Particle and Field Dynamics

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In this chapter we shall study the dynamics of particles and fields. For a particle, the relativistically correct equation of motion is

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{1}$$

where $\mathbf{p} = m\gamma\mathbf{u}$; the corresponding equation for the time rate of change of the particle's energy is

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{u}. \tag{2}$$

The dynamics of the electromagnetic field is given by the Maxwell equations,

$$\begin{array}{cc} \nabla\cdot\mathbf{E}(\mathbf{x},t) = 4\pi\rho(\mathbf{x},t) & \nabla\cdot\mathbf{B}(\mathbf{x},t) = 0 \\ \nabla\times\mathbf{E}(\mathbf{x},t) + \dfrac{1}{c}\dfrac{\partial\mathbf{B}(\mathbf{x},t)}{\partial t} = 0 & \nabla\times\mathbf{B}(\mathbf{x},t) - \dfrac{1}{c}\dfrac{\partial\mathbf{E}(\mathbf{x},t)}{\partial t} = \dfrac{4\pi}{c}\mathbf{J}(\mathbf{x},t). \end{array} \tag{3}$$

These are tied together by the Lorentz force which gives $\mathbf{F}$ in terms of the electromagnetic fields

$$\mathbf{F} = q\left[\mathbf{E} + \frac{1}{c}(\mathbf{u}\times\mathbf{B})\right] \tag{4}$$

and by the expressions for $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ in terms of the particles' coordinates and velocities

$$\begin{array}{c} \rho(\mathbf{x},t) = \sum_i q_i\delta(\mathbf{x}-\mathbf{x}_i(t)) \\ \mathbf{J}(\mathbf{x},t) = \sum_i q_i\mathbf{u}_i(t)\delta(\mathbf{x}-\mathbf{x}_i(t)). \end{array} \tag{5}$$

In view of the fact that we already know all of this, what further do we want to do? Two things: (1) Formulate appropriate covariant Lagrangians and Hamiltonians from which covariant dynamical equations can be derived; and (2) applications.

1 Lagrangian and Hamiltonian of a Charged Particle in an External Field

We want to devise a Lagrangian for a charged particle in the presence of given applied fields which are treated as parameters and not as dynamic variables. This Lagrangian is to yield the equations of motion Eqs. (1) and (2) with **F** given by the Lorentz force. These equations can be written as

$$\frac{dp^\alpha}{dt} = \frac{q}{mc\gamma} F^{\alpha\beta} p_\beta \tag{6}$$

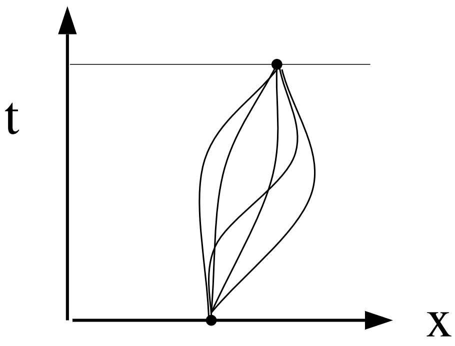
which is almost in Lorentz covariant form. A more obviously Lorentz covariant form can be obtained by using the fact that the infinitesimal time element dt can be related to an infinitesimal proper time element $d\tau$ by $dt = \gamma d\tau$. Then we have

$$\frac{dp^\alpha}{d\tau} = \frac{q}{mc} F^{\alpha\beta} p_\beta. \tag{7}$$

To get a Lagrangian from which these equations follow, we postulate the existence of the action A which may be expressed as an integral,

$$A = \int_a^b dA, \tag{8}$$

over possible "paths" from a to b. The action is an extremum for the actual motion of the system.



Possible paths which contribute to the action.

In this case, the system consists of a single particle. The paths have the constraint that they start at given $(\mathbf{x}_a, t_a)$ and end at $(\mathbf{x}_b, t_b)$.

Next comes a delicate point. We could say that the first postulate of relativity requires that A be the same in all inertial frames[1], which elevates the action and its

[1] This argument is (elegantly) made in *The Classical Theory of Fields*

consequences to "law of nature" status. It seems better to regard the invariance of A as an assumption or postulate in its own right and to see where that leads us.

Rewrite Eq. (8) as follows:

$$A = \int_a^b dA = \int_{t_a}^{t_b} \frac{dA}{dt} dt \equiv \int_{t_a}^{t_b} L dt. \tag{9}$$

This equation expresses nothing more than the parametrization of the integral using the time and the definition of the *Lagrangian* L as the derivative of A with respect to t. Let us further parametrize the integral using the proper time τ of the particle,

$$A = \int_{\tau_a}^{\tau_b} L\gamma d\tau \tag{10}$$

where we use $dt = \gamma d\tau$, $\gamma = 1/\sqrt{1 - u^2/c^2}$, $\mathbf{u}$ being the particle's velocity as measured in the lab frame or the frame in which the time t is measured. *The proper time is an invariant, so if we believe that A is one also, we have to conclude that $L\gamma$ is an invariant.* This statement of invariance greatly limits the possible forms of L.

1.1 Lagrangian of a Free Particle

Consider first the case of a free particle. What invariants may we construct from the properties of a free particle? We have only the four-vectors $\bar{p}$ and $\bar{x}$. The presumed translational invariance of space rules out the use of the latter. That leaves only the four-momentum and the single invariant $p^\alpha p_\alpha = m^2c^2$ which is a constant. Hence we are led to $L\gamma = C$ where C is a constant. Hence, $L = C/\gamma$ and

$$A = C \int_{\tau_a}^{\tau_b} d\tau = C \int_{t_a}^{t_b} dt \sqrt{1 - u^2/c^2}. \tag{11}$$

We may find the constant C by appealing to the nonrelativistic limit and expanding in powers of u^2/c^2.

$$A \approx C \int_{t_a}^{t_b} dt \left(1 - \frac{u^2}{2c^2} + \cdots\right). \tag{12}$$

The term proportional to u^2 should be the usual nonrelativistic Lagrangian of a free particle, $mu^2/2$. This condition leads to

$$C = -mc^2 \tag{13}$$

and so

$$L_f = -mc^2\sqrt{1 - u^2/c^2} \tag{14}$$

is the free-particle Lagrangian.

