v^2}{\dot{v}_\perp} \tag{79}$$

Since the angular width of the pulse is $\sim 1/\gamma$, the particle will travel a distance

$$d \sim \rho\Delta\theta \sim \frac{\rho}{\gamma} \tag{80}$$

while illuminating the observer for a time

$$\Delta t \sim \frac{d}{v} \sim \frac{\rho}{\gamma v}. \tag{81}$$

If we assume that pattern of radiation is roughly that of a coiled beam of rectangular cross-section then the front edge of the beam will travel

$$D = c\Delta t \sim \frac{\rho c}{\gamma v} = \frac{\rho}{\gamma \beta} \tag{82}$$

in time Δt; whereas, the trailing edge will be a distance

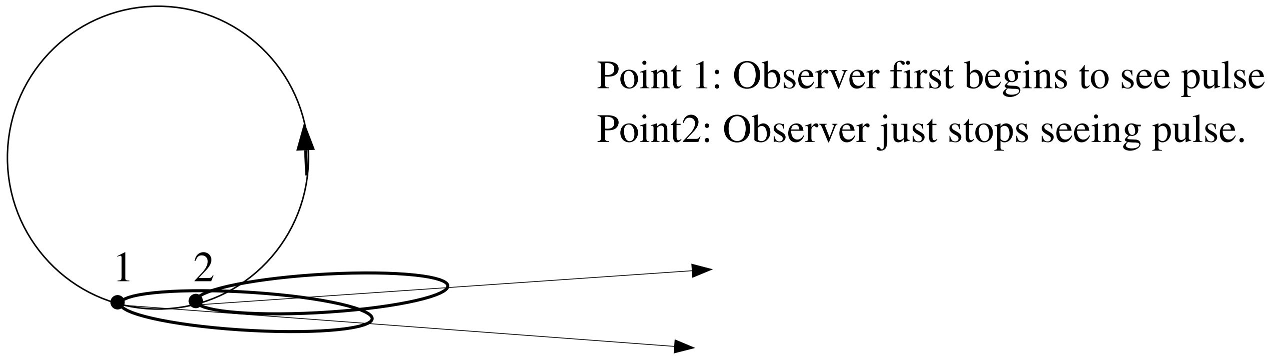
$$L = D - d \approx \left(\frac{\beta - \beta^2}{\beta^2}\right)\frac{\rho}{\gamma} \approx \left(1 - \frac{1}{2\gamma^2} - 1 + \frac{1}{\gamma^2}\right)\frac{\rho}{\gamma} = \frac{\rho}{2\gamma^3} \tag{83}$$

behind the front edge since the charge moves the distance d in the same time interval. Thus the pulse width is roughly L in space or L/c in time.

This derivation (straight out of Jackson) raises about as many questions as it answers. Perhaps the most fundamental is this: why is Δt different than the observed width of the pulse L/c. The difference is that Δt is really a difference of retarded times; whereas, L/c is in the time frame of the observer. Lets repeat this calculation more carefully, distinguishing between the observer's time and the particles (retarded) time.

The pulse width for a distant observer would be

$$\frac{L}{c} = t_2 - t_1 = (t_2' + R_2/c) - (t_1' + R_1/c) = \Delta t - (R_1 - R_2)/c\,. \tag{84}$$

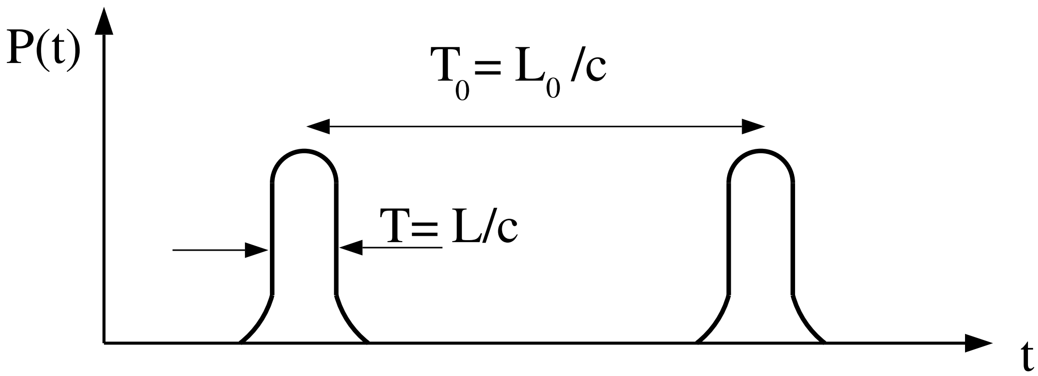


Since $R_1 - R_2 \approx v'\Delta t$, we have

$$\frac{L}{c} = t_2 - t_1 = \Delta t(1 - v/c) \approx \frac{\Delta t}{2\gamma^2} = \frac{\rho}{2\gamma^3},\,. \tag{85}$$

where the right hand side is to be evaluated in the retarded time.

An observer would receive periodic pulses of width L/c in time.



From the principles of Fourier decomposition, we can also estimate the type of radiation the observer would receive. From the uncertainty principle $\Delta t \Delta\omega \sim 1$, the pulse would contain components up to a cutoff[3] of roughly

$$\omega_c \sim \frac{c}{L} \sim \frac{c}{\rho}\gamma^3 \tag{86}$$

[3]Higher frequency components would yield a narrower pulse

However, the frequency ω_0 of the circular motion is $v/\rho \approx c/\rho$. Thus, a broad spectrum of radiation is emitted up to γ^3 times the fundamental frequency of the rotation. This kind of radiation, called *synchrotron radiation* provides a good, more or less continuous, source of radiation in the range from visible and ultraviolet to soft X-rays.

Thus in a 200 MeV electron synchrotron where $\gamma \approx 400$ and the fundamental frequency is $\omega_0 \approx 3 \times 10^8$, the frequency of the emitted radiation extends to 2×10^{16}. In a 10 GeV synchrotron, x-rays can be produced. Thus synchrotrons can be used as high intensity radiation sources. Synchrotrons are now even being used in industry as radiation sources for x-ray lithography[4].

4 Frequency Distribution of the Radiated Energy

The radiation produced by rotating charges is predominantly at the fundamental frequency ω_0 of the motion with much smaller amounts at integral multiples, or harmonics, of this frequency. The expansion parameter in the problem is $k_0 a = \omega_0 a/c \sim v/c = \beta$, and the energy emitted at higher frequencies than the fundamental is proportional to some power of this parameter. For $\beta << 1$, this energy will be relatively small. But if $\beta \sim 1$, there will be a significant fraction of the total radiated energy appearing at higher frequencies.

There is a simple and instructive way to see roughly how may harmonics will contribute to the radiation. For a relativistic particle, $d\mathcal{P}/d\Omega$ is a peaked distribution which has a width in angle $\delta\theta \sim 1/\gamma$. If the particle is, e.g., travelling in a circle with a frequency of motion ω_0, then the particle sweeps through an angle $\delta\theta$ in a time $\delta t' \sim \delta\theta/\omega_0 = 1/\gamma\omega_0$. This is of the order of the duration of the pulse observed at some fixed point in space, but measured in units of the time at the source. The time

[4] **Physics Today**, October 1991.

which passes at the location of the observer is

$$\delta t = \delta t' \frac{dt}{dt'} = \kappa \delta t'. \tag{87}$$

Further, during the pulse, $\mathbf{n}$ is parallel to $\boldsymbol{\beta}$, $\mathbf{n} \cdot \boldsymbol{\beta} = \beta$ with corrections of order $1/\gamma^2$ and so $\kappa \sim 1 - \beta[1 + \mathcal{O}(1/\gamma^2)] \sim 1/\gamma^2$. Hence $\delta t \sim 1/\gamma^3 \omega_0$. Now, a pulse which lasts a time δt at a point must contain in it frequencies of order[5] $1/\delta t$. Thus the pulse which our observer sees must have in particular frequencies of order $\omega \sim \gamma^3 \omega_0$. If the particle is highly relativistic, $\gamma >> 1$, then the typical frequencies in the pulse will be much larger than the frequency of the particle's motion, or ω_0; they will be γ^3 times as large, meaning that many harmonics must contribute to the pulse.

4.1 Continuous Frequency Distribution

To make our analysis of the radiation more quantitative we are going to consider the Fourier transforms in time of the fields at an observation point $\mathbf{x}$. Start from the expression for the angular distribution of radiated power:

$$\frac{d\mathcal{P}}{d\Omega} = \frac{c}{4\pi} \left(\mathbf{E}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t) \right) \cdot [\mathbf{n} R^2]_{ret}. \tag{88}$$

Using just the radiation fields, noting that they are transverse to $\mathbf{n}$ far away from the source, and making use the fact that $\mathbf{B}(\mathbf{x}, t) = [\mathbf{n}]_{ret} \times \mathbf{E}(\mathbf{x}, t)$, we can write

$$\frac{d\mathcal{P}}{d\Omega} = \frac{c}{4\pi} \left\{ [R]_{ret} \mathbf{E}(\mathbf{x}, t) \right\}^2. \tag{89}$$

For simplicity of notation, define

$$\mathbf{a}(t) \equiv \sqrt{\frac{c}{4\pi}} [R]_{ret} \mathbf{E}(\mathbf{x}, t). \tag{90}$$

Then, in terms of $\mathbf{a}(t)$ the angular distribution of radiated power is

$$\frac{d\mathcal{P}}{d\Omega} = [\mathbf{a}(t)]^2. \tag{91}$$

[5] In quantum theory this statement would be called the uncertainty principle.

Further, from Eq. (29),

$$\mathbf{a}(t) = \sqrt{\frac{e^2}{4\pi c}} \left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\kappa^3}\right]_{ret} . \tag{92}$$

Introduce also the Fourier transform[6] $\mathbf{a}(\omega)$ of $\mathbf{a}(t)$:

$$\mathbf{a}(t) \equiv \frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega t}\mathbf{a}(\omega); \qquad \mathbf{a}(\omega) = \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}\mathbf{a}(t). \tag{93}$$

Using the Fourier integral (and its complex conjugate) for $\mathbf{a}(t)$ in Eq. (91), we can write the power distribution as

$$\frac{d\mathcal{P}(t)}{d\Omega} = \frac{1}{2\pi}\int d\omega d\omega'\, \mathbf{a}(\omega)e^{-i\omega t}\cdot\mathbf{a}^*(\omega')e^{i\omega' t}. \tag{94}$$

Integrating over all t, and thereby generating an expression for the angular distribution of the total radiated energy, we find

$$\begin{aligned}\frac{dW}{d\Omega} &\equiv \int_{-\infty}^{\infty} dt\, \frac{d\mathcal{P}(t)}{d\Omega} = \int d\omega d\omega'\, \delta(\omega-\omega')\mathbf{a}(\omega)\cdot\mathbf{a}^*(\omega') \\ &= \int_{-\infty}^{\infty} d\omega\, |\mathbf{a}(\omega)|^2 = \int_0^{\infty} d\omega\, \left\{|\mathbf{a}(\omega)|^2 + |\mathbf{a}(-\omega)|^2\right\}.\end{aligned} \tag{95}$$

The reality of $\mathbf{a}(t)$ demands that $\mathbf{a}^*(-\omega) = \mathbf{a}(\omega)$, and so the two terms in the integrand are identical:

$$\frac{dW}{d\Omega} = 2\int_0^{\infty} d\omega\, |\mathbf{a}(\omega)|^2 \equiv \int_0^{\infty} d\omega\, \frac{dI(\omega)}{d\Omega}. \tag{96}$$

The integrand, $dI/d\Omega$, is interpreted as the total radiation received per unit frequency per unit solid angle during the entire pulse of radiation. It is simply the square of $\mathbf{a}(\omega)$; further $\mathbf{a}(\omega)$ is

$$\begin{aligned}\mathbf{a}(\omega) &= \sqrt{\frac{e^2}{4\pi c}}\frac{1}{\sqrt{2\pi}}\int dt\, e^{i\omega t}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\kappa^3}\right]_{ret} \\ &= \sqrt{\frac{e^2}{8\pi^2 c}}\int dt'\, \left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\kappa^2}\right] e^{i\omega(t'+R(t')/c)}\end{aligned} \tag{97}$$

[6]The field $\mathbf{a}(t)$ depends on $\mathbf{x}$ as well as on t; we suppress the former dependence as it is not of interest at present.

It's *déja vu* all over again. Recalling the typical equations generated in Chapter 9, we can see that this one probably can be obtained without great difficulty from what we did there. Proceeding in familiar fashion, then, let us look at this integral in the far zone which means approximate R in the exponent by

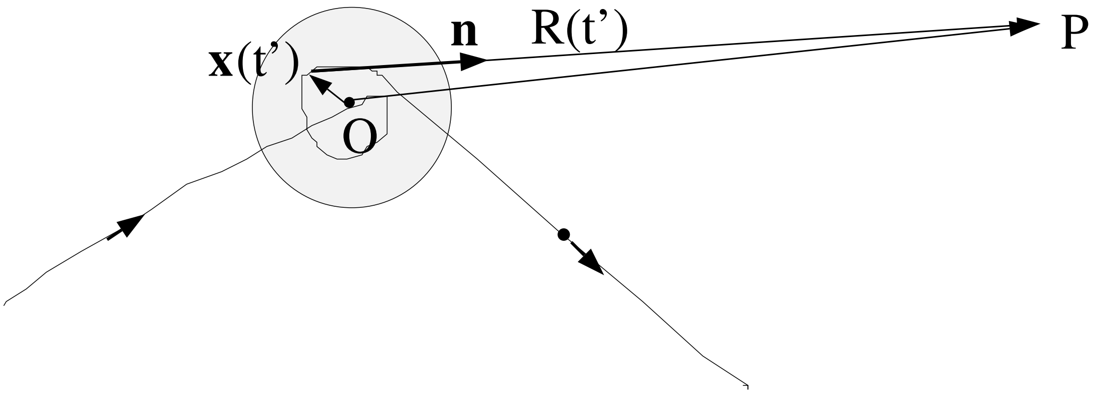
$$R = |\mathbf{x} - \mathbf{x}(t')| \approx r - \mathbf{n} \cdot \mathbf{x}(t') \tag{98}$$

and so

$$\mathbf{a}(\omega) = \sqrt{\frac{e^2}{8\pi^2 c}} \int dt'\, e^{i\omega t'} e^{i\omega r/c} e^{-i\omega \mathbf{n}\cdot\mathbf{x}(t')/c} \left[\frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{\kappa^2} \right] \tag{99}$$

or

$$\mathbf{a}(\omega) = e^{i\omega r/c} \sqrt{\frac{e^2}{8\pi^2 c}} \int dt'\, e^{i\omega(t' - \mathbf{n}\cdot\mathbf{x}(t')/c)} \left[\frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{\kappa^2} \right]. \tag{100}$$



If we confine the integration to times when the particle is accelerating, i.e. confine it to the shaded region above, then we can put this integral into a form such that the integrand involves $\boldsymbol{\beta}$ but not $\dot{\boldsymbol{\beta}}$; that can be done by, in essence, a parts integration. First, consider

$$\frac{d}{dt'} \left(\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{\kappa} \right) = -\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{\kappa^2} (-\mathbf{n} \cdot \dot{\boldsymbol{\beta}}) + \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}})}{\kappa^2} (1 - \mathbf{n} \cdot \boldsymbol{\beta}) \tag{101}$$

where we have not kept derivatives of $\mathbf{n}$ because they give corrections of relative order $|\mathbf{x}(t')/R|$. Now group terms as follows:

$$\frac{d}{dt'} \left(\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{\kappa} \right) = \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}})}{\kappa^2} + \frac{\mathbf{n} \times \left\{ \mathbf{n} \times [(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})\boldsymbol{\beta} - (\mathbf{n} \cdot \boldsymbol{\beta})\dot{\boldsymbol{\beta}}] \right\}}{\kappa^2}$$

$$
\begin{aligned}
&= \frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol\beta})}{\kappa^2} + \frac{\mathbf{n}\times\left\{\mathbf{n}\times[\mathbf{n}\times(\boldsymbol\beta\times\dot{\boldsymbol\beta})]\right\}}{\kappa^2} \\
&= \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol\beta)\times\dot{\boldsymbol\beta}]}{\kappa^2} \qquad (102)
\end{aligned}
$$

where we have used the fact that

$$
\mathbf{n}\times\left\{\mathbf{n}\times\left[\mathbf{n}\times(\boldsymbol\beta\times\dot{\boldsymbol\beta})\right]\right\} = -\mathbf{n}\times(\boldsymbol\beta\times\dot{\boldsymbol\beta}). \qquad (103)
$$

Using this identity to do a parts integration of the expression for $\mathbf{a}(\omega)$ we find (again, drop terms proportional to powers of $1/R$)

$$
\begin{aligned}
\mathbf{a}(\omega) &= -e^{i\omega r/c}\sqrt{\frac{e^2}{8\pi^2 c}}\int dt'\, i\omega(1-\mathbf{n}\cdot\boldsymbol\beta)\left(\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol\beta)}{\kappa}\right)e^{i\omega(t'-\mathbf{n}\cdot\mathbf{x}(t')/c)} \\
&= -\sqrt{\frac{e^2}{8\pi^2 c}}i\omega e^{i\omega r/c}\int dt'[\mathbf{n}\times(\mathbf{n}\times\boldsymbol\beta)]e^{i\omega(t'-\mathbf{n}\cdot\mathbf{x}(t')/c)}. \qquad (104)
\end{aligned}
$$

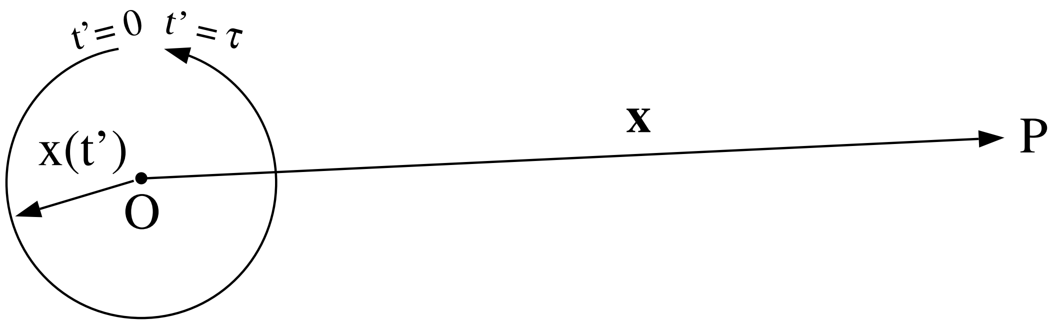
Combining this result and Eq. (96), we find for the radiated energy per unit frequency per unit solid angle

$$
\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\,[\mathbf{n}\times(\mathbf{n}\times\boldsymbol\beta)]e^{i\omega(t'-\mathbf{n}\cdot\mathbf{x}(t')/c)}\right|^2. \qquad (105)
$$

In this derivation we were not careful when doing the parts integration, meaning that we did not worry about whether the terms involving the integrand evaluated at the endpoints of the interval of integration contribute to the result. Consequently, in any application of Eq. (105), one should check to see whether this assumption is justified; there are occasions when it is not and then, naturally, the contributions from the endpoint(s) must be included. Our formulation of $dI/d\Omega$ is most appropriate for a source with an open orbit in which case the natural limits on the integration are $\pm\infty$, and there is usually no difficulty in ignoring the contributions from the endpoints where the particle is far away and unaccelerated; however, that is not enough to guarantee that the endpoints contribute nothing.

4.2 Discrete Frequency Distribution

For truly cyclic motion, it is more convenient and perhaps more sensible from a physical point of view to set up a Fourier series to express the frequency distribution of the radiation.



The point is that for non-cyclic motion, the distribution of radiation in frequency will be continuous and the integral formula we derived above is appropriate for expressing this distribution. But for cyclic motion, the radiation will be distributed in frequency space only at harmonics or multiples of the fundamental frequency of the motion and so a sum or series expansion of $dI(\omega)/d\Omega$ is more appropriate for expressing the distribution. We shall now set up this sum. To get started, suppose that the period of the motion is $\tau' = 2\pi/\omega_0$. Then, letting the period measured by an observer at a point $\mathbf{x}$ be τ, one can show that $\tau = \tau'$. That is, a time t for the observer and the corresponding retarded time t' are related by

$$t = t' + R(t')/c. \tag{106}$$

One period later in the life of the source, its time has increased to $t' + \tau'$ and the signal emitted by the source at this time will reach the observer at a time $t + \tau$ which is

$$t + \tau = t' + \tau' + R(t' + \tau')/c. \tag{107}$$

But $R(t' + \tau') = R(t')$ for the cyclic motion so

$$t + \tau = t' + \tau' + R(t')/c. \tag{108}$$

Comparing Eqs. (106) and (108), we see that $\tau = \tau'$.

The radiation fields produced by this cyclic motion of a charged particle will also be periodic. Hence where we formerly had a Fourier integral for $\mathbf{a}(t)$, we now have a Fourier series, a sum over frequencies which are integral multiples of the sources's frequency,

$$\mathbf{a}(t) = \sqrt{\frac{c}{4\pi}}[R\mathbf{E}(t)] \equiv \sum_{n=-\infty}^{\infty} \mathbf{a}_n e^{-in\omega_0 t} \tag{109}$$

with

$$\mathbf{a}_n = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dt\, \mathbf{a}(t) e^{in\omega_0 t}. \tag{110}$$

The energy received during one cycle, per unit solid angle, at some point $\mathbf{x}$ is the integral of $d\mathcal{P}/d\Omega$ over one period. We shall write it as $dW/d\Omega$,

$$\begin{aligned}\frac{dW}{d\Omega} \equiv \int_0^{2\pi/\omega_0} dt \frac{d\mathcal{P}(t)}{d\Omega} &= \int_0^{2\pi/\omega_0} dt\, |\mathbf{a}(t)|^2 \\ &= \sum_{n,m}\int_0^{2\pi/\omega_0} dt\, (\mathbf{a}_n \cdot \mathbf{a}_m^*) e^{-i(n-m)\omega_0 t} = \frac{2\pi}{\omega_0}\sum_{n=-\infty}^{\infty}|\mathbf{a}_n|^2 \\ &= \frac{4\pi}{\omega_0}\sum_{n=1}^{\infty}|\mathbf{a}_n|^2. \end{aligned} \tag{111}$$

Notice that the $n = 0$ term has been discarded in the final expression; that is okay because there is no radiation at zero frequency. The radiated energy at the frequency $n\omega_0$ is determined by $\mathbf{a}_n$, which is

$$\begin{aligned}\mathbf{a}_n &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dt \left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\kappa^3}\right]_{ret}\sqrt{\frac{e^2}{4\pi c}}e^{in\omega_0 t} \\ &= \frac{\omega_0}{2\pi}\sqrt{\frac{e^2}{4\pi c}}\int_0^{2\pi/\omega_0} dt'\, \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{\kappa^2}e^{in\omega_0(t'+R(t')/c)}. \end{aligned} \tag{112}$$

We can now do the same integration by parts that led to Eq. (104) and find

$$\mathbf{a}_n = -\frac{\omega_0}{2\pi}\sqrt{\frac{e^2}{4\pi c}}in\omega_0\int_0^{2\pi/\omega_0} dt'\, [\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})]e^{in\omega_0(t'+R(t')/c)}, \tag{113}$$

with no contribution ever from the end points of the interval as they correspond to the same point on the periodic orbit of the particle. From Eq. (113), $|\mathbf{a}_n|^2$ follows easily and hence $dW/d\Omega$ from Eq. (111). Notice also that the time-averaged power distribution is just

$$\left(\frac{d\mathcal{P}}{d\Omega}\right)_{ave} = \frac{dW}{d\Omega} \Big/ \left(\frac{2\pi}{\omega_0}\right) \tag{114}$$

4.3 Examples

4.3.1 A Particle in Instantaneous Circular Motion

Suppose we have a particle in instantaneous circular motion, meaning that its acceleration is, at least temporarily, perpendicular to its velocity. Any particle subjected to a magnetic field but no electric field will satisfy this condition. We might do the calculation by supposing that the motion is truly periodic and circular and using the relatively easily applied Fourier series approach just developed. It is more difficult and therefore more challenging to use the Fourier integral approach. Let's try the latter.

First, we want to characterize the orbit of the particle. A circular orbit at constant speed can be described as

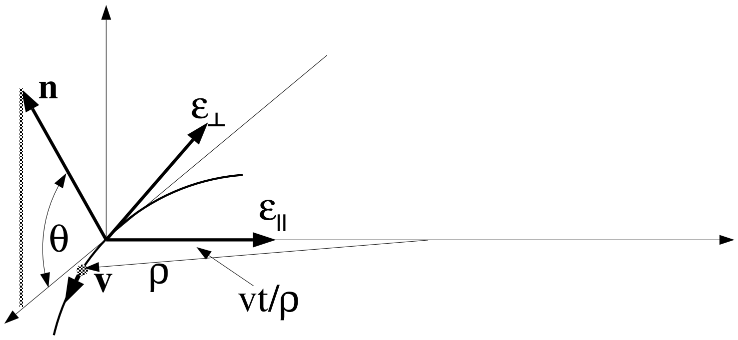
$$\boldsymbol{\beta} = \beta[(\cos\omega_0 t')\boldsymbol{\epsilon_1} + (\sin\omega_0 t')\boldsymbol{\epsilon_2}] \tag{115}$$

and, for sufficiently small times t', meaning $\omega_0 t' << 1$, we have

$$\boldsymbol{\beta} \approx \beta(\boldsymbol{\epsilon_1} + \omega_0 t'\boldsymbol{\epsilon_2}) \tag{116}$$

with corrections of order $(\omega_0 t')^2$. Let the observer be located in the x-z plane; then $\mathbf{n} = \cos\theta\,\boldsymbol{\epsilon_1} + \sin\theta\,\boldsymbol{\epsilon_3}$ where $\theta << 1$ if the observer is to experience the strong pulse of radiation that the particle emits in the forward direction. We know that the times of importance at the source for this pulse at the position of the observer are of order $t' \sim 1/\omega_0\gamma$. Consequently, in our approximation for $\boldsymbol{\beta}$, we lose corrections of relative order $(\omega_0 t')^2 \sim 1/\gamma^2$ in each of the components.