1.1.1 Equations of Motion

The equations of motion are found by requiring that A be an extremum,

$$\delta A = \delta\left(\int_{t_a}^{t_b} dt\, L\right) = 0. \tag{15}$$

The path $\mathbf{x}(t)$ is to be fixed at the end points t_a and t_b, $\delta\mathbf{x}(t_a) = \delta\mathbf{x}(t_b) = 0$. Writing L as a function of the Cartesian components of the position and velocity, we have, allowing for possible position-dependence which will appear if the particle is not free, $L = L(x_i, u_i, t)$, and

$$\delta A = \sum_i \int_{t_a}^{t_b} dt\,\left[\left(\frac{\partial L}{\partial x_i}\right)\delta x_i + \left(\frac{\partial L}{\partial u_i}\right)\delta u_i\right]. \tag{16}$$

But δu_i is related to δx_i through $u_i = dx_i/dt$, so

$$\begin{aligned}\delta A &= \sum_i \int_{t_a}^{t_b} dt\,\left[\left(\frac{\partial L}{\partial x_i}\right)\delta x_i + \left(\frac{\partial L}{\partial u_i}\right)\delta\left(\frac{dx_i}{dt}\right)\right] \\ &= \left(\frac{\partial L}{\partial u_i}\right)\delta x_i\bigg|_{t_a}^{t_b} + \int_{t_a}^{t_b} dt\,\left[\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right)\right]\delta x_i(t)\end{aligned} \tag{17}$$

where we have integrated by parts to achieve the last step. The first term in the final expression vanishes because $\delta x_i = 0$ at the endpoints of the interval of integration. Arguing that $\delta x_i(t)$ is arbitrary elsewhere, we conclude that the factor [...] in the final expression must vanish everywhere,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right) - \frac{\partial L}{\partial x_i} = 0 \tag{18}$$

for each $i = 1, 2, 3$.

These are the *Euler-Lagrange* equations of motion. Let's apply them to the free-particle Lagrangian L_f,

$$\frac{\partial L_f}{\partial x_i} = 0 \qquad \text{and} \qquad \frac{\partial L_f}{\partial u_i} = m\gamma u_i, \tag{19}$$

so

$$\frac{d}{dt}(m\gamma \mathbf{u}) = 0 \tag{20}$$

is the equation of motion. It is the same as

$$\frac{d\mathbf{p}}{dt} = 0 \tag{21}$$

which is correct for a free particle.

1.2 Lagrangian of a Charged Particle in Fields

Next suppose that there are electric and magnetic fields of roughly the same order of magnitude present so that the particle experiences some force and acceleration. Then $L = L_f + L_{int}$ where L_{int} is the "interaction" Lagrangian and contains the information about the fields and forces. For the action to be an invariant, it must be the case that

$$A_{int} \equiv \int_{t_a}^{t_b} L_{int} dt \tag{22}$$

is an invariant which means $L_{int}\gamma$ has to be an invariant. Now, in the nonrelativistic limit one has, to lowest order, $L = T - V$ with $V = q\Phi$, so we have in this limit $L_{int}\gamma = -q\Phi\gamma = -qE\Phi/mc^2 = -(q/mc)p_0A^0$. This is not an invariant but can be made one by including the rest of $\bar{p}\cdot\bar{A}$[2], and we expect that the result is valid not just in the nonrelativistic limit but in general:

$$L_{int}\gamma = -\left(\frac{q}{mc}\right) p_\alpha A^\alpha. \tag{23}$$

[2]We have little choice other than this form since we only have the p, x and A four-vectors to work with

This choice of L_{int} gives the desired invariant and reduces to the correct static limit. It is the simplest choice of the interaction Lagrangian with the following properties:

1. Translationally invariant (in the sense that it is independent of explicit dependence on $\mathbf{x}$; the potentials do depend on $\mathbf{x}$)
2. Linear in the charge (as are the forces on the particle)
3. Linear in the momenta (as are the forces)
4. Linear in the fields (as are the equations of motion of the particle)
5. A function of no time derivatives of p^α (appropriate for the equations of motion)

1.2.1 Equations of Motion

Let us proceed to the Euler-Lagrange equations of motion. The total Lagrangian is

$$L = -mc^2\sqrt{1-u^2/c^2} + \frac{q}{c}\mathbf{u}\cdot\mathbf{A} - q\Phi; \tag{24}$$

$$\begin{aligned}\frac{\partial L}{\partial x_i} &= \frac{q}{c}\mathbf{u}\cdot\frac{\partial \mathbf{A}}{\partial x_i} - q\frac{\partial \Phi}{\partial x_i} \\ \frac{\partial L}{\partial u_i} &= \frac{mc^2}{\sqrt{1-u^2/c^2}}\frac{u_i}{c^2} + \frac{q}{c}A_i,\end{aligned} \tag{25}$$

so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right) = \frac{d}{dt}(m\gamma u_i) + \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}(\mathbf{u}\cdot\nabla)A_i. \tag{26}$$

Notice the last term on the right-hand side of this equation. It is there because when we take the total time derivative, we must remember that the position variable $\mathbf{x}$ on which $\mathbf{A}$ depends is really the position of the particle at time t, so, by application of the chain rule, we pick up a sum of terms, each of which is the derivative of $\mathbf{A}$ with respect to a component of $\mathbf{x}$ times the derivative of that component of $\mathbf{x}$ with respect to t; the last is a component of the velocity of the particle.

Finally, the equations of motion are

$$\frac{d}{dt}(m\gamma u_i) = -\frac{q}{c}\frac{\partial A_i}{\partial t} - \frac{q}{c}(\mathbf{u}\cdot\nabla)A_i - q\frac{\partial\Phi}{\partial x_i} + \frac{q}{c}\mathbf{u}\cdot\frac{\partial\mathbf{A}}{\partial x_i}$$
$$= qE_i + \frac{q}{c}\mathbf{u}\cdot\frac{\partial\mathbf{A}}{\partial x_i} - \frac{q}{c}(\mathbf{u}\cdot\nabla)A_i. \tag{27}$$

These are supposed to be familiar; consider

$$(\mathbf{u}\times\mathbf{B})_i = [\mathbf{u}\times(\nabla\times\mathbf{A})]_i = [\nabla(\mathbf{u}\cdot\mathbf{A}) - (\mathbf{u}\cdot\nabla)\mathbf{A}]_i = \mathbf{u}\cdot\frac{\partial\mathbf{A}}{\partial x_i} - (\mathbf{u}\cdot\nabla)A_i. \tag{28}$$

Comparison of this expansion with Eq. (27) demonstrates that the latter can be written as

$$\frac{dp_i}{dt} = qE_i + \frac{q}{c}(\mathbf{u}\times\mathbf{B})_i. \tag{29}$$

1.3 Hamiltonian of a Charged Particle

One can also make a Hamiltonian description of the system. Introduce the canonical three-momentum $\boldsymbol{\pi}$ with components

$$\pi_i \equiv \frac{\partial L}{\partial u_i} = \gamma m u_i + \frac{q}{c}A_i = p_i + \frac{q}{c}A_i. \tag{30}$$

Then the Hamiltonian[3] is

$$H = \boldsymbol{\pi}\cdot\mathbf{u} - L = \boldsymbol{\pi}\cdot\mathbf{u} + mc^2\sqrt{1-u^2/c^2} + q\Phi - \frac{q}{c}(\mathbf{u}\cdot\mathbf{A}). \tag{31}$$

We want H to depend on $\mathbf{x}$ and $\boldsymbol{\pi}$ but not on $\mathbf{u}$. To this end consider how to write $\mathbf{u}$ in terms of $\boldsymbol{\pi}$,

$$\boldsymbol{\pi} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} + \frac{q}{c}\mathbf{A}, \tag{32}$$

or

$$\left(\boldsymbol{\pi} - \frac{q}{c}\mathbf{A}\right)^2\left(1 - \frac{u^2}{c^2}\right) = m^2u^2 \tag{33}$$

[3]The hamiltonian $H(q,p)$ obtained from the Lagrangian through a Legendre transformation $H = \sum_i p_i\dot{q}_i - L(q,\dot{q})$

which may be solved for $\mathbf{u}$ to give

$$\mathbf{u} = c\frac{c\boldsymbol{\pi} - q\mathbf{A}}{\sqrt{m^2c^4 + (c\boldsymbol{\pi} - q\mathbf{A})^2}} \tag{34}$$

Use of this result in the expression for the Hamiltonian leads to

$$H = \sqrt{m^2c^4 + (c\boldsymbol{\pi} - q\mathbf{A})^2} + q\Phi. \tag{35}$$

The development of Hamilton's equations will be left as an exercise.