Let's work out $\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})$. Define a unit vector $\boldsymbol{\epsilon}_\perp \equiv \mathbf{n}\times\boldsymbol{\epsilon_2} = \cos\theta\,\boldsymbol{\epsilon_3} - \sin\theta\,\boldsymbol{\epsilon_1}$; it will prove to be useful.



$$\mathbf{n}\times\boldsymbol{\beta} = \beta[(\mathbf{n}\times\boldsymbol{\epsilon_1}) + \omega_0 t'(\mathbf{n}\times\boldsymbol{\epsilon_2})] = \beta(\sin\theta\,\boldsymbol{\epsilon_2} + \omega_0 t'\boldsymbol{\epsilon}_\perp); \tag{117}$$

and

$$\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) = \beta(\sin\theta\,\boldsymbol{\epsilon}_\perp - \omega_0 t'\boldsymbol{\epsilon_2}). \tag{118}$$

Further, the interesting range of θ is of order $1/\gamma << 1$, so let us replace $\sin\theta$ by θ. Further, let $\boldsymbol{\epsilon_2}$ be designated $\boldsymbol{\epsilon}_\parallel$. Then, using this equation in Eq. (105) and setting $n\omega_0 = \omega$, we find

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\,(\beta\theta\boldsymbol{\epsilon}_\perp - \omega_0 t'\boldsymbol{\epsilon}_\parallel)e^{i\omega(t'-\mathbf{n}\cdot\mathbf{x}(t')/c)}\right|^2. \tag{119}$$

We need to evaluate the exponent in order to complete the integral. Consider

$$\begin{aligned} \mathbf{n}\cdot\mathbf{x}(t') &= \cos\theta\,(\boldsymbol{\epsilon_1}\cdot\mathbf{x}(t')) = \frac{c\beta}{\omega_0}\cos\theta\,\sin\omega_0 t' \\ &\approx \frac{c\beta}{\omega_0}\left(1-\frac{\theta^2}{2}\right)\left(\omega_0 t' - \frac{1}{6}(\omega_0 t')^3\right); \end{aligned} \tag{120}$$

Notice that we have kept the leading term and corrections to it of order $1/\gamma^2$. Basically, we have kept all phases of order unity when $\omega_0 t' \sim 1/\gamma$, $\theta \sim 1/\gamma$, and $\omega \sim \omega_0\gamma^3$, these being what we believe to be the important ranges of t', ω, and θ. Hence the total phase is, to the order indicated,

$$\begin{aligned} \omega[t' - \mathbf{n}\cdot\mathbf{x}(t')/c] &= \omega t' - \frac{\omega\beta}{\omega_0}\omega_0 t'\left(1-\frac{\theta^2}{2}\right) + \frac{\omega\beta}{\omega_0}\frac{1}{6}(\omega_0 t')^3\left(1-\frac{\theta^2}{2}\right) \\ &= \omega t'\left(1-\beta+\frac{\beta\theta^2}{2}\right) + \frac{\omega\beta}{6\omega_0}(\omega_0 t')^3\left(1-\frac{\theta^2}{2}\right) \\ &\approx \omega t'\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right) + \frac{\omega}{6\omega_0}(\omega_0 t')^3 \end{aligned}$$

$$= \left(\frac{\omega}{2\omega_0}\right)\left[\omega_0 t'\left(\frac{1}{\gamma^2}+\theta^2\right)+\frac{1}{3}(\omega_0 t')^3\right]. \tag{121}$$

With the foregoing expression for the phase, we are able to write the frequency and angle distribution of the intensity as

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt'\,(\theta\boldsymbol{\epsilon}_\perp - \omega_0 t'\boldsymbol{\epsilon}_\parallel)e^{i\frac{\omega}{2\omega_0}\left[\omega_0 t'\left(\frac{1}{\gamma^2}+\theta^2\right)+\frac{1}{3}(\omega_0 t')^3\right]}\right|^2. \tag{122}$$

The integration has been extended to $\pm\infty$ even though the integrand is accurate only for $|\omega_0 t'| \sim 1/\gamma$ (or less). The extended range of integration is sensible only if there is no important contribution coming from other regimes of t'. That is the case because the term in the phase proportional to t'^3 produces rapid oscillations of the integrand at larger $|t'|$ which yield a very small net contribution to the integral.

Introduce x such that $\omega_0 t' \equiv x(1/\gamma^2+\theta^2)^{1/2}$. The important range of t' corresponds to $|x| \sim 1$ and the frequency distribution of the intensity is given by

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\frac{1}{\omega_0}\sqrt{\frac{1}{\gamma^2}+\theta^2}\int_{-\infty}^{\infty} dx\,\left[\theta\boldsymbol{\epsilon}_\perp - \sqrt{\frac{1}{\gamma^2}+\theta^2}\,x\boldsymbol{\epsilon}_\parallel\right]e^{i\frac{\omega}{2\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\left(x+\frac{1}{3}x^3\right)}\right|^2. \tag{123}$$

Let

$$\eta = \frac{\omega}{3\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}. \tag{124}$$

Then

$$\begin{aligned}\frac{dI(\omega)}{d\Omega} &= \frac{e^2\omega^2}{4\pi^2 c\,\omega_0^2}\left|\int_{-\infty}^{\infty} dx\,\left[\sqrt{\frac{1}{\gamma^2}+\theta^2}\,\theta\boldsymbol{\epsilon}_\perp + \left(\frac{1}{\gamma^2}+\theta^2\right)x\boldsymbol{\epsilon}_\parallel\right]e^{i\frac{3}{2}\eta(x+x^3/3)}\right|^2 \\ &\equiv \frac{e^2\omega^2}{4\pi^2 c\,\omega_0^2}\left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\,\theta I_\perp\boldsymbol{\epsilon}_\perp + \left(\frac{1}{\gamma^2}+\theta^2\right)I_\parallel\boldsymbol{\epsilon}_\parallel\right|^2. \end{aligned} \tag{125}$$

where

$$I_\perp = \int_{-\infty}^{\infty} dt\, e^{i(3\eta t+\eta t^3)/2} \tag{126}$$

and

$$I_\parallel = \int_{-\infty}^{\infty} dt\, t e^{i(3\eta t+\eta t^3)/2}. \tag{127}$$

These integrals may be expressed in terms of *Airy functions* which are modified Bessel functions of order 1/3 and 2/3. An integral representation[7] of the function Ai is

$$\frac{\pi}{(3a)^{1/3}} Ai[x/(3a)^{1/3}] = \int_0^\infty dt\, \cos(xt + at^3) = \frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{i(xt+at^3)}. \tag{128}$$

From this representation one can see that

$$I_\perp = \frac{2\pi}{(3\eta/2)^{1/3}} Ai[(3\eta/2)^{2/3}]. \tag{129}$$

As for $I_\parallel$, it is

$$\begin{aligned} I_\parallel &= \int_{-\infty}^{\infty} dt\, t e^{i(3\eta t+\eta t^3)/2} = \int_{-\infty}^{\infty} dt\, t e^{i(xt+\eta t^3/2)}\Big|_{x=3\eta/2} \\ &= \frac{1}{i}\frac{d}{dx}\left(\int_{-\infty}^{\infty} dt e^{i(xt+\eta t^3/2)}\right)\Big|_{x=3\eta/2} = \frac{1}{i}\frac{d}{dx}\left(\frac{2\pi}{(3\eta/2)^{1/3}} Ai[x/(3\eta/2)^{1/3}]\right)\Big|_{x=3\eta/2} \\ &= \frac{2\pi}{i(3\eta/2)^{2/3}} Ai'[(3\eta/2)^{2/3}]. \end{aligned} \tag{130}$$

The prime on the Airy function denotes differentiation with respect to the argument.

The connection between Airy functions and modified Bessel functions is[8]

$$Ai[(3\eta/2)^{2/3}] = \frac{1}{\pi}\left[\frac{(3\eta/2)^{2/3}}{3}\right]^{1/2} K_{1/3}(\eta). \tag{131}$$

Also[9],

$$-Ai'[(3\eta/2)^{2/3}] = \frac{1}{\pi}\frac{(3\eta/2)^{2/3}}{\sqrt{3}} K_{2/3}(\eta). \tag{132}$$

Thus we may express the result for the frequency distribution of intensity in terms of modified Bessel functions as

$$\begin{aligned} \frac{dI(\omega)}{d\Omega} &= \frac{e^2}{4\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2 \left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\,\theta\frac{2}{\sqrt{3}} K_{1/3}(\eta)\boldsymbol{\epsilon}_\perp + \left(\frac{1}{\gamma^2}+\theta^2\right)\frac{2}{i\sqrt{3}} K_{2/3}(\eta)\boldsymbol{\epsilon}_\parallel\right|^2 \\ &= \frac{e^2}{3\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2 \left(\frac{1}{\gamma^2}+\theta^2\right)^2 \left[K_{2/3}^2(\eta) + \frac{\theta^2}{1/\gamma^2+\theta^2} K_{1/3}^2(\eta)\right] \\ &= \frac{3e^2\gamma^2}{\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left[K_{2/3}^2(\eta) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K_{1/3}^2(\eta)\right] \end{aligned} \tag{133}$$

[7] Abramowitz and Stegun, 10.4.32.

[8] Abramowitz and Stegun, 10.4.14.

[9] Abramowitz and Stegun, 10.4.16.

with

$$\omega_c = 3\gamma^3\omega_0 \quad \text{and} \quad \eta = \frac{\omega}{3\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2} = \frac{\omega}{\omega_c}(1+\theta^2\gamma^2)^{3/2}. \tag{134}$$

This is a fairly transparent, if not simple, result. The variable η is proportional to ω and is scaled by $3\gamma^3\omega_0 \equiv \omega_c$ which we believe, on the basis of arguments given earlier, to be the appropriate or characteristic scale of frequency in this radiating system. For $\eta << 1$,

$$K_\nu(\eta) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{\eta}\right)^\nu \tag{135}$$

and for $\eta >> 1$,

$$K_\nu(\eta) \sim \sqrt{\frac{\pi}{2\eta}}e^{-\eta}. \tag{136}$$

Thus, for $\eta << 1$ and at $\theta = 0$,

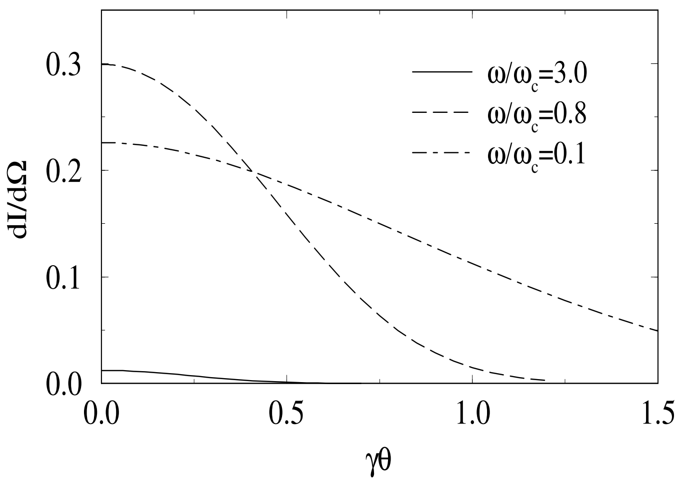
$$\frac{dI(\omega)}{d\Omega} = \frac{3e^2}{\pi^2 c}\gamma^2\left(\frac{\omega}{\omega_c}\right)^2\left(\frac{\Gamma(2/3)}{2}\right)^2\left(\frac{2}{\omega/\omega_c}\right)^{4/3} = \frac{e^2}{c}\left(\frac{\Gamma(2/3)}{\pi}\right)^2\left(\frac{3}{4}\right)^{1/3}\left(\frac{\omega}{\omega_0}\right)^{2/3}. \tag{137}$$

If $\theta \neq 0$, we pick up a contribution proportional to $K_{1/3}^2$, leading to an additional term which is proportional to $\omega^{4/3}$. Note also that the term we do have is produced by waves with the electric field polarized in the plane of the particle's orbit.

In the large frequency regime, $\omega >> \omega_c$ and $\theta = 0$, we find

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2}{\pi^2 c}3\gamma^2\left(\frac{\omega}{\omega_c}\right)^2\frac{\pi}{2\omega/\omega_c}e^{-2\omega/\omega_c} = \frac{e^2}{2\pi c\gamma}\frac{\omega}{\omega_c}e^{-2\omega/3\gamma^3\omega_0}. \tag{138}$$

The accompanying figure shows the angular distribution of radiation at several values



of ω/ω_c.

This figure also very clearly makes the point that at a given frequency, except for the very low ones, the intensity is greatest at the smallest angles.