The Hamiltonian is the 0^{th} component of a four-vector. Notice, from Eq. (35), that

$$(H - q\Phi)^2 - (c\boldsymbol{\pi} - q\mathbf{A})^2 = m^2c^4, \tag{36}$$

is an invariant. This invariant is the inner product of a four-vector with itself. The spacelike components are $c\boldsymbol{\pi} - q\mathbf{A} = c\mathbf{p}$, and the timelike component is $H - q\Phi$. The vector is just the energy-momentum four-vector in the presence of fields,

$$p^\alpha = (E/c, \mathbf{p}) = \left(\frac{1}{c}(H - q\Phi), \boldsymbol{\pi} - \frac{e}{c}\mathbf{A}\right). \tag{37}$$

1.4 Invariant Forms

Next we are going to repeat everything that we have just done, but in a manner that is "manifestly" covariant. That is, we want to rewrite the Lagrangian in terms of invariant 4-vector products. We can write the free-particle Lagrangian as

$$L_f = -mc^2\sqrt{1 - u^2/c^2} = -\frac{1}{\gamma}\sqrt{E^2 - p^2c^2} = -\frac{c}{\gamma}\sqrt{p_\alpha p^\alpha} = -\frac{mc}{\gamma}\sqrt{U_\alpha U^\alpha} \tag{38}$$

where

$$U^\alpha = (E/mc, \mathbf{p}/m) \equiv dx^\alpha/d\tau \tag{39}$$

is a four-vector we shall call the *four-velocity*. The action of the free particle is

$$A = -\int_{t_a}^{t_b} dt\, \gamma^{-1} mc\sqrt{U^\alpha U_\alpha} = -mc\int_{\tau_a}^{\tau_b} d\tau\, \sqrt{U^\alpha U_\alpha} \tag{40}$$

Note the manifestly invariant form. However, note that we must also impose the constraint $U^\alpha U_\alpha = c^2$. Thus we may not freely vary this action to find the equations of motion. There are two ways to overcome this. First, we could introduce a Lagrange multiplier to impose the constraint[4], or we could introduce an additional degree of freedom into our equations, and use it to impose the constraint *a posteriori*. Following Jackson, we will follow the later (less conventional) route. To this end we rewrite the action, and introduce s.

$$A = -mc \int_{\tau_a}^{\tau_b} \sqrt{dx_\alpha dx^\alpha} = -mc \int_{s_a}^{s_b} ds \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}} \tag{41}$$

where the path of integration has been parametrized using some (invariant) s. We shall now **treat** each dx^α/ds as an independent generalized velocity, and the Lagrangian takes on the functional form $L(x^\alpha, dx^\alpha/ds, s)$. This (more general) parametrization of the action integral is just as good as the standard one using the time; the Euler-Lagrange equations of motion, found by demanding that A be an extremum, are familiar in appearance,

$$\frac{d}{ds}\left(\frac{\partial \tilde{L}}{\partial(dx_\alpha/ds)}\right) - \frac{\partial \tilde{L}}{\partial x_\alpha} = 0 \tag{42}$$

In this case, we obtain the equation of motion

$$mc\frac{d}{ds}\left[\frac{dx^\alpha/ds}{\sqrt{\frac{dx^\beta}{ds}\frac{dx_\beta}{ds}}}\right] = 0 \tag{43}$$

These velocities are constrained by the condition

$$\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}}ds = cd\tau \tag{44}$$

because there are really only three independent generalized velocities, so that

$$m\frac{d^2x^\alpha}{d\tau^2} = 0 \tag{45}$$

[4]This approach is discussed in **Electrodynamics and Classical Theory of Fields and Particles** by A.O. Barut, Dover, page 65

Analyzing and including the interaction Lagrangian in the same manner leads to a total Lagrangian and an action which is

$$A = -\int_{s_a}^{s_b} ds \left[mc\sqrt{g^{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}} + \frac{e}{c}\frac{dx_\alpha}{ds}A^\alpha(x) \right] \equiv -\int_{s_a}^{s_b} ds\, \tilde{L}. \tag{46}$$

The equation of motion may be found in the same manner and in the present application these turn out to be

$$m\frac{d^2x^\alpha}{d\tau^2} = \frac{e}{c}(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\frac{dx_\beta}{d\tau}, \tag{47}$$

and they are correct, as one may show by comparing them with the standard forms. The corresponding canonical momenta are

$$\pi^\alpha = \frac{\partial \tilde{L}}{\partial(dx_\alpha/ds)} = mU^\alpha + \frac{e}{c}A^\alpha. \tag{48}$$

Hence the Hamiltonian is

$$\tilde{H} = \pi_\alpha U^\alpha - \tilde{L} = \frac{1}{2m}\left(\pi_\alpha - \frac{eA_\alpha}{c}\right)\left(\pi^\alpha - \frac{eA^\alpha}{c}\right) - \frac{1}{2}mc^2. \tag{49}$$

Hamilton's equations of motion[5] are

$$\begin{aligned} \frac{dx^\alpha}{d\tau} &= \frac{\partial \tilde{H}}{\partial \pi_\alpha} = \frac{1}{m}\left(\pi^\alpha - \frac{e}{c}A^\alpha\right) \\ \frac{d\pi^\alpha}{d\tau} &= -\frac{\partial \tilde{H}}{\partial x_\alpha} = \frac{e}{mc}\left(\pi_\beta - \frac{eA_\beta}{c}\right)\partial^\alpha A^\beta \end{aligned} \tag{50}$$

2 Lagrangian for the Electromagnetic Field

The electromagnetic field and fields in general have continuous degrees of freedom. The analog of a generalized coordinate q_i is the value of a field ϕ_k at a point $\bar{x}$. There are an infinite number of such points and so we have an infinite number of generalized

[5]For $H(p,q)$ Hamiltons equations are $\dot{q}_i = \frac{\partial H}{\partial p_i}$, are $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, and $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$

coordinates. The corresponding "generalized velocities" are derivatives of the field with respect to the variables, $\partial\phi_k(\bar{x})/\partial x^\alpha$ or $\partial\phi_k(\bar{x})\partial x_\alpha$ with $\alpha = 0, 1, 2, 3$.

$$\begin{aligned} q_i &\rightarrow \phi_k(\bar{x}) \\ \dot{q}_i &\rightarrow \frac{\partial\phi_k(\bar{x})}{\partial x^\alpha} \end{aligned} \tag{51}$$

Instead of a Lagrangian L which depends on the coordinates and velocities q_i and $\dot{q}_i$, one now has a *Lagrangian density* $\mathcal{L}$, and the Lagrangian is obtained by integrating this density over position space,

$$L = \int d^3x\ \mathcal{L}(\phi_k(\bar{x}), \partial^\alpha\phi_k(\bar{x})); \tag{52}$$

The action is the integral of this over time, or

$$A = \int d^4x\, \mathcal{L}(\phi_k(\bar{x})\,,\, \partial^\alpha\phi_k(\bar{x})). \tag{53}$$

Given that A and d^4x are invariants, $\mathcal{L}$ must also be an invariant.