4.3.2 A Particle in Circular Motion

Let's do the same sort of calculation for truly circular motion. Then, from Eqs. (111) and (113), we find that the energy emitted per cycle per unit solid angle is

$$\frac{dW}{d\Omega} = \sum_{n=0}^{\infty} \frac{dW_n}{d\Omega} = \frac{4\pi}{\omega_0} \sum_{n=1}^{\infty} |\mathbf{a}_n|^2 \tag{139}$$

with

$$|\mathbf{a}_n| = \sqrt{\frac{e^2}{16\pi^3 c}} n\omega_0^2 \left| \int_0^{2\pi/\omega_0} dt' \, (\mathbf{n} \times \boldsymbol{\beta}) e^{in\omega_0(t' - \mathbf{n}\cdot\mathbf{x}(t')/c)} \right|. \tag{140}$$

An individual term in the sum, $dW_n/d\Omega$, is the energy per cycle per unit solid angle radiated at frequency $\omega_n = n\omega_0$. It can be written as

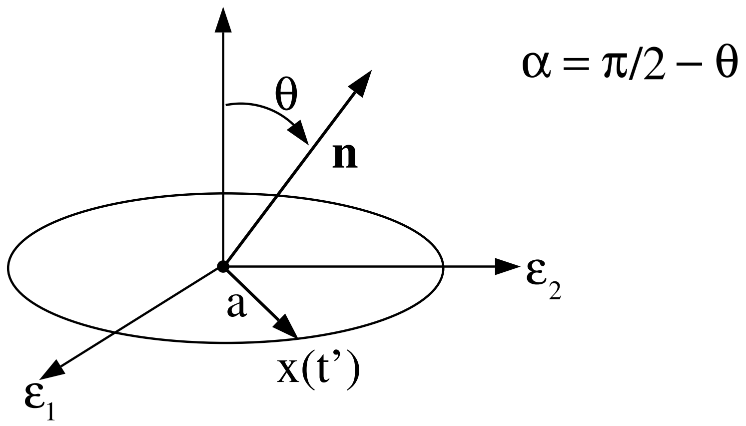
$$\frac{dW_n}{d\Omega} = \frac{4\pi}{\omega_0} \frac{e^2 n^2 \omega_0^4}{16\pi^3 c} \left| \int_0^{2\pi/\omega_0} dt' \, (\mathbf{n} \times \boldsymbol{\beta}) e^{in\omega_0(t' - \mathbf{n}\cdot\mathbf{x}(t')/c)} \right|^2. \tag{141}$$

If we divide by the period, $2\pi/\omega_0$, we find the time-averaged power at frequency $n\omega_0$ per unit solid angle,

$$\frac{d\mathcal{P}_n}{d\Omega} = \frac{e^2 n^2 \omega_0^4}{8\pi^3 c} \left| \int_0^{2\pi/\omega_0} dt' \, (\mathbf{n} \times \boldsymbol{\beta}) e^{in\omega_0(t' - \mathbf{n}\cdot\mathbf{x}(t')/c)} \right|^2. \tag{142}$$

Let the motion be in the x-y plane as before,

$$\mathbf{x}(t') = a[\boldsymbol{\epsilon_1}\cos(\omega_0 t') + \boldsymbol{\epsilon_2}\sin(\omega_0 t')] \tag{143}$$



so that

$$\boldsymbol{\beta}(t') = \frac{\omega_0 a}{c}[-\boldsymbol{\epsilon_1}\sin(\omega_0 t') + \boldsymbol{\epsilon_2}\cos(\omega_0 t')]. \tag{144}$$

Also, write $\mathbf{n}$ in terms of the usual[10] spherical coordinates,

$$\mathbf{n} = \boldsymbol{\epsilon_3}\cos\theta + \boldsymbol{\epsilon_1}\sin\theta\,\cos\phi + \boldsymbol{\epsilon_2}\sin\theta\,\sin\phi. \tag{145}$$

Then

$$\mathbf{n}\cdot\mathbf{x}(t') = a[\sin\theta\,\cos\phi\,\cos(\omega_0 t') + \sin\theta\,\sin\phi\,\sin(\omega_0 t')] = a\sin\theta\,\cos(\omega_0 t' - \phi), \tag{146}$$

and

$$\begin{aligned}\mathbf{n}\times\boldsymbol{\beta}(t') &= \frac{a\omega_0}{c}[-\boldsymbol{\epsilon_2}\cos\theta\sin(\omega_0 t') - \boldsymbol{\epsilon_1}\cos\theta\cos(\omega_0 t') \\ &\quad +\boldsymbol{\epsilon_3}\sin\theta(\cos\phi\cos\omega_0 t' + \sin\phi\sin\omega_0 t')] \\ &= \frac{a\omega_0}{c}[-\boldsymbol{\epsilon_2}\cos\theta\sin(\omega_0 t') - \boldsymbol{\epsilon_1}\cos\theta\cos(\omega_0 t') + \boldsymbol{\epsilon_3}\sin\theta\cos(\omega_0 t' - \phi)]\end{aligned} \tag{147}$$

[10]But be aware that the angle θ in this example is the polar angle and not the latitude as in the previous example; the angle α introduced below is the latitude, i.e., the same as the θ of the previous example.

We must do the integral

$$\mathbf{I} = \int_0^{2\pi/\omega_0} dt' (\mathbf{n}\times\boldsymbol{\beta}) e^{in\omega_0(t'-\mathbf{n}\cdot\mathbf{x}(t')/c)} \equiv \frac{\beta}{\omega_0}[-\boldsymbol{\epsilon}_\mathbf{1} K\cos\theta - \boldsymbol{\epsilon}_\mathbf{2} J\cos\theta + \boldsymbol{\epsilon}_\mathbf{3} L'\sin\theta] \quad (148)$$

where $\omega_0 t' = y$, and

$$\begin{aligned} K &= \int_0^{2\pi} dy\ \cos(y+\phi) e^{i(ny - n\beta\sin\theta\cos y)} e^{in\phi} \\ J &= \int_0^{2\pi} dy\ \sin(y+\phi) e^{i(ny - n\beta\sin\theta\cos y)} e^{in\phi} \\ L' &= \int_0^{2\pi} dy\ \cos y e^{i(ny - n\beta\sin\theta\cos y)} e^{in\phi} \end{aligned} \quad (149)$$

with $\beta = a\omega_0/c$. Thus

$$\begin{aligned} \left\{ \begin{array}{c} J \\ K \end{array} \right\} &= \int_0^{2\pi} dy \left\{ \begin{array}{c} \sin(y+\phi) \\ \cos(y+\phi) \end{array} \right\} e^{in\phi} e^{iny} e^{-in\beta\sin\theta\cos y} \\ &= e^{in\phi} \int_0^{2\pi} dy \left[\sin\phi \left\{ \begin{array}{c} \cos y \\ -\sin y \end{array} \right\} + \cos\phi \left\{ \begin{array}{c} \sin y \\ \cos y \end{array} \right\} \right] e^{i(ny - n\beta\sin\theta\cos y)} \\ &\equiv e^{in\phi} \left[\sin\phi \left\{ \begin{array}{c} L \\ M \end{array} \right\} + \cos\phi \left\{ \begin{array}{c} M \\ L \end{array} \right\} \right] \end{aligned} \quad (150)$$

where

$$\begin{aligned} M &= \int_0^{2\pi} dy\ \sin y\, e^{iny - in\beta\sin\theta\cos y} \\ &= \frac{1}{in\beta\sin\theta} \int_0^{2\pi} dy\, e^{iny} \frac{d}{dy} \left(e^{-in\beta\sin\theta\cos y} \right) \\ &= -\frac{1}{\beta\sin\theta} \int_0^{2\pi} dy\, e^{iny} e^{-in\beta\sin\theta\cos y} = -\frac{2\pi}{\beta\sin\theta} \frac{J_n(n\beta\sin\theta)}{i^n}, \end{aligned} \quad (151)$$

and

$$\begin{aligned} L &= \int_0^{2\pi} dy\ \cos y\, e^{iny} e^{-in\beta\sin\theta\cos y} \\ &= \frac{1}{-i} \frac{d}{d(n\beta\sin\theta)} \int_0^{2\pi} dy\, e^{i(ny - n\beta\sin\theta\cos y)} \\ &= -\frac{2\pi}{i} \frac{1}{i^n} \frac{dJ_n(n\beta\sin\theta)}{d(n\beta\sin\theta)} \end{aligned} \quad (152)$$

Hence

$$\begin{aligned} |\mathbf{I}|^2 &= \frac{\beta^2}{\omega_0^2}\left[\cos^2\theta\,(|K|^2+|J|^2)+\sin^2\theta\,|L|^2\right] \\ &= \frac{4\pi^2\beta^2}{\omega_0^2}\left[\cos^2\theta\left(\frac{J_n(n\beta\sin\theta)}{\beta\sin\theta}\right)^2+(\cos^2\theta+\sin^2\theta)\left(\frac{dJ_n(n\beta\sin\theta)}{d(n\beta\sin\theta)}\right)^2\right] \end{aligned} \tag{153}$$

Also,

$$\frac{d\mathcal{P}_n}{d\Omega} = \frac{e^2n^2\beta^2\omega_0^2}{2\pi c}\left[\left(\frac{dJ_n(n\beta\sin\theta)}{d(n\beta\sin\theta)}\right)^2+\frac{\cot^2\theta}{\beta^2}\left(J_n(n\beta\sin\theta)\right)^2\right]. \tag{154}$$

For better comparison of this result with things we already know, let us look at some limiting cases. First, in the nonrelativistic limit, $\beta << 1$, we expect that only the $n = 1$ term will contribute appreciably and, in addition, we can use the small argument approximation to the Bessel function,

$$J_1(x)\approx x/2 \quad \text{and} \quad J_1'(x)\approx 1/2 \tag{155}$$

so that

$$\frac{d\mathcal{P}_1}{d\Omega} = \frac{e^2\beta^2\omega_0^2}{2\pi c}\left(\frac{1}{4}+\frac{\cot^2\theta\sin^2\theta}{4}\right) = \frac{e^2\omega_0^4a^2}{8\pi c^3}(1+\cos^2\theta) \tag{156}$$

which may be compared with the result of a completely nonrelativistic calculation as in, e.g., Jackson, Problem 14.2(b). In the highly relativistic limit, on the other hand, we have $\beta\approx 1$ and we also know that most of the radiation is close to the equatorial plane or $\theta\approx\pi/2$. Let us introduce $\alpha\equiv\pi/2-\theta << 1$. Then

$$\frac{d\mathcal{P}_n}{d\Omega} = \frac{e^2n^2\beta^2\omega_0^2}{2\pi c}\left.\left[\left(\frac{dJ_n(x)}{dx}\right)^2+\frac{\tan^2\alpha}{\beta^2}\left(J_n(x)\right)^2\right]\right|_{x=n\beta\cos\alpha}. \tag{157}$$

The argument of the Bessel function J_n is comparable to n but is always less than n. For this particular range of argument, it is the case[11] that

$$J_n(x) = \frac{1}{\pi}\sqrt{\frac{2(n-x)}{3x}}K_{1/3}\left(\frac{2\sqrt{2}(n-x)^{3/2}}{3\sqrt{x}}\right), \tag{158}$$

[11] See, e.g., Watson, *Bessel Functions*, p. 249.

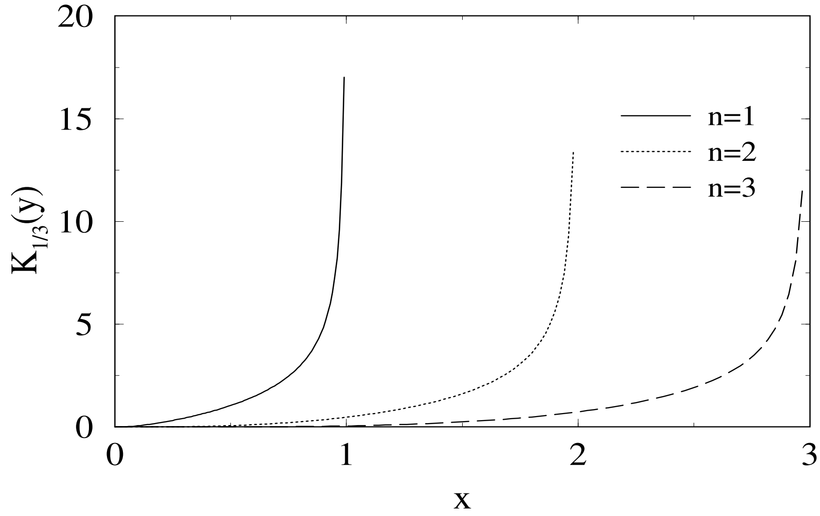
or

$$J_n(x) = \frac{1}{\pi}\frac{y^{1/3}}{3^{1/6}x^{1/3}}K_{1/3}(y) \qquad \text{where} \qquad y = \frac{2\sqrt{2}(n-x)^{3/2}}{3\sqrt{x}}. \tag{159}$$

Also,

$$\frac{dJ_n(x)}{dx} = \frac{1}{\pi 3^{1/6}}\frac{d}{dx}\left(\frac{y^{1/3}K_{1/3}(y)}{x^{1/3}}\right). \tag{160}$$

As shown in the figure below, for the interesting values of x, $K_{1/3}(y)$ varies much more than x.



Thus, the preceding derivative may be approximated by

$$\begin{aligned}\frac{dJ_n(x)}{dx} &\approx \frac{1}{\pi 3^{1/6}x^{1/3}}\frac{d[y^{1/3}K_{1/3}(y)]}{dy}\frac{dy}{dx} \\ &= -\frac{y^{1/3}K_{2/3}(y)}{\pi 3^{1/6}x^{1/3}}\frac{dy}{dx} = \frac{y^{1/3}K_{2/3}(y)}{3^{1/6}x^{1/3}}\frac{\sqrt{2}(n-x)^{1/2}}{3x^{3/2}}(2x+n). \end{aligned}\tag{161}$$

Hence

$$\frac{d\mathcal{P}_n}{d\Omega} = \frac{e^2\beta^2 n^2\omega_0^2 y^{2/3}}{2\pi c\pi^2 3^{1/3}x^{2/3}}\left[\frac{n^2\sin^2\alpha}{x^2}K_{1/3}^2(y) + \frac{2(n-x)(2x+n)^2}{9x^3}K_{2/3}^2(y)\right]. \tag{162}$$

Now let's see what can be said about y and x. Expanding in powers of α and $1/\gamma$, we have

$$y \approx \frac{n}{3}\left(\frac{1}{\gamma^2}+\alpha^2\right)^{3/2} \qquad \text{and} \qquad x \approx n\left(1-\frac{1}{2\gamma^2}-\frac{\alpha^2}{2}\right), \tag{163}$$

so

$$\frac{d\mathcal{P}_n}{d\Omega} = \frac{e^2\omega_0^2 n^2}{6\pi^3 c}\left(\frac{1}{\gamma^2}+\alpha^2\right)^2\left[K_{2/3}^2(y) + \frac{\alpha^2\gamma^2}{1+\alpha^2\gamma^2}K_{1/3}^2(y)\right]. \tag{164}$$

This term describes radiation at the particular frequency $\omega = n\omega_0$, so we can also write it as

$$\frac{d\mathcal{P}_n}{d\Omega} = \frac{\omega_0^2}{2\pi}\frac{e^2(\omega^2/\omega_0^2)}{3\pi^2 c\gamma^6}(1+\alpha^2\gamma^2)^2\gamma^2\left[K_{2/3}^2(y) + \frac{\alpha^2\gamma^2}{1+\alpha^2\gamma^2}K_{1/3}^2(y)\right] \tag{165}$$

with

$$y = \frac{\omega}{3\omega_0\gamma^3}(1+\alpha^2\gamma^2)^{3/2} \equiv \frac{\omega}{\omega_c}(1+\alpha^2\gamma^2)^{3/2}. \tag{166}$$

By grouping terms appropriately in Eq. (165) we can write

$$\frac{d\mathcal{P}_n}{d\Omega} = \frac{\omega_0^2}{2\pi}\frac{3e^2}{\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2(1+\alpha^2\gamma^2)^2\gamma^2\left[K_{2/3}^2(y) + \frac{\alpha^2\gamma^2}{1+\alpha^2\gamma^2}K_{1/3}^2(y)\right]. \tag{167}$$

This is the time-averaged power received at latitude α, at frequency $\omega = n\omega_0$, per unit solid angle. The energy/cycle received at this frequency is obtained if we multiply by $2\pi/\omega_0$, and, finally, the energy per unit frequency is found it we multiply by a factor of the inverse spacing of the harmonics, or $1/\omega_0$. Hence we conclude that for a single pulse, meaning one cycle of the particle,

$$\frac{dI(\omega)}{d\Omega} = \frac{2\pi}{\omega_0^2}\frac{d\mathcal{P}_n}{d\Omega} = \frac{3e^2}{\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2(1+\alpha^2\gamma^2)^2\gamma^2\left[K_{2/3}^2(y) + \frac{\alpha^2\gamma^2}{1+\alpha^2\gamma^2}K_{1/3}^2(y)\right] \tag{168}$$

which is precisely what we found when we calculated the radiated intensity from a particle in instantaneous circular motion.

5 Thomson Scattering; Blue Sky

It's time for a change of pace to something with less nineteenth-century analysis. *Thomson scattering* provides just such a respite. It is the scattering of radiation by a free charge. The mechanism involved, classically, is the coupling of the charge to the incident electric field $\mathbf{E}_0$; this produces acceleration of the charge and consequent radiation by the accelerated charge which becomes the "scattered" radiation. We saw in Chapter 9 how one can describe scattering of electromagnetic radiation in this way. We didn't actually do Thomson scattering at that time, so we'll do it now.

Thomson scattering is to be distinguished from *Compton scattering* in which the same phenomenon is treated using quantum theory. One must use quantum theory, when λ, the wavelength of the radiation, is comparable to the *Compton wavelength* of the scatterer, $\lambda = 2\pi c/\omega \sim \hbar/mc \equiv \lambda_c$; λ_c is the Compton wavelength. This condition may also be written as $\hbar\omega \sim 2\pi mc^2$ which says that if the photon energy $\hbar\omega$ ($\sim$ eV) is comparable to the particle's rest energy mc^2 ($\sim$ MeV), then the calculation must be done using quantum theory. For, e.g., visible light the photon energy is much smaller than an electron's rest energy so the Thomson scattering, or classical, calculation is quite adequate.

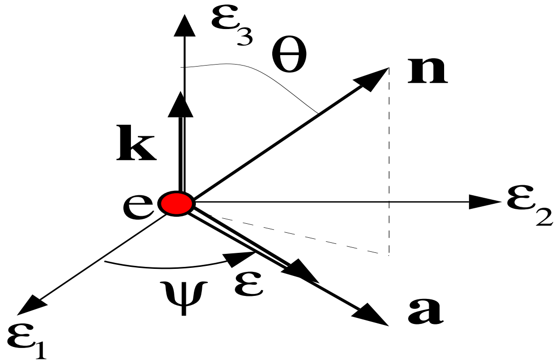
Assuming a nonrelativistic particle, we can make use of the Larmor formula for the scattered radiation,

$$\frac{d\mathcal{P}}{d\Omega} = \frac{e^2}{4\pi c^3} a^2 \sin^2\Theta \qquad \text{where} \qquad \Theta = \angle(\mathbf{n}, \mathbf{a}) \tag{169}$$

for the instantaneous radiated power. To find the acceleration $\mathbf{a}$ of the particle, we must solve the equation of motion of the scatterer. The force on it is provided by the incident electric field,

$$\mathbf{E}_0 = E_0 \boldsymbol{\epsilon} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{170}$$

where for convenience we shall let $\mathbf{k} = k\boldsymbol{\epsilon_3}$ and assume linear polarization of the incident plane wave $\boldsymbol{\epsilon} = \cos\psi\,\boldsymbol{\epsilon_1} + \sin\psi\,\boldsymbol{\epsilon_2}$.



Similarly, we shall suppose for simplicity that the charged particle has no velocity component along the direction of $\mathbf{k}$ and shall ignore the magnetic force which is reasonable so long as the particle is nonrelativistic. Letting the particle be located in the $z = 0$ plane, we find that it experiences an incident field

$$\mathbf{E}_0 = \boldsymbol{\epsilon} E_0 e^{-i\omega t}; \tag{171}$$

given that it has a mass m and charge e, the force acting on it is $\mathbf{F} = e\mathbf{E}_0 = m\mathbf{a}$, so

$$\mathbf{a} = \boldsymbol{\epsilon}\frac{eE_0}{m}e^{-i\omega t}, \tag{172}$$

and

$$\mathbf{a}\cdot\mathbf{n} = \frac{eE_0}{m}\sin\theta(\cos\psi\cos\phi + \sin\psi\sin\phi)e^{-i\omega t}. \tag{173}$$

where θ and ϕ specify, in spherical coordinates, the direction to the observation point.

The time average of the power emitted in some particular direction will be proportional to

$$< a^2\sin^2\Theta >=< a^2(1-\cos^2\Theta) >=< a^2 - (\mathbf{a}\cdot\mathbf{n})^2 >= \frac{e^2E_0^2}{2m^2}[1-\sin^2\theta\cos^2(\psi-\phi)] \tag{174}$$

where the brackets $< ... >$ denote a time average. If the incident radiation is not polarized, then we must average over ψ with the result that

$$< a^2\sin^2\Theta >= \frac{e^2E_0^2}{2m^2}\left(1-\frac{1}{2}\sin^2\theta\right) = \frac{e^2E_0^2}{4m^2}(1+\cos^2\theta) \tag{175}$$

and

$$\frac{d\mathcal{P}}{d\Omega} = \frac{e^4}{16\pi c^3 m^2}|E_0|^2(1+\cos^2\theta) \tag{176}$$

is the time-averaged power distribution when the incident wave is unpolarized.

We define the scattering cross-section in the usual way, i.e., the time-averaged power per unit solid angle divided by the time-averaged incident power per unit area,

$$\frac{d\sigma}{d\Omega} = \frac{d\mathcal{P}/d\Omega}{(c/8\pi)|E_0|^2} = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2(1+\cos^2\theta). \tag{177}$$

This is J. J. Thomson's formula for the scattering of light by a charged particle. The total cross-section is

$$\sigma = \int d\Omega\left(\frac{d\sigma}{d\Omega}\right) = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2 4\pi(1+1/3) = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \equiv \frac{8\pi}{3}r_c^2 \tag{178}$$

where r_c is the classical radius of the particle. For an electron it is $\sim 3\times 10^{-13}\,cm$ and σ is about $0.7\times 10^{-24}\,cm^2$.

We may also calculate the scattering of radiation by a bound charge using the model of a damped harmonic oscillator. Let a charge e with mass m be bound at the origin of coordinates with a natural frequency of oscillation ω_0 and damping constant Γ. Then, given an applied electric field

$$\mathbf{E}_0(\mathbf{x},t) = \boldsymbol{\epsilon}E_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{179}$$

with $\mathbf{k} = k\boldsymbol{\epsilon}_3$, we know from earlier calculations that the charge will respond with a displacement

$$\mathbf{x}(t) = \mathbf{x}_0 e^{-i\omega t} \qquad \text{where} \qquad \mathbf{x}_0 = \frac{eE_0/m}{\omega_0^2-\omega^2-i\omega\Gamma}\boldsymbol{\epsilon} \tag{180}$$

if we approximate $\mathbf{E}_0(\mathbf{x},t)$ by $\mathbf{E}_0(0,t)$ as is reasonable when the particle's displacement from the origin is small compared to the wavelength of the incident radiation. That is certainly true for visible light and an atomic electron.

From $\mathbf{x}(t)$ it is a simple matter to compute the acceleration,

$$\mathbf{a} = -\boldsymbol{\epsilon}\frac{eE_0\omega^2/m}{\omega_0^2-\omega^2-i\omega\Gamma}e^{-i\omega t}, \tag{181}$$

and then to find the radiated, or scattered, time- and incident-polarization-averaged power per unit solid angle,

$$\frac{d\mathcal{P}}{d\Omega} = \frac{e^2}{16\pi c^3}\left(\frac{eE_0}{m}\right)^2\left(\frac{1+\cos^2\theta}{(1-\omega_0^2/\omega^2)^2+\Gamma^2/\omega^2}\right); \tag{182}$$

the scattering cross-section follows:

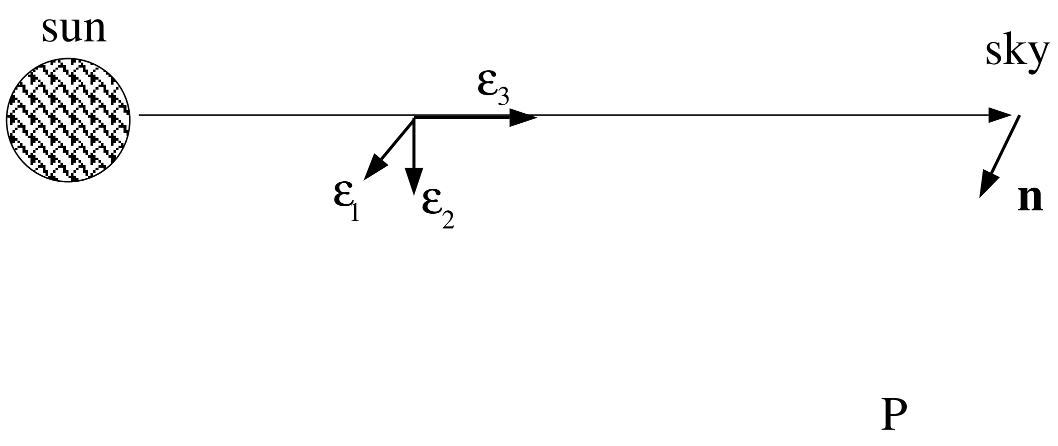
$$\frac{d\sigma}{d\Omega} = \frac{1}{2}r_c^2\left(\frac{1+\cos^2\theta}{(1-\omega_0^2/\omega^2)^2+\Gamma^2/\omega^2}\right). \tag{183}$$

For sufficiently large ω, $\omega >> \omega_0, \Gamma$, the cross-section reduces to the Thomson result as it should since for very large ω the particle will respond to the field as though it were a free particle. Also, for $\omega << \omega_0$ and ω_0^2/Γ, we find the Rayleigh scattering result,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}r_c^2\left(\frac{\omega}{\omega_0}\right)^4(1+\cos^2\theta) \tag{184}$$

with the characteristic ω^4 behavior indicating dipole scattering.