The Euler-Lagrange equations of motion are obtained as usual by demanding that A be an extremum with respect to variation of the fields, or

$$\delta A/\delta\phi_k(\bar{x}) = 0 \tag{54}$$

for each field ϕ_k. The resulting equations are, explicitly,

$$\partial^\beta\left(\frac{\partial\mathcal{L}}{\partial(\partial^\beta\phi_k)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_k} = 0. \tag{55}$$

Now let's turn to the question of an appropriate Lagrangian density for the electromagnetic field. The things we have to work with are $F^{\alpha\beta}$, A^α, and J^α, if we rule out explicit dependence on space and time (a translationally invariant universe). We must make an invariant out of these. One which practically suggests itself is

$$\mathcal{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha. \tag{56}$$

The various constants are a matter of definition; otherwise we have something which is linear in components of $\bar{A}$ and of $\bar{J}$, and bilinear derivatives of components of $\bar{A}$. Let's write it out in detail:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{16\pi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \frac{1}{c}J_\alpha A^\alpha \\ &= -\frac{1}{16\pi}g_{\alpha\gamma}g_{\beta\delta}(\partial^\gamma A^\delta - \partial^\delta A^\gamma)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \frac{1}{c}J_\alpha A^\alpha \end{aligned} \tag{57}$$

The generalized fields (called ϕ_k above) are the components of $\bar{A}$. Hence the functional derivatives of $\mathcal{L}$ which enter the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = \frac{1}{4\pi}F_{\alpha\beta} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c}J_\alpha \tag{58}$$

and so the equations of motion are

$$\frac{1}{4\pi}\partial^\beta F_{\beta\alpha} = \frac{1}{c}J_\alpha. \tag{59}$$

These are indeed the four[6] inhomogeneous Maxwell equations. The homogeneous equations are automatically satisfied because we have constructed the Lagrangian in terms of the potentials. The charge continuity equation follows from taking the contravariant derivative of the equation above,

$$\frac{1}{4\pi}\partial^\alpha\partial^\beta F_{\beta\alpha} = \frac{1}{c}\partial^\alpha J_\alpha; \tag{60}$$

the left-hand side is zero when summed because $F_{\alpha\beta} = -F_{\beta\alpha}$ and so we have

$$\partial^\alpha J_\alpha = 0. \tag{61}$$

3 Stress Tensors and Conservation Laws

Conservation of energy emerges from the usual Lagrangian formulation if L has no explicit dependence on the time; then $dH/dt = 0$ which means that the Hamiltonian

[6] Four equations come from the one scalar and one vector inhomogeneous Maxwell's equations

is a constant of the motion. If we want to carry this sort of thing over to our field theory, we need to construct a *Hamiltonian density* $\mathcal{H}$ whose integral over all position space, H, is interpreted as the energy. If one proceeds in analogy with the particle case, he would take a Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi_k(\bar{x}), \partial^\alpha \phi_k(\bar{x})); \tag{62}$$

introduce momentum fields

$$\Pi_k(\bar{x}) \equiv \partial \mathcal{L} / \partial(\partial \phi_k / \partial t); \tag{63}$$

and a Hamiltonian density

$$\mathcal{H} = \sum_k \Pi_k(\bar{x})(\partial \phi_k(\bar{x})/\partial t) - \mathcal{L}. \tag{64}$$

We are going to generalize this procedure by introducing a rank-two tensor instead of a simple Hamiltionian density. The reason is that if one has a simple density and introduces H as

$$H = \int d^3x \, \mathcal{H} = \int dx_0 \frac{d^3x}{dx_0} \mathcal{H}, \tag{65}$$

and if one wants this to be an energy, which, as we have seen, transforms as the 0^{th} component of a four-vector, then $\mathcal{H}$ should be the (0,0) component of a rank-two tensor. To this end, let us introduce

$$\psi_k^\alpha(\bar{x}) \equiv \partial \mathcal{L} / \partial(\partial_\alpha \phi_k) \tag{66}$$

and

$$T^{\alpha\beta} \equiv \sum_k \psi_k^\alpha(\bar{x}) \partial^\beta \phi_k - g^{\alpha\beta} \mathcal{L}. \tag{67}$$

This rank-two tensor is called the *canonical stress tensor.*

3.1 Free Field Lagrangian and Hamiltonian Densities

Let's see what form the Lagrangian density and canonical stress tensor take in the absence of any sources $\mathbf{J}^\alpha$. In this case the Lagrangian density becomes $\mathcal{L}_{ff}$, the free-field Lagrangian density.

$$\mathcal{L}_{ff} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} \tag{68}$$

By carrying out the implied manipulations we find

$$T^{\alpha\beta} = -\frac{1}{4\pi} g^{\alpha\gamma} F_{\gamma\delta} \partial^\beta A^\delta - g^{\alpha\beta} \mathcal{L}_{ff}. \tag{69}$$

Look in particular at T^{00}:

$$\begin{aligned} T^{00} &= -\frac{1}{4\pi}(F_{0\gamma}\partial^0 A^\gamma) - \frac{1}{8\pi}(E^2 - B^2) \\ &= -\frac{1}{4\pi}\left[E_x \frac{1}{c}\frac{\partial A_x}{\partial t} + E_y \frac{1}{c}\frac{\partial A_y}{\partial t} + E_z \frac{1}{c}\frac{\partial A_z}{\partial t}\right] - \frac{1}{8\pi}(E^2 - B^2) \\ &= \frac{1}{4\pi}E^2 + \mathbf{E}\cdot\nabla\Phi - \frac{1}{8\pi}(E^2 - B^2) = \frac{1}{8\pi}(E^2 + B^2) + \frac{\mathbf{E}\cdot\nabla\Phi}{4\pi}. \end{aligned} \tag{70}$$

This contains the expected and desired term $(E^2 + B^2)/8\pi$, which is the feild energy density, but there is an additional term $\mathbf{E}\cdot\nabla\Phi$. Because $\nabla\cdot\mathbf{E} = 0$ for free fields, it is the case that $\mathbf{E}\cdot\nabla\Phi = \nabla\cdot(\mathbf{E}\Phi)$ and so the integral over all space of this part of T^{00} will vanish for a localized field distribution. Hence we find that

$$\int d^3x\, T^{00} = \frac{1}{8\pi}\int d^3x\,(E^2 + B^2) \tag{71}$$

is indeed the field energy.

And what of the other components of the stress tensor? These too have some unexpected properties. For example, one can show that

$$T^{0i} = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})_i + \frac{1}{4\pi}\nabla\cdot(A_i\mathbf{E}) \tag{72}$$

and

$$T^{i0} = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})_i + \frac{1}{4\pi}\left[(\nabla\times(\Phi\mathbf{B}))_i - \frac{\partial}{\partial x_0}(\Phi E_i)\right]. \tag{73}$$

Evidently, this tensor is not symmetric. Also, one would have hoped that these components of the tensor would have turned out to be components of the Poynting vector, with appropriate scaling, so that we would have found an equation $0 = \partial_\alpha T^{0\alpha}$ which would have been equivalent to the Poynting theorem,

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0. \tag{74}$$

Although this is not going to happen, there is some sort of conservation law contained in our stress tensor. One can show that

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{75}$$

which gives not one but four conservation laws. To demonstrate this equation, consider the following:

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= \sum_k \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k \right] - \partial^\beta \mathcal{L} \\ = \sum_k \Bigg[\partial_\alpha &\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \right) \partial^\beta \phi_k + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \right) \partial^\beta (\partial_\alpha \phi_k) \Bigg] - \partial^\beta \mathcal{L} \\ &= \sum_k \left[\left(\frac{\partial \mathcal{L}}{\partial \phi_k} \right) \partial^\beta \phi_k + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \right) \partial^\beta (\partial_\alpha \phi_k) \right] - \partial^\beta \mathcal{L} \end{aligned} \tag{76}$$

where we have used the Euler-Lagrange equations of motion Eq. (55) on the first term in the middle line. Now we can recognize that the terms summed over k in the last line are $\partial^\beta \mathcal{L}$ since $\mathcal{L}$ is a function of the fields ϕ_k and their derivatives $\partial_\alpha \phi_k$. Hence we have demonstrated that

$$\partial_\alpha T^{\alpha\beta} = \partial^\beta \mathcal{L} - \partial^\beta \mathcal{L} = 0 \tag{77}$$

These give familiar global conservation laws when integrated over all space for a localized set of fields. Consider

$$0 = \int d^3x \, \partial_\alpha T^{\alpha\beta} = \frac{\partial}{\partial x^0} \left(\int d^3x \, T^{0\beta} \right) + \int d^3x \, \frac{\partial}{\partial x^i} (T^{i\beta}). \tag{78}$$

The last term on the right-hand side is zero as one shows by integrating over that coordinate with respect to which the derivative is taken and appealing to the fact that we have localized fields which vanish as $|x^i|$ becomes large. Hence our conclusion is that

$$\frac{d}{dt}\left(\int d^3x\, T^{0\beta}\right) = 0 \tag{79}$$

If one looks at the explicit components of the tensor, one finds that these simply say the total energy and total momentum are constant, using our identifications (from chapter 6) of u and $\mathbf{g}$ as the energy and momentum density.

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right) \quad \mathbf{g} = \frac{1}{4\pi c}\left(\mathbf{E}\times\mathbf{B}\right) \tag{80}$$

3.2 Symmetric Stress Tensor

It is troubling that the canonical stress tensor is not symmetric. This becomes a serious problem when one examines the angular momentum. Consider the rank-three tensor

$$M^{\alpha\beta\gamma} \equiv T^{\alpha\beta}x^\gamma - T^{\alpha\gamma}x^\beta. \tag{81}$$

If this is to represent the angular momentum in some way we would like it to provide a conservation law in the form $\partial_\alpha M^{\alpha\beta\gamma} = 0$. But that doesn't happen. Rather,

$$\partial_\alpha M^{\alpha\beta\gamma} = T^{\gamma\beta} - T^{\beta\gamma} + (\partial_\alpha T^{\alpha\beta})x^\gamma - (\partial_\alpha T^{\alpha\gamma})x^\beta = T^{\gamma\beta} - T^{\beta\gamma} \tag{82}$$

which doesn't vanish because $\bar{T}$ is not symmetric.

The standard way out of this and other difficulties associated with the asymmetry of the canonical stress tensor is to define a different stress tensor which works. By regrouping terms in the canonical stress tensor one can write

$$T^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\gamma}F_{\gamma\delta}F^{\delta\beta} + \frac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right) - \frac{1}{4\pi}g^{\alpha\gamma}F_{\gamma\delta}\partial^\delta A^\beta. \tag{83}$$

Now, the second term is

$$\begin{aligned}-\frac{1}{4\pi}g^{\alpha\gamma}F_{\gamma\delta}\partial^\delta A^\beta = -\frac{1}{4\pi}F^{\alpha\delta}\partial_\delta A^\beta = \frac{1}{4\pi}F^{\delta\alpha}\partial_\delta A^\beta = \\ \frac{1}{4\pi}(F^{\delta\alpha}\partial_\delta A^\beta + A^\beta\partial_\delta F^{\delta\alpha}) = \frac{1}{4\pi}\partial_\delta(F^{\delta\alpha}A^\beta).\end{aligned} \tag{84}$$

This is a four-divergence, so for fields of finite extent, it must be the case that

$$\int d^3x\,\partial_\delta(F^{\delta 0}A^\beta) = 0. \tag{85}$$

Moreover, it has a vanishing four-divergence,

$$\partial_\alpha\partial_\delta(F^{\delta\alpha}A^\beta) = 0, \tag{86}$$

which follows from the antisymmetric character of the field tensor. Hence, if we simply remove this piece from the stress tensor, leaving a new tensor $\bar{\theta}$, known as the *symmetric stress tensor*,

$$\theta^{\alpha\beta} \equiv \frac{1}{4\pi}(g^{\alpha\gamma}F_{\gamma\delta}F^{\delta\beta} + \frac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}), \tag{87}$$

then this tensor is such that

$$\frac{d}{dt}\left(\int d^3x\,\theta^{0\beta}\right) = 0 \qquad \text{and} \qquad \partial_\alpha\theta^{\alpha\beta} = 0. \tag{88}$$

It is easy to work out the components of this tensor; they are $(i,j = 1,2,3)$

$$\begin{aligned} \theta^{00} &= \tfrac{1}{8\pi}(E^2+B^2) \\ \theta^{i0} &= \theta^{0i} = \tfrac{1}{4\pi}(\mathbf{E}\times\mathbf{B}) \\ \theta^{ij} &= -\tfrac{1}{4\pi}[E_iE_j + B_iB_j - \tfrac{1}{2}\delta_{ij}(E^2+B^2)]. \end{aligned} \tag{89}$$

Hence in block matrix form,

$$\theta^{\alpha\beta} = \begin{pmatrix} u & c\mathbf{g} \\ c\mathbf{g} & -T^{(M)}_{ij} \end{pmatrix} \tag{90}$$

where $T^{(M)}_{ij}$ is the ij component of the Maxwell stress tensor. The conservation laws

$$\partial_\alpha\theta^{\alpha\beta} = 0 \tag{91}$$

are well-known to us. They are the Poynting theorem, for $\beta = 0$, and the momentum conservation laws

$$\frac{\partial g_i}{\partial t} - \sum_j \frac{\partial T^{(M)}_{ij}}{\partial x_j} = 0. \tag{92}$$

when $\beta = i$

Now consider once again the question of angular momentum. Define

$$M^{\alpha\beta\gamma} \equiv \theta^{\alpha\beta} x^{\gamma} - \theta^{\alpha\gamma} x^{\beta}. \tag{93}$$

Then the equations

$$\partial_{\alpha} M^{\alpha\beta\gamma} = 0 \tag{94}$$

express angular momentum conservation as well as some other things.