Why is the sky blue? Why is the light polarized when we look at the sky perpendicular to the line of sight from us to the sun?



6 Cherenkov Radiation Revisited

While studying the energy loss of a charged particle traversing a material, we derived an expression for the rate of energy loss through Cherenkov radiation. Specifically, for a charge q moving at speed β through a medium with a real dielectric function $\epsilon(\omega)$, we found that the energy loss per unit path length is

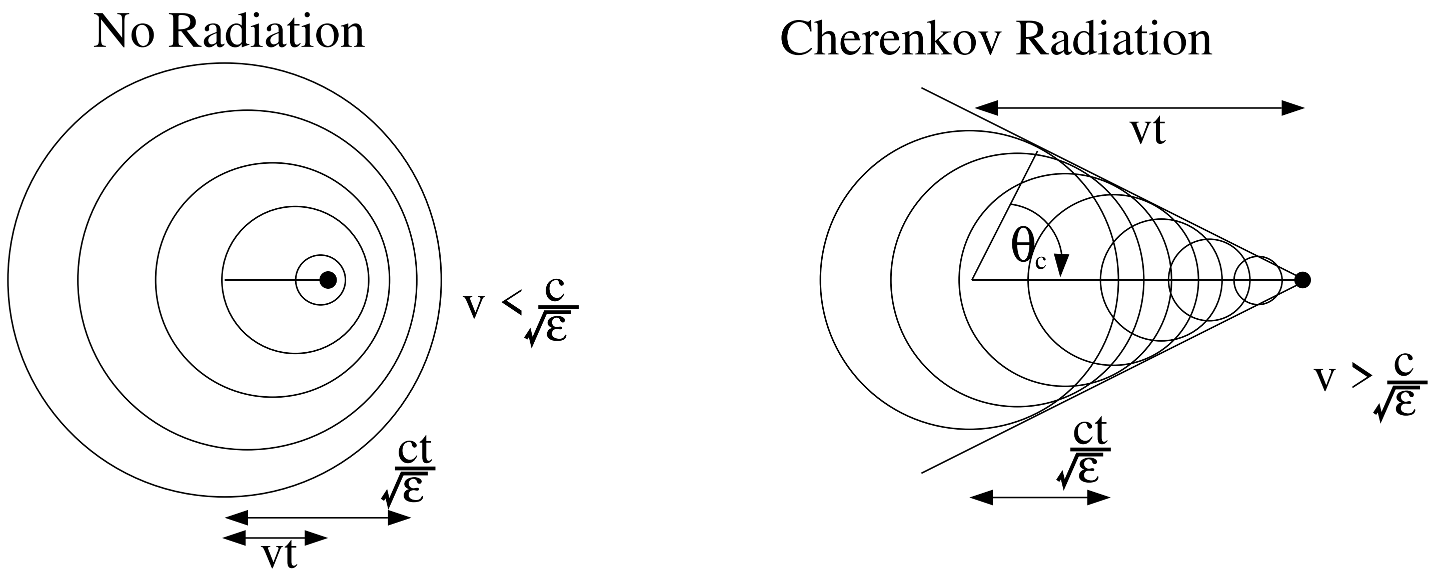
$$\frac{dE}{dx} = \frac{q^2}{c^2}\int_{\epsilon\beta^2>1} d\omega\,\omega(1-\frac{1}{\epsilon(\omega)\beta^2}). \tag{185}$$

The form of this expression suggests that the radiation per unit length of path per unit frequency is given by the integrand,

$$\frac{dI(\omega)}{dx} = \omega\frac{q^2}{c^2}\left(1-\frac{1}{\epsilon(\omega)\beta^2}\right). \tag{186}$$

What we did not do is look at the form of the potentials and fields in real space and time. That is interesting and revealing, so we are going to do it now using retarded potentials.

Before starting, lets be sure that we understand the physical mechanism giving rise to the radiation. It is not the incident particle, which may be reasonably described as having constant velocity, that is doing the radiating. Rather, the incident particle produces fields which act on the particles in the medium, causing them to be accelerated in various ways. They then produce radiation fields which, when the incident particle moves more rapidly than the speed of light in the medium, but not when it moves more slowly, add in a coherent fashion to give Cherenkov radiation.



It is not easy to see why the fields must cancel when the particle is moving slower than the speed of light, but it *is* easy to see why they must not when it is moving faster than light in the medium. Consider the right hand side of the figure above. Before the wake of the radiation hits a particle in the medium, it does not feel the incident particle. Once the wake hits a particular particle in the medium, not only it, but all of its neighbors accelerate in a direction perpendicular to the wake (depending upon the relative charge). Clearly these particles will radiate coherently.

If we are going to produce a calculation of Cherenkov radiation, then, we have to find the total field produced by all of the particles and not just the incident particle. We in fact did that in Chapter 13 in the space of $\mathbf{k}$ and ω. Here we want to determine appropriate Liénard-Wieckert potentials for this field so as to find it in real space and time. That turns out not to be very hard if we ignore the frequency-dependence of the dielectric function.

Consider the Fourier-transformed potentials $\mathbf{A}(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$ produced by the incident charge in the medium where it is taken to have constant velocity. As we have seen, these obey the equations

$$\begin{aligned}\left(k^2 - \frac{\omega^2}{c^2}\epsilon\right)\Phi(\mathbf{k},\omega) &= 4\pi\frac{\rho(\mathbf{k},\omega)}{\epsilon}\\ \left(k^2 - \frac{\omega^2}{c^2}\epsilon\right)\mathbf{A}(\mathbf{k},\omega) &= \frac{4\pi}{c}\mathbf{J}(\mathbf{k},\omega)\end{aligned} \tag{187}$$

where ρ and $\mathbf{J}$ are the macroscopic sources, i.e., the charge and current density of the incident particle. Given that ϵ is independent of ω, then these are the same as the equations obeyed by the potentials of a fictitious system consisting of a point particle with charge $q/\sqrt{\epsilon}$ moving at constant velocity $\mathbf{v}$ in a 'vacuum' where the speed of light is $c' = c/\sqrt{\epsilon}$. Let's rewrite them in such a way as to see this correspondence more clearly:

$$\begin{aligned}\left(k^2 - \frac{\omega^2}{c'^2}\right)[\sqrt{\epsilon}\Phi(\mathbf{k},\omega)] &= 4\pi\frac{\rho(\mathbf{k},\omega)}{\sqrt{\epsilon}}\\ \left(k^2 - \frac{\omega^2}{c'^2}\right)\mathbf{A}(\mathbf{k},\omega) &= \frac{4\pi}{c'}\frac{\mathbf{J}(\mathbf{k},\omega)}{\sqrt{\epsilon}}.\end{aligned} \tag{188}$$

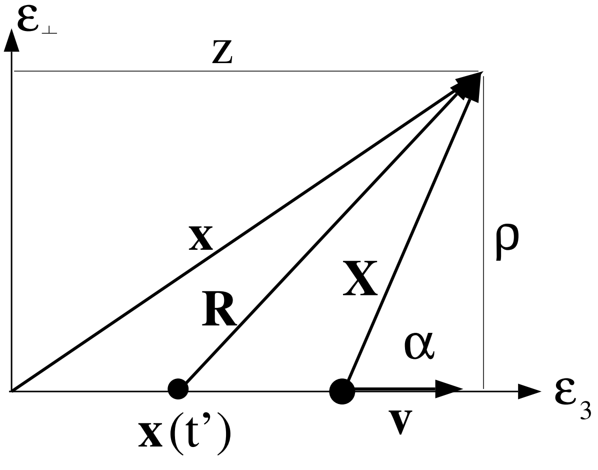
Now let's think about these potentials in real space and time. Because they are the same as a particle with a renormalized charge $q/\sqrt{\epsilon}$ moving in a vacuum with a renormalized speed of light is c', we can write them as Liénard-Wieckert potentials using the renormalized charge and velocity of signal propagation,

$$\sqrt{\epsilon}\Phi(\mathbf{x},t) = \frac{q}{\sqrt{\epsilon}}\left[\frac{1}{\kappa R}\right]_{ret} \tag{189}$$

and similarly for $\mathbf{A}(\mathbf{x},t)$. In this case the retardation means that κR should be evaluated at the time $t' = t - R(t')/c'$. Further, if $\mathbf{x}(t') = \mathbf{v}t'$, $\mathbf{v} = v\boldsymbol{\epsilon_3}$, and $\mathbf{x} = z\boldsymbol{\epsilon_3} + \rho\boldsymbol{\epsilon}_\perp$, then

$$R(t') = (z - vt')\boldsymbol{\epsilon_3} + \rho\boldsymbol{\epsilon}_\perp. \tag{190}$$

Also, for this system, $\kappa = 1 - \mathbf{n}\cdot\mathbf{v}/c'$. Notice that κ can be negative; the absolute value of κ should be employed in evaluating the potential because the potential really involves $|\kappa R|$ as one may see by going back to its derivation, especially Eq. (4).



It is useful to introduce a vector $\mathbf{X} \equiv \mathbf{x} - \mathbf{v}t$ which is the relative displacement of the observation point and the particle at time t. Then

$$\mathbf{R} = \mathbf{x} - \mathbf{v}t' = \mathbf{x} - \mathbf{v}t + \mathbf{v}(t - t') = \mathbf{X} + \mathbf{v}(t - t'), \tag{191}$$

and

$$t - t' = R/c' = |\mathbf{X} + \mathbf{v}(t - t')|/c'. \tag{192}$$

Square this relation to find

$$(t-t')^2 = \frac{X^2}{c'^2} + \frac{v^2}{c'^2}(t-t')^2 + \left(\frac{2\mathbf{X}\cdot\mathbf{v}}{c'^2}\right)(t-t'). \tag{193}$$

This is a quadratic equation that can be solved for $t-t'$; there are two solutions which are

$$t-t' = \frac{-\mathbf{X}\cdot\mathbf{v} \pm \sqrt{(\mathbf{X}\cdot\mathbf{v})^2 - (v^2-c'^2)X^2}}{v^2-c'^2}. \tag{194}$$

Acceptable solutions must be real and positive. Given that $v^2 > c'^2$, which we know to be the regime where there is Cherenkov radiation, we find that there are either no such solutions or there are two of them. The conditions under which there are two are

$$\mathbf{X}\cdot\mathbf{v} < 0 \qquad \text{and} \qquad (\mathbf{X}\cdot\mathbf{v})^2 > (v^2-c'^2)X^2. \tag{195}$$

Let the angle between $\mathbf{X}$ and $\mathbf{v}$ be α. Then we require, first, that α be larger than $\pi/2$ and, second, that $X^2v^2\cos^2\alpha > (v^2-c'^2)X^2$ or $\cos^2\alpha > 1-c'^2/v^2$. Hence there is a cutoff angle α_0 given by

$$\alpha_0 = \arccos(-\sqrt{1-c'^2/v^2}) \tag{196}$$

such that for $\alpha < \alpha_0$ there is no potential. There can thus be potentials and fields at time t only within a cone whose apex is the current position of the particle and which has an apex angle of $\pi - \alpha_0$. Within this cone the potential is the sum of two terms, $\Phi = \Phi_1 + \Phi_2$, corresponding to the two allowed values of $t-t'$. Making use of Eq. (191), and the fact that $\mathbf{R} \parallel \mathbf{n}$, we see that we can write, for either case,

$$[\kappa R]_{ret} = |(1-\mathbf{n}\cdot\mathbf{v}/c)\cdot\mathbf{R}| = R-\mathbf{v}\cdot\mathbf{R}/c = |\mathbf{X}+\mathbf{v}(t-t')| - \mathbf{X}\cdot\mathbf{v}/c' - v^2(t-t')/c'. \tag{197}$$

Using Eqs. (189) and (197) with Eq. (194) for $t-t'$, one finds that $\Phi_1 \equiv \Phi_2$ and that

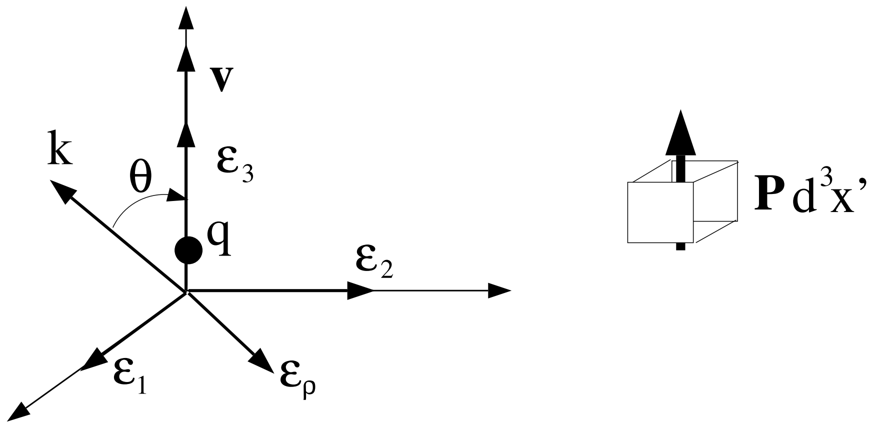
$$\Phi(\mathbf{x},t) = \left(\frac{2q}{\epsilon}\right)\frac{1}{X\sqrt{1-(v^2/c'^2)\sin^2\alpha}}. \tag{198}$$

A similar expression can be found for $\mathbf{A}(\mathbf{x},t)$ and with a bit more work one can compute the radiated power, recovering the same equations as found in chapter 13 for the particular case that $\epsilon(\omega)$ is a constant. Such a dielectric function never exists, of course, and so our conclusions are flawed in some respects. One of them has to do with the form of Φ close to $\alpha = \alpha_0$; Eq. (198) indicates that it in fact diverges here. That would indeed happen if signals with all frequencies traveled at speed c' so that such a singular non-dispersing wave front could be built by superposing waves with many different wavelengths including ones approaching zero. In reality, there is no such singularity although the amplitude of the wave does have a strong maximum at the leading edge.