3.3 Conservation Laws in the Presence of Sources

Finally, what happens if there are sources? Then we won't find the same form for the conservation laws. Consider

$$\begin{aligned} \partial_{\alpha}\theta^{\alpha\beta} &= \frac{1}{4\pi}\left[\partial^{\gamma}(F_{\gamma\delta}F^{\delta\beta}) + \frac{1}{4}\partial^{\beta}(F_{\gamma\delta}F^{\gamma\delta})\right] \\ = \frac{1}{4\pi}&\left[(\partial^{\gamma}F_{\gamma\delta}F^{\delta\beta} + F_{\gamma\delta}(\partial^{\gamma}F^{\gamma\beta}) + \frac{1}{2}F_{\gamma\delta}(\partial^{\beta}F^{\gamma\delta})\right]. \end{aligned} \tag{95}$$

Making use of the Maxwell equations $\partial^{\gamma}F_{\gamma\delta} = \frac{4\pi}{c}J_{\delta}$, we can rewrite this as

$$\partial_{\alpha}\theta^{\alpha\beta} + \frac{1}{c}F^{\beta\delta}J_{\delta} = \frac{1}{8\pi}\left[F_{\gamma\delta}(\partial^{\gamma}F^{\delta\beta} + \partial^{\gamma}F^{\delta\beta} + \partial^{\beta}F^{\gamma\delta})\right]. \tag{96}$$

Now recall that (these are the homogeneous Maxwell's equations)

$$\partial^{\gamma}F^{\delta\beta} + \partial^{\beta}F^{\gamma\delta} + \partial^{\delta}F^{\beta\gamma} = 0, \tag{97}$$

so Eq. (90) may be written as

$$\partial_{\alpha}\theta^{\alpha\beta} + \frac{1}{c}F^{\beta\delta}J_{\delta} = \frac{1}{8\pi}F_{\gamma\delta}(\partial^{\gamma}F^{\delta\beta} - \partial^{\delta}F^{\beta\gamma}). \tag{98}$$

However,

$$(\partial^{\gamma}F^{\delta\beta} - \partial^{\delta}F^{\beta\gamma})F_{\gamma\delta} = (\partial^{\gamma}F^{\delta\beta} + \partial^{\delta}F^{\gamma\beta})F_{\gamma\delta} \tag{99}$$

is a contraction of an object symmetric in the indices γ and δ and one which is antisymmetric; therefore it is zero. Hence we conclude that

$$\partial_{\alpha}\theta^{\alpha\beta} = -\frac{1}{c}F^{\beta\delta}J_{\delta}. \tag{100}$$

The four equations contained in this conservation law are the familiar ones

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} \quad \text{when } \beta = 0 \tag{101}$$

and

$$\frac{\partial g_i}{\partial t} - \sum_j \frac{\partial}{\partial x^j} T_{ij}^{(M)} = -[\rho E_i + \frac{1}{c}(\mathbf{J} \times \mathbf{B})_i] \quad \text{when } \beta = i\,. \tag{102}$$

4 Examples of Relativistic Particle Dynamics

4.1 Motion in a Constant Uniform Magnetic Induction

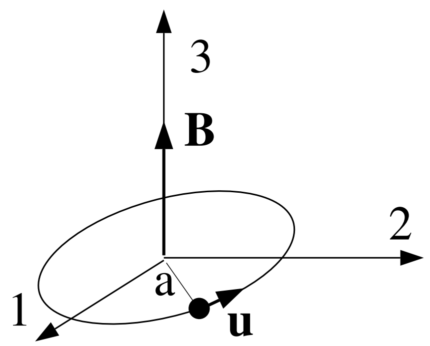
Given an applied constant magnetic induction, the equations of motion for a particle of charge q are

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{u} = 0, \qquad \frac{d\mathbf{p}}{dt} = \frac{q}{c}(\mathbf{u} \times \mathbf{B}) = m\gamma \frac{d\mathbf{u}}{dt} \tag{103}$$

where the last step follows from the fact that $\mathbf{p} = m\gamma\mathbf{u}$ and the fact that γmc^2, the particle's energy, is constant because magnetic forces do no work. Hence the equations reduce to

$$\frac{d\mathbf{u}}{dt} = \mathbf{u} \times \omega_B \tag{104}$$

where $\omega_B = q\mathbf{B}/m\gamma c$. Notice that this frequency depends on the energy of the particle. For definiteness, let $\mathbf{B} = B\boldsymbol{\epsilon}_3$. Also, write $\mathbf{u} = u_{\|}\boldsymbol{\epsilon}_3 + \mathbf{u}_{\perp}$ where $\mathbf{u}_{\perp} \cdot \boldsymbol{\epsilon}_3 = 0$.



From the equations of motion, one can see that $u_{\|}$ is a constant while $\mathbf{u}_{\perp}$ obeys

$$\frac{d\mathbf{u}_{\perp}}{dt} = \omega_B(\mathbf{u}_{\perp} \times \boldsymbol{\epsilon}_3), \tag{105}$$

or

$$\frac{du_x}{dt} = \omega_B u_y \qquad \text{and} \qquad \frac{du_y}{dt} = -\omega_B u_x. \tag{106}$$

Combining these we find, e.g.,

$$\frac{d^2 u_x}{dt^2} = -\omega_B^2 u_x \tag{107}$$

with the general solution

$$u_x = u_0 e^{-i\omega_B t} \tag{108}$$

where u_0 is a complex constant. Further,

$$u_y = \frac{1}{\omega_B}\frac{du_x}{dt} = -iu_x, \tag{109}$$

so

$$\mathbf{u}_\perp = u_0(\boldsymbol{\epsilon_1} - i\boldsymbol{\epsilon_2})e^{-i\omega_B t}. \tag{110}$$

We may integrate over time to find the trajectory:

$$\frac{d\mathbf{x}}{dt} = u_\| \boldsymbol{\epsilon_3} + \mathbf{u}_\perp \tag{111}$$

and so

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}(0) + \int_0^t dt' \left[u_\| \boldsymbol{\epsilon_3} + u_0(\boldsymbol{\epsilon_1} - i\boldsymbol{\epsilon_2})e^{-i\omega_B t'} \right] \\ &= \mathbf{x}(0) + u_\| t\boldsymbol{\epsilon_3} + i\frac{u_0}{\omega_B}(\boldsymbol{\epsilon_1} - i\boldsymbol{\epsilon_2})(e^{-i\omega_B t} - 1). \end{aligned} \tag{112}$$

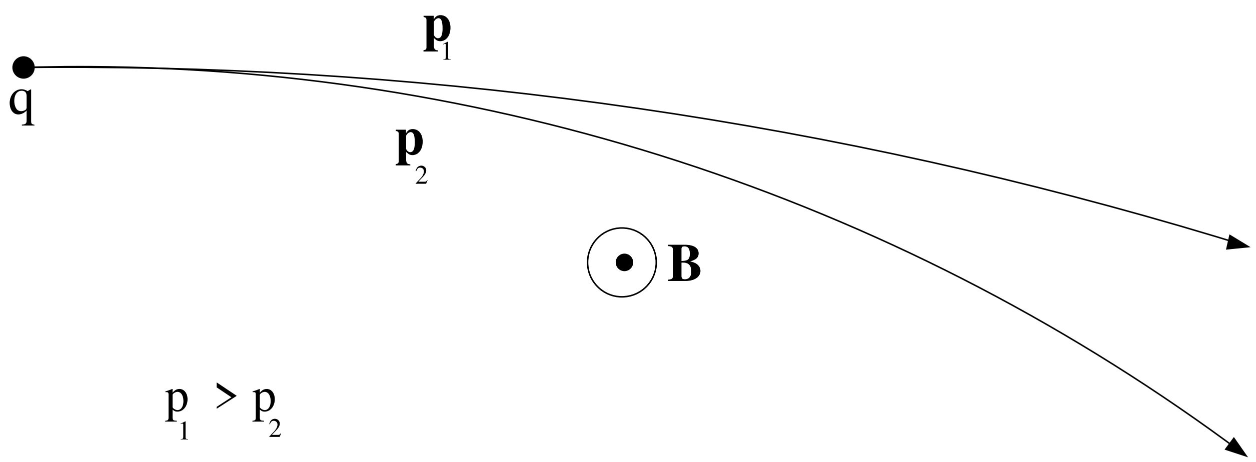
The physical trajectory is the real part of this and is, for real u_0,

$$\mathbf{x}(t) = \mathbf{x}(0) + u_\| t\boldsymbol{\epsilon_3} + \frac{u_0}{\omega_B}[\sin(\omega_B t)\boldsymbol{\epsilon_1} + (\cos(\omega_B t) - 1)\boldsymbol{\epsilon_2}]. \tag{113}$$

This equation describes helical motion with the helix axis parallel to the z-axis. The radius of the axis is a, where $a = u_0/\omega_B$.

It is worthwhile to establish the connection betwen a and $|\mathbf{p}_\perp|$ where $\mathbf{p}_\perp = m\gamma\mathbf{u}_\perp$ is the momentum in the plane perpendicular to the direction of $\mathbf{B}$.

$$p_\perp = m\gamma u_0 = m\gamma\omega_B a = m\gamma\frac{qB}{m\gamma c}a = \frac{qBa}{c}. \tag{114}$$



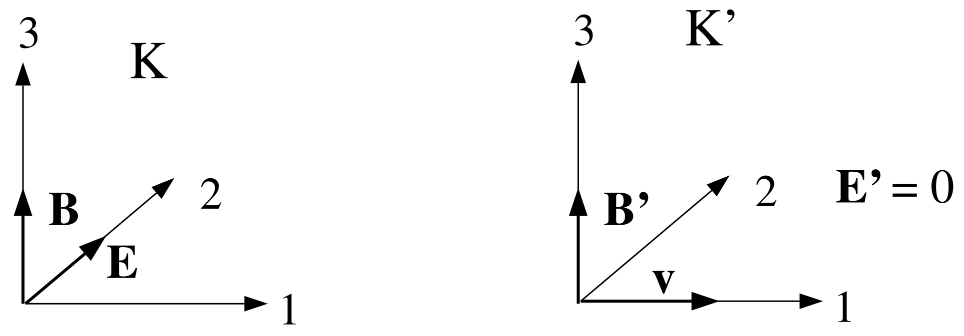
This relation, $p_\perp = qBa/c$, tells us the radius of curvature in the plane perpendicular to $\mathbf{B}$ (which is not the same as the radius of curvature of the orbit) is a linear function of $p_\perp$, and it suggests a simple way to select particles of a given momentum out of a beam containing particles with many momenta. One simply passes the beam through a region of space where there is some $\mathbf{B}$ applied transverse to the direction of the beam. The amount by which a particle is deflected will increase with decreasing $p_\perp$ and so the beam is spread out much as a prism separates the different frequency components of a beam of light. The device is a momentum selector.

4.2 Motion in crossed E and B fields, $E < B$.

For $\mathbf{E}\cdot\mathbf{B} = 0$ in frame K, we can find a frame K' where $\mathbf{E}' = 0$, provided $E < B$. This may be seen from the form of the field transforms.

$$\begin{array}{ll} \mathbf{E}'_\parallel = \mathbf{E}_\parallel & \mathbf{E}'_\perp = \gamma[\mathbf{E}_\perp + (\boldsymbol{\beta}\times\mathbf{B})] \\ \mathbf{B}'_\parallel = \mathbf{B}_\parallel & \mathbf{B}'_\perp = \gamma[\mathbf{B}_\perp - (\boldsymbol{\beta}\times\mathbf{E})] \end{array} \tag{115}$$

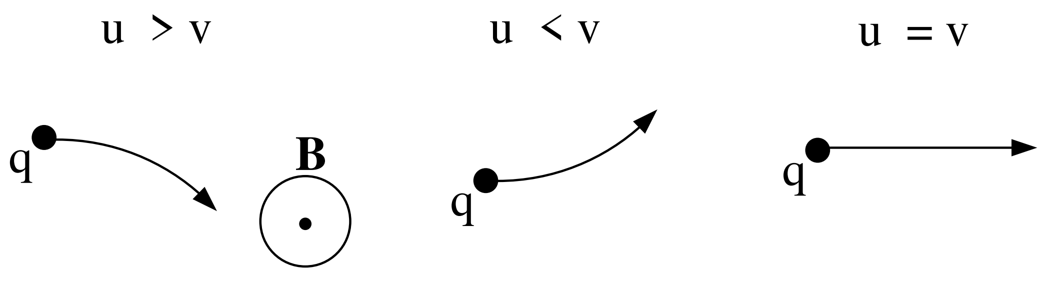
In fact, we have already solved exactly this problem in chapter 11 where we found that K' moves relative to K with a velocity which is $\mathbf{v} = c(\mathbf{E}\times\mathbf{B})/B^2$. If we let $\mathbf{E} = E\boldsymbol{\epsilon}_2$ and $\mathbf{B} = B\boldsymbol{\epsilon}_3$, then $\mathbf{v} = c(E/B)\boldsymbol{\epsilon}_1$, and $\mathbf{B}' = B\sqrt{1 - E^2/B^2}\boldsymbol{\epsilon}_3$.