7 Cherenkov Radiation; Transition Radiation

This time we will do a calculation using perturbation theory much the way we did scattering *via* perturbation theory. We will learn a little bit more about the character of Cherenkov radiation and will also derive a new (to us) phenomenon. The basic requirement for validity of the calculation is to have $|\epsilon(\omega) - 1| << 1$ so that we can get away with calculating the true macroscopic fields as a correction to the fields produced by the incident particle in vacuum. We have shown that, when expressed as a function of $\mathbf{x}$ and ω, the electric field of a particle with charge q moving at constant velocity $\mathbf{v} = v\boldsymbol{\epsilon}_3$ on a trajectory $\mathbf{x}(t) = \mathbf{v}t$ is, in vacuum,

$$\mathbf{E}_i(\mathbf{x}', \omega) = \sqrt{\frac{2}{\pi}} \frac{q\omega}{\gamma v^2} e^{i\omega z'/v} [K_1(\omega\rho'/\gamma v)\boldsymbol{\epsilon}_{\rho'} - (i/\gamma)K_0(\omega\rho'/\gamma v)\boldsymbol{\epsilon}_3]. \qquad (199)$$



This field produces a polarization in the medium which is

$$\mathbf{P}(\mathbf{x}'\omega) = \frac{\epsilon(\omega) - 1}{4\pi}\mathbf{E}_i(\mathbf{x}', \omega) \tag{200}$$

and a dipole moment $\mathbf{P}(\mathbf{x}', \omega)d^3x'$ in a volume element d^3x'. From chapter 9 (Eq. (43)), we know that such a harmonic dipole moment gives rise to a radiation field

$$d\mathbf{E}_{rad}(\mathbf{x}, \omega) = \frac{e^{ikR}}{R}k^2[\mathbf{n} \times \{\mathbf{P}(\mathbf{x}', \omega)d^3x'\}] \times \mathbf{n}. \tag{201}$$

In the radiation zone we can expand $R = |\mathbf{x} - \mathbf{x}'|$ as $R = r - \mathbf{n} \cdot \mathbf{x}'$, leading to

$$\mathbf{E}_{rad}(\mathbf{x}, \omega) = \frac{e^{ikr}}{r}\left(\frac{\epsilon(\omega) - 1}{4\pi}\right)k^2 \int_V d^3x'\, e^{-ik\mathbf{n}\cdot\mathbf{x}'}\{[\mathbf{n} \times \mathbf{E}_i(\mathbf{x}', \omega)] \times \mathbf{n}\} \tag{202}$$

$$= \frac{e^{ikr}}{r}\left(\frac{\epsilon(\omega) - 1}{4\pi}\right)k^2 \int d^2x'_\perp\{[\mathbf{n} \times \mathbf{E}_i(\mathbf{x}'_\perp, 0, \omega)] \times \mathbf{n}\}e^{-ikx'\sin\theta}\int dz' e^{i(\omega z'/v - kz'\cos\theta)} \tag{203}$$

where we have specified $\mathbf{k} = k(\boldsymbol{\epsilon_3}\cos\theta + \boldsymbol{\epsilon_1}\sin\theta)$ without loss of generality since the fields are invariant under rotation around the direction of $\mathbf{v}$. Also, $k = \omega\sqrt{\epsilon(\omega)}/c = \omega/v_p$ where v_p is the phase velocity of a wave with frequency ω. If the medium is infinite in the z direction, we can complete the integration over z' with ease and find

$$\mathbf{E}_{rad}(\mathbf{x}, \omega) = \frac{e^{ikr}}{r}\left(\frac{\epsilon(\omega) - 1}{4\pi}\right)k^2\delta(\omega/v - k\cos\theta)\int d^2x'_\perp\{[\mathbf{n} \times \mathbf{E}_i(\mathbf{x}'_\perp, \omega)] \times \mathbf{n}\}e^{ikx'\sin\theta}. \tag{204}$$

Notice that k and ω are related by $k = \omega\sqrt{\epsilon(\omega)}/c = \omega/v_p$ so that we find no radiation field unless $\cos\theta = v_p/v$ which can only happen if $v > v_p$. We seem to be on the right track. The implication is that there is radiation coming out of this system in a direction $\mathbf{n}$ which makes an angle α_0 with the z axis where $\alpha_0 = \arccos(v_p/v)$. What we have shown in chapter 13 is that this is the direction of the outcoming radiation, which is Cherenkov radiation, is perpendicular to the "bow wave" shown in the figure earlier.

We can get something new out of this calculation also. Suppose that $v < v_p$ and that the medium is finite in extent in the z direction. If it begins at $z = 0$ and ends at some arbitrary, large, value of z, then the integral over z' is not from $-\infty$ to $+\infty$ but rather starts from zero and will be non-zero even though $v < v_p$.

The corresponding radiation, known as *transition radiation*, comes about because of the presence of the boundary, in this case between vacuum and dielectric. The integral we must do can be evaluated with the use of a convergence factor (which emulates the damping present in most materials):

$$\begin{aligned}\int_0^\infty dz'\, e^{iz'(\omega/v - k\cos\theta)} &= \lim_{\eta\to 0}\int_0^\infty dz'\, e^{[-\eta + i(\omega/v - k\cos\theta)]z'} \\ &= \lim_{\eta\to 0}\left(\frac{-1}{-\eta + i(\omega/v - k\cos\theta)}\right) = \frac{i}{\omega/v - k\cos\theta},\end{aligned} \tag{205}$$

and so

$$\mathbf{E}_{rad}(\mathbf{x},\omega) = \frac{e^{ikr}}{r}\frac{\epsilon(\omega)-1}{4\pi}k^2\left(\frac{i}{\omega/v - k\cos\theta}\right)\int d^2x'_\perp \left\{[\mathbf{n}\times\mathbf{E}_i(\mathbf{x}'_\perp,\omega)]\times\mathbf{n}\right\}e^{-ikx'\sin\theta}. \tag{206}$$

More generally, a slab of material of finite thickness d produces an integral over z' which is

$$\int_0^d dz'\, e^{i(\omega/v - k\cos\theta)z'} = \frac{i\left(1 - e^{i(\omega/v-k\cos\theta)d}\right)}{\omega/v - k\cos\theta}. \tag{207}$$

Whether we treat a finite slab or no, we have to do the integral over the other two coordinates. The integrand involves $(\mathbf{n}\times\mathbf{E}_i)\times\mathbf{n}$. For a given $\mathbf{n} = \mathbf{k}/k$, $\mathbf{E}_i$, and $\mathbf{x}'$, and noting that $\boldsymbol{\epsilon}_{\rho'} = \cos\phi'\,\boldsymbol{\epsilon}_1 + \sin\phi'\,\boldsymbol{\epsilon}_2$, we have

$$(\mathbf{n}\times\mathbf{E}_i)\times\mathbf{n} = (E_\rho\cos\phi'\cos\theta - E_z\sin\theta)(\boldsymbol{\epsilon}_2\times\mathbf{n}) + E_\rho\sin\phi'\boldsymbol{\epsilon}_2. \tag{208}$$

The second term is an odd function of ϕ' and will give zero when integrated over $\mathbf{x}'_\perp$. Thus we must do the integral

$$\mathbf{I} = \int d^2x'_\perp(\mathbf{n}\times\mathbf{E}_i)\times\mathbf{n}e^{-ikx'\sin\theta} \tag{209}$$

$$= \sqrt{\frac{2}{\pi}}\frac{q\omega}{\gamma v^2}(\boldsymbol{\epsilon}_2\times\mathbf{n})\int dx'dy'\,[K_1(\omega\rho'/\gamma v)\cos\theta\cos\phi' + (i/\gamma)K_0(\omega\rho'/\gamma v)\sin\theta]e^{-ikx'\sin\theta} \tag{210}$$

Now, $\cos\phi' = x'/\rho'$ and

$$\cos\phi'\,K_1(\omega\rho'/\gamma v) = \frac{x'}{\rho'}K_1(\omega\rho'/\gamma v) = -\frac{\gamma v}{\omega}\frac{\partial K_0(\omega\rho'/\gamma v)}{\partial x'}. \tag{211}$$

Using this in the expression for $\mathbf{I}$, we can do an integration by parts and find

$$\begin{aligned}\mathbf{I} &= \sqrt{\frac{2}{\pi}}\frac{q\omega}{\gamma v^2}(\boldsymbol{\epsilon}_2\times\mathbf{n})\int dx'dy'\left[-\frac{ik\gamma v\sin\theta}{\omega}K_0(\omega\rho'/\gamma v)\cos\theta + \frac{i}{\gamma}K_0(\omega\rho'/\gamma v)\sin\theta\right] \\ &= \sqrt{\frac{2}{\pi}}\frac{q}{v}(-i\sin\theta)\left(k\cos\theta - \frac{\omega}{\gamma^2 v}\right)(\boldsymbol{\epsilon}_2\times\mathbf{n})\int dx'dy'\,K_0(\omega\rho'/\gamma v)e^{-ikx'\sin\theta}.\end{aligned} \tag{212}$$

The integral over x' can be done using the identity

$$\int_0^\infty dz\,K_0(\beta\sqrt{z^2+t^2})\cos(\alpha z) = \frac{\pi}{2\sqrt{\alpha^2+\beta^2}}e^{-|t|\sqrt{\alpha^2+\beta^2}}. \tag{213}$$

Thus,

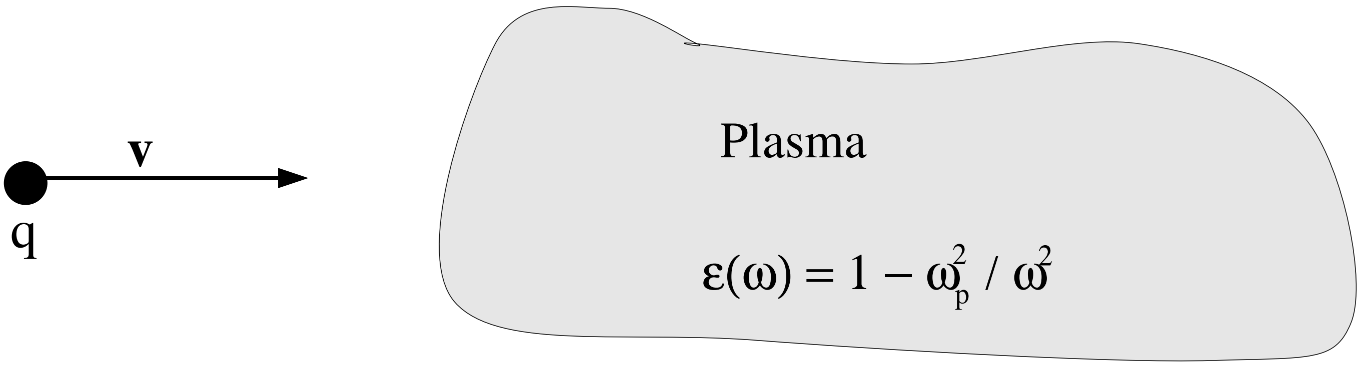
$$\begin{aligned}\mathbf{I} &= \sqrt{\frac{2}{\pi}}\frac{q}{v}\boldsymbol{\epsilon}_{\mathbf{2}} \times \mathbf{n}(-i\sin\theta)\left(k\cos\theta - \frac{\omega}{\gamma^2 v}\right)\int_{-\infty}^{\infty}\frac{\pi dy' \exp(-|y'|\sqrt{k^2\sin^2\theta + \omega^2/\gamma^2 v^2})}{\sqrt{k^2\sin^2\theta + \omega^2/\gamma^2 v^2}} \\ &= \sqrt{\frac{2}{\pi}}\frac{q}{v}\boldsymbol{\epsilon}_{\mathbf{2}} \times \mathbf{n}(-i\sin\theta)\left(k\cos\theta - \frac{\omega}{\gamma^2 v}\right)\frac{2\pi}{k^2\sin^2\theta + \omega^2/\gamma^2 v^2}. \qquad (214)\end{aligned}$$

Hence

$$\mathbf{E}_{rad} = e^{ikr}\frac{\epsilon(\omega) - 1}{4\pi r}\frac{k^2\sin\theta}{\omega/v - k\cos\theta}\frac{2\sqrt{2\pi}(q/v)\boldsymbol{\epsilon}_{\mathbf{2}} \times \mathbf{n}}{k^2\sin^2\theta + \omega^2/\gamma^2 v^2}\left(k\cos\theta - \frac{\omega}{\gamma^2 v}\right). \qquad (215)$$

7.1 Cherenkov Radiation in a Dilute Collisionless Plasma

This is a general result. Let us look at some specific example in a simple limit in order to extract the salient qualitative features of transition radiation. First, we shall choose a dielectric function. A very simple one, which has the added virtue that for some frequencies $\epsilon < 1$ so that it is impossible to have Cherenkov radiation, is that for a dilute collisionless plasma, $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$.