Now imagine a particle is injected into this system with an initial velocity[7] $\mathbf{u}(0) = u_0 \boldsymbol{\epsilon_1}$. In the frame K', its initial velocity is

$$\mathbf{u}'(0) = \frac{u_0 - v}{1 + u_0 v/c^2} \boldsymbol{\epsilon_1}. \tag{116}$$

From our first example, we know that the particle will proceed to execute circular motion in this frame, always with the same speed $u'(0)$. What then is its motion in frame K? Superposed on the circular motion will be a drift velocity $\mathbf{v}$. If $q > 0$ and $u_0 > v$, we get the first motion shown below. But if $u_0 < v$, we get the second motion. For the special case of $u_0 = v$, the particle is at rest in K' which means it moves in K at a constant velocity $\mathbf{u}(t) = \mathbf{v}$.



[7]We could be more general and include a component of $\mathbf{u}$ parallel to $\mathbf{B}$; this would not lead to anything significantly different from what we are about to find.

Such a device can be employed as a velocity selector and so it complements the device described in the first example which was a momentum selector. The idea is that a particle coming in with a speed u_0 greater than v will experience a magnetic force greater than the electric force and so it will be deflected accordingly. But one coming in with a speed smaller than v will experience an electric force greater than the magnetic one, and it will be deflected in the other direction.

The picture chages after a while, however, because the particle will speed up and slow down under the influence of the electric field. Suppose that initially $u_0 > v$ ($u_0 < v$). Then the B-field (the E-field) force dominates, and the particle is deflected in such a way that it moves against (with) the electric field. This causes it to slow down (speed up) so that after some time $u_0 < v$ ($u_0 > v$). Then the electric (magnetic) field force dominates, causing the particle to swing around so that it eventually moves with (against) the electric field force. And so on. The end result is a trajectory that produces a time-averaged velocity equal to $\mathbf{v}$ or $c(\mathbf{E} \times \mathbf{B})/B^2$. This is called the *$E$ cross B drift velocity*. It is in the direction of $\mathbf{E} \times \mathbf{B}$ no matter what is the sign of the charge.

4.3 Motion in crossed E and B fields, $E > B$

This time we want to consider the motion in a frame K' moving at velocity $\mathbf{v} = c(\mathbf{E} \times \mathbf{B})/E^2 \to c(B/E)\boldsymbol{\epsilon}_1$, if we keep the same directions of the fields as in the preceding example. In this frame there is only an electric field $\mathbf{E}' = \mathbf{E}\sqrt{1 - B^2/E^2}$ which will cause the particle to move away in the direction of $\mathbf{E}'$. The equations of motion in K' are

$$mc^2 \frac{d\gamma'}{dt'} = qE' \frac{dy'}{dt'} \quad \text{and} \quad \frac{dp'_y}{dt'} = qE'; \tag{117}$$

the components of the momentum in the other directions are constant. One easily solves to find

$$\mathbf{p}'(t') = \mathbf{p}'(0) + qE't'\boldsymbol{\epsilon}_2 \tag{118}$$

and we can then find γ' directly from the dispersion relation,

$$\gamma' = \frac{1}{mc^2}\sqrt{m^2c^4 + \mathbf{p}'(t')\cdot\mathbf{p}'(t')c^2}. \tag{119}$$

The speed u'_y is found easily from the equation of motion for γ' which integrates trivially to produce

$$\begin{aligned} y'(t') &= y'(0) + \frac{mc^2}{qE'}(\gamma'(t') - \gamma'(0)) \\ &= y'(0) + \frac{\sqrt{m^2c^4 + (\mathbf{p}'(t'))^2c^2} - \sqrt{m^2c^4 + (\mathbf{p}'(0))^2c^2}}{qE'}. \end{aligned} \tag{120}$$

Consider also $\mathbf{x}'_\perp$, the component of $\mathbf{x}'$ perpendicular to the electric field. Because $\mathbf{p}'_\perp/dt = 0$, it is true that $\gamma'\mathbf{y}'_\perp = \gamma'(0)\mathbf{u}'_\perp(0)$, a constant. Hence

$$\mathbf{u}'_\perp(t') = \mathbf{u}'_\perp(0)\sqrt{1 + (\mathbf{p}'(0))^2/m^2c^2}/\sqrt{1 + (\mathbf{p}'(t'))^2/m^2c^2}. \tag{121}$$

We can integrate the velocity over time to find the displacement of the particle. For the special case that there is no component of $\mathbf{p}'(0)$ in the direction of the field, one finds that

$$\mathbf{x}'_\perp(t) - \mathbf{x}'_\perp(0) = \frac{\mathbf{p}'_\perp(0)}{qE'}\ln\left[\frac{qE't'}{m\gamma(0)} + \sqrt{1 + \left(\frac{qE't'}{m\gamma(0)}\right)^2}\right]. \tag{122}$$

We can combine Eqs. (113) and (115) to remove the time and so have an equation that determines the shape of the trajectory. For simplicity, let $\mathbf{x}'_\perp(0) = y'(0) = 0$. Then one finds

$$\frac{x'_\perp qE'}{m\gamma'(0)u'(0)} = \ln\left[\sqrt{\left(1 + \frac{qE'y'}{m\gamma'(0)}\right)^2 - 1} + 1 + \frac{qE'y'}{m\gamma'(0)}\right]. \tag{123}$$

For short times satisfying the condition $|qE'y'/m\gamma'(0)| << 1$, the trajectory is a parabola,

$$\frac{\mathbf{x}'_\perp qE'}{m\gamma'(0)u'(0)} \approx \sqrt{\frac{2qE'y'}{m\gamma'(0)}} \tag{124}$$

or

$$y' = \frac{qE'x'^2_\perp}{2m\gamma'(0)(u'(0))^2}. \tag{125}$$

The long time behavior is displayed for $|qE'y'/m\gamma'(0)| >> 1$, and it is such that

$$y' = \frac{m\gamma'(0)}{2qE'} \exp\left(\frac{x'_{\perp} qE'}{m\gamma'(0)u'(0)}\right). \tag{126}$$

4.4 Motion for general uniform E and B.

Then we cannot find a frame where one of the fields can be made to vanish. But there is a frame where the electric field and magnetic induction are parallel; here the solution of the equations of motion is relatively simple and is left as an exercise.

4.5 Motion in slowly spatially varying $\mathbf{B}(\mathbf{x})$

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This problem is greatly simplified by (1) the fact that then energy, or γ, is a constant and by (2) the assumption that $\mathbf{B}(\mathbf{x})$ does not vary much relative to its magnitude over distances on the order of the radius of the particle's orbit.