For ω large enough, this function is less than, but close to, unity so that it satisfies the criteria for validity of the perturbation theory. Then

$$(\epsilon(\omega) - 1)k^2/4\pi = -\omega_p^2 k^2/4\pi\omega^2 = -\omega_p^2/4\pi c^2 \qquad (216)$$

where we approximate ω by ck wherever it is not important.[12] Suppose also that

[12] This can be done in some places because $\epsilon(\omega) \approx 1$; in other places the difference $\epsilon(\omega) - 1$ is needed and here we cannot set $\epsilon(\omega)$ equal to 1.

$\gamma >> 1$, i.e., that the particle is highly relativistic. These conditions allow us to approximate as follows:

$$\frac{\omega}{v}-k\cos\theta = \frac{\omega}{v}-\frac{\omega}{v_p}\cos\theta = \frac{\omega}{\beta c}-\frac{\omega}{c}\sqrt{\epsilon}\cos\theta \approx \frac{\omega}{c(1-1/2\gamma^2)}-\frac{\omega}{c}\left(1-\frac{\omega_p^2}{2\omega^2}\right)\left(1-\frac{\theta^2}{2}\right) \tag{217}$$

$$\approx \frac{\omega}{c}\left(\frac{1}{2\gamma^2}+\frac{\omega_p^2}{2\omega^2}+\frac{\theta^2}{2}\right) = \frac{\omega}{2\gamma^2 c}\left(1+\frac{\gamma^2\omega_p^2}{\omega^2}+\gamma^2\theta^2\right) \equiv \frac{\omega}{2\gamma^2 c}\left(1+\frac{1}{\nu^2}+\eta\right) \tag{218}$$

where

$$\nu \equiv \omega/\gamma\omega_p \quad \eta \equiv \gamma^2\theta^2 \quad \beta = \sqrt{1-\gamma^{-2}} \approx 1-1/2\gamma^2 \quad \text{and} \quad \sqrt{\epsilon} = \sqrt{1-\frac{\omega_p^2}{\omega^2}} \approx 1-\frac{\omega_p^2}{2\omega^2}. \tag{219}$$

Also,

$$\frac{\omega^2}{\gamma^2 v^2}+k^2\sin^2\theta = \frac{\omega^2}{\gamma^2\beta^2c^2}+\frac{\omega^2\epsilon\sin^2\theta}{c^2} = \frac{\omega^2}{\gamma^2\beta^2c^2}[1+\gamma^2\beta^2\epsilon\sin^2\theta] \approx \frac{\omega^2}{\gamma^2c^2}(1+\eta) \tag{220}$$

while

$$k\cos\theta-\frac{\omega}{v\gamma^2} = \frac{\omega}{c}\sqrt{\epsilon(\omega)}\cos\theta-\frac{\omega}{\beta c\gamma^2} = \frac{\omega}{c}\left(\sqrt{\epsilon(\omega)}\cos\theta-\frac{1}{\beta\gamma^2}\right) \approx \frac{\omega}{c}. \tag{221}$$

Hence,

$$\begin{aligned}
\mathbf{E}_{rad} &= \frac{e^{ikr}}{r}\left(\frac{-\omega_p^2}{4\pi c^2}\right)\frac{\theta(2\sqrt{2\pi})(q/c)(\omega/c)\boldsymbol{\epsilon_2}\times\mathbf{n}}{(\omega/2\gamma^2c)(1+\eta+1/\nu^2)(\omega^2/\gamma^2c^2)(1+\eta)} \\
&= -\frac{e^{ikr}}{r}\frac{\omega_p^2}{4\pi}\frac{2\gamma^4\theta 2\sqrt{2\pi}q\boldsymbol{\epsilon_2}\times\mathbf{n}\omega}{c\omega(1+\eta+1/\nu^2)\omega^2(1+\eta)} \\
&= -\frac{e^{ikr}}{r}\sqrt{\frac{2}{\pi}}\frac{\gamma\sqrt{\eta}}{\nu^2}\frac{q\boldsymbol{\epsilon_2}\times\mathbf{n}}{c(1+\eta+1/\nu^2)(1+\eta)}.
\end{aligned} \tag{222}$$

The radiated energy per unit frequency per unit solid angle is[13]

$$\frac{d^2I(\omega)}{d\Omega d\omega} = 2\frac{c}{4\pi}|r\mathbf{E}_{rad}(\mathbf{x},\omega)|^2 = \frac{q^2\gamma^2}{\pi^2c}\frac{\eta}{\nu^4(1+\eta+1/\nu^2)^2(1+\eta)^2}. \tag{223}$$

[13]Previously, we just called this $dI/d\Omega$.

We can write $d\Omega = \sin\theta d\theta d\phi \approx \theta d\theta d\phi = d\eta d\phi/2\gamma^2$ and integrate over ϕ to find the distribution per unit η. Also, let's write the frequency in terms of ν, $d\nu = d\omega/\gamma\omega_p$, to find

$$\frac{d^2I}{d\eta d\nu} = \int d\phi \frac{d^2I}{d\Omega d\omega}\frac{d\omega}{d\nu}\frac{d\Omega}{d\eta} = \frac{d^2I}{d\Omega d\omega}\frac{\gamma\omega_p\pi}{\gamma^2} = \frac{q^2\gamma\omega_p}{\pi c}\frac{\eta}{\nu^4(1+\eta+1/\nu^2)^2(1+\eta)^2}. \quad (224)$$

This expression fails at small ν ($\omega \sim \omega_p$). It falls off as η^{-3} at large η. It peaks as a function of η around $\eta = 1$ for small ν and at $\eta = 1/3$ for large ν. If one integrates over η the result is

$$\frac{dI}{d\nu} = \frac{q^2\gamma\omega_p}{\pi c}[(1+2\nu^2)\ln(1+1/\nu^2) - 2]. \quad (225)$$

Further, the total energy radiated is

$$I = \int_0^\infty d\nu \frac{dI}{d\nu} = \frac{q^2\gamma\omega_p}{3c} = \frac{(q/e)^2\gamma\hbar\omega_p}{3}\frac{e^2}{\hbar c} = \frac{(q/e)^2}{3(137)}(\gamma\hbar\omega_p). \quad (226)$$

A typical photon energy is $\gamma\hbar\omega_p/3$ ($\nu = 1/3$), so the number of photons emitted on average is $(q/e)^2/137$ which is quite a bit smaller than one. However, a sizable amount of transition radiation can be obtained by employing a stack of thin slabs of material with adjacent slabs having significantly different dielectric constants. Then there is some radiation produced at each interface between different materials.

8 Example Problems

8.1 A Relativistic Particle in a Capacitor

A particle of charge e and mass m initially at rest is accelerated across a parallel plate capacitor held at (stat) voltage V; the distance between the plates is d. Assuming nonrelativistic motion, find the total energy radiated by the particle during this process. Then, *without calculation, answer or estimate the following:*

1. The angular distribution of the radiated energy of the particle,

2. the order of magnitude of typical frequencies emitted from the charge, and

3. the angular distribution of radiated energy from the charge if it were highly relativistic.

Solution. The field in a parallel plate capacitor is constant, thus so is the acceleration of the charged particle and its radiation.

$$P = \frac{2}{3}\frac{e^2}{c^3}a^2 \quad W = \int P dt = \frac{2}{3}\frac{e^2}{c^3}t$$

where t is the duration of the pulse.

$$a = F/m = eE/m = eV/md \quad \text{and} \quad t = \sqrt{2d/a}$$

Hence, solving for W

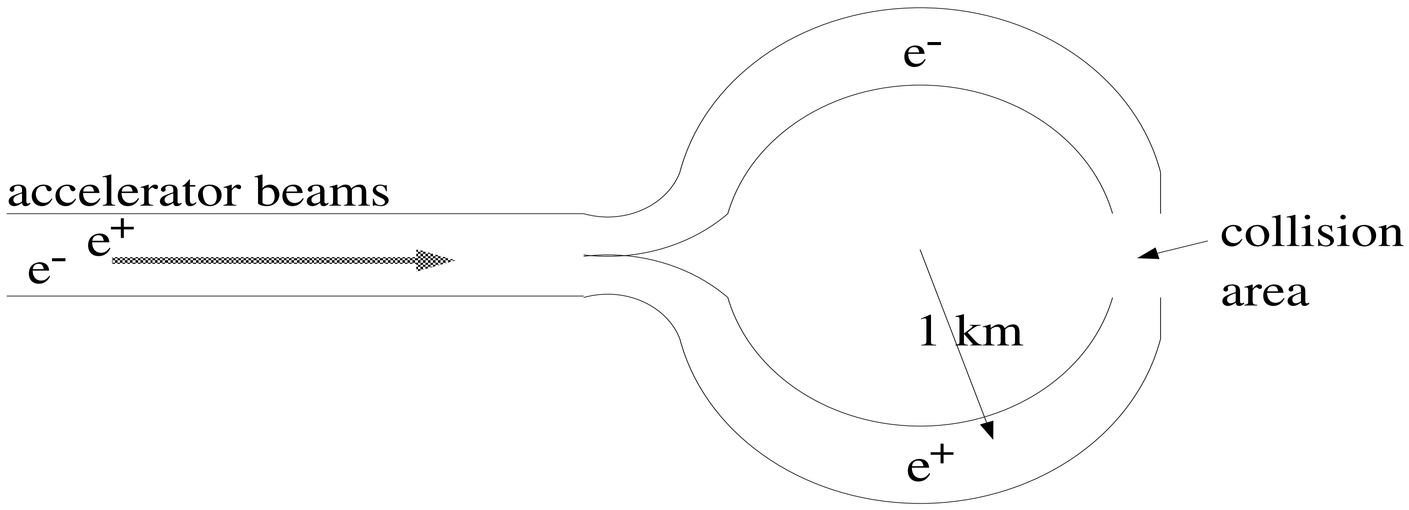
$$W = \frac{2\sqrt{2}e^{7/2}V^{3/2}}{3c^3m^{3/2}d}$$

In the non relativistic limit, the angular distribution of the radiation is given by $\sin^2(\theta)$ where the angle is measured relative to the velocity vector. The typical frequencies can be found using the Fourier uncertainty principle. Since the retarded duration of the pulse is the same as the observer's duration for a nonrelativistic particle, we have $\omega \sim 1/t = \sqrt{eV/md^2}$. In the relativistic limit the angular distribution of the radiation will be strongly pitched in the direction of the velocity/acceleration, but will be zero along the axis. The maximum of the pulse of radiation will be at an angle of $\theta \sim 1/\gamma$ away from the axis defined by the velocity vector.

8.2 Relativistic Electrons at SLAC

At the Stanford linear accelerator, devices have been added at the end of the accelerator to guide electrons and positrons around roughly semicircular paths until they collide head-on as shown in the sketch below. If each particle has a total energy of 50GeV and rest energy of 0.5 MeV, while the circular paths have radii of about

1km, roughly what is the fraction of the particles energy lost to radiation before the collision takes place?



Solution. To determine the power radiated, we must use the relativistic Larmor formula. For $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$, $|\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}| = |\beta\dot{\beta}|$, so the power radiated is

$$P = \frac{2e^2}{3c}\gamma^6 \left[\dot{\boldsymbol{\beta}}^2 - \left(\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}\right)^2\right] = \frac{2e^2}{3c}\gamma^6\dot{\beta}^2\left(1-\beta^2\right) = \frac{2e^2}{3c}\gamma^4\dot{\beta}^2$$

The acceleration is centripetal, so $\dot{\beta} = v^2/cr \approx c/r$, so

$$P = \frac{2e^2}{3c}\gamma^4\frac{c^2}{r^2}$$

Counting the initial deflection of the particles into the circular region, both the electron and the positron travel about 3/4 of a circle. The time it takes to do this is roughly $\tau = 3\pi r/2c$ (assuming that the particles travel roughly at velocity c), so the energy radiated is

$$\Delta E = \frac{\pi e^2}{r}\gamma^4$$

Since $\gamma = E/mc^2$, the relative energy loss is

$$\frac{\Delta E}{E} = \frac{\pi e^2/r}{mc^2}\gamma^3$$

For $E = 50$ GeV and $mc^2 = 0.511$ MeV, $\gamma \approx 10^5$, and the relative energy loss is

$$\frac{\Delta E}{E} = \frac{\pi 23\,(10^{-20})\,(10^{15})}{0.911\,(10^{-27})\,(9)\,(10^{20})\,(10^5)} \approx 0.9 \times 10^{-2}$$

or roughly only one percent of the energy is lost.