Introduction to Electrostatics

Charles Augustin de Coulomb
(1736 - 1806)

December 22, 2000

Contents

We shall follow the approach of Jackson, which is more or less historical. Thus we start with classical electrostatics, pass on to magnetostatics, add time dependence, and wind up with Maxwell's equations. These are then expressed within the framework of special relativity. The remainder of the course is devoted to a broad range of interesting and important applications.

This development may be contrasted with the more formal and elegant approach which starts from the Maxwell equations plus special relativity and then proceeds to work out electrostatics and magnetostatics - as well as everything else - as special cases. This is the method of *e.g.,* Landau and Lifshitz, *The Classical Theory of Fields.*

The first third of the course, *i.e.,* Physics 707, deals with physics which should be familiar to everyone; what will perhaps not be familiar are the mathematical techniques and functions that will be introduced in order to solve certain kinds of problems. These are of considerable usefulness and therefore will be important to us.

1 Coulomb's Law

By performing experiments on small charged bodies (ideally, point charges), Charles Augustin de Coulomb, working around the time of the American and French revolutions (1785), was able to empirically infer that the force between two static charged particles is proportional to the inverse square of the distance between them. The following has since become known[1] have as *Coulomb's Law*: Given two static charges q_1 and q_2, there is a force acting on each of them which is:

1. Proportional to the product of the magnitudes of the charges, being attractive for unlike charges and repulsive for like charges
2. Inversely proportional to the square of the distance between the charges.
3. Directed along the line between the charges.

In the form of an equation, the law states that

$$\mathbf{F}_{21} = k\frac{q_1 q_2}{|\mathbf{x}_2 - \mathbf{x}_1|^2}\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|} \tag{1}$$

where charge q_i is located at $\mathbf{x}_i$, $\mathbf{F}_{21}$ is the force on charge 2 produced by charge 1, and k is a positive constant; vectors are denoted by boldface type.

In addition, the force satisfies a superposition law (or principle) in that the force $\mathbf{F}$ on a charge q in the presence of a number of other charges q_i at $\mathbf{x}_i$, $i = 1,\ldots,n$, is simply the sum of the forces arising from each of the latter as though it were the only other charge present[2]. Thus,

$$\mathbf{F} = kq\sum_{i=1}^{n}\frac{q_i(\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3}, \tag{2}$$

[1] Numerous others, such as Henry Cavendish, also may legitimately have some claim to the law.

[2] As we shall see, the principle of superposition follows from the linearity of Maxwell's Equations

given that q is at $\mathbf{x}$.

The constant k has units and magnitude which depend on the system of units employed. We shall adopt cgs Gaussian units. The units of mass length and time are, respectively grams (g), centimeters (cm), and seconds (sec). The unit of charge is the statcoulomb ($statcoul$) which is defined by the statement that the force between two charges, each of one $statcoul$, one cm apart is one dyne (dyn). Then $k = 1\, dyn - cm^2/(statcoul)^2$. In practice one may treat k as having dimension unity while charge has dimension of $M^{1/2}L^{3/2}/T$.

2 Electric Field

It is customary and useful to introduce the concept of the electric field at this point. This is a vector field, *i.e.,* a vector function of $\mathbf{x}$. It is written as $\mathbf{E}(\mathbf{x})$ and is defined as the force that would be experienced by a charge q at $\mathbf{x}$, divided by q[3]. Thus, for a distribution of charges q_i at $\mathbf{x}_i$, i=1,2,...,n,

$$\mathbf{E}(\mathbf{x}) = \sum_{i=1}^{n} \frac{q_i(\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3} \tag{3}$$

The electric field has the property of being independent of the 'test' charge q; it is a function of the charge distribution which gives rise to the force on the test charge, and, of course, of the test charge's position. This object has dimension Q/L^2 or $M^{1/2}/L^{1/2}T$.

[3] This definition is not complete. The field has other attributes as well since it carries momentum and energy: i.e. photons

At this point let us introduce the charge density $\rho(\mathbf{x})$ which is the charge per unit volume at, or very close to, $\mathbf{x}$. This object is needed if we would like to integrate over a source distribution instead of summing over its constituent charges. Thus a sum is replaced by an equivalent integral,

$$\sum_{i=1}^{n} q_i \rightarrow \int d^3x\, \rho(\mathbf{x}). \tag{4}$$

The charge density has dimension Q/L^3. In terms of ρ, the expression for the electric field can be written as

$$\mathbf{E}(\mathbf{x}) = \int d^3x'\, \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{5}$$

In the particular case of a distribution of discrete point charges, it is possible to recover the sum in Eq. (3) by writing the charge density in an appropriate way. To do so we introduce the Dirac delta function $\delta(x-a)$. It is defined by the integral

$$f(a) = \int dx f(x) \delta(x-a) \tag{6}$$

where $f(x)$ is an arbitrary continuous function of x, and the range of integration includes the point $x = a$. A special case is $f(x) = 1$ which leads to

$$\int dx \delta(x-a) = 1, \tag{7}$$

demonstrating the normalization of the delta function. From the arbitrariness of $f(x)$, we may conclude that $\delta(x-a)$ is zero when x is not a and sufficiently singular at $x = a$ to give the normalization property.

In other words, it is in essence the charge density of a point charge (in one dimension) located at $x = a$.

Some important relations involving delta functions are as follows:

$$\int_{a_1}^{a_2} f(x)\frac{d\delta(x-a)}{dx}dx = -\left.\frac{df(x)}{dx}\right|_{x=a} \tag{8}$$

and

$$\int_{a_1}^{a_2} \delta[f(x)]dx = \sum_{i=1}^{N}\left[1/\left|\frac{df(x)}{dx}\right|_{x_i}\right] \tag{9}$$

In the final expression the x_i are the 0's of $f(x)$ between a_1 and a_2.

A delta function in three dimensions may be built as a product of three one-dimensional delta functions. In Cartesian coordinates,

$$\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z) \tag{10}$$

This function has the property that

$$\int d^3x\, f(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0) = f(\mathbf{x}_0) \tag{11}$$

Returning to electrostatics, we can see that the charge density of a collection of point charges can be written as a sum of delta functions:

$$\rho(\mathbf{x}) = \sum_{1=i}^{n} q_i\delta(\mathbf{x}-\mathbf{x}_i) \tag{12}$$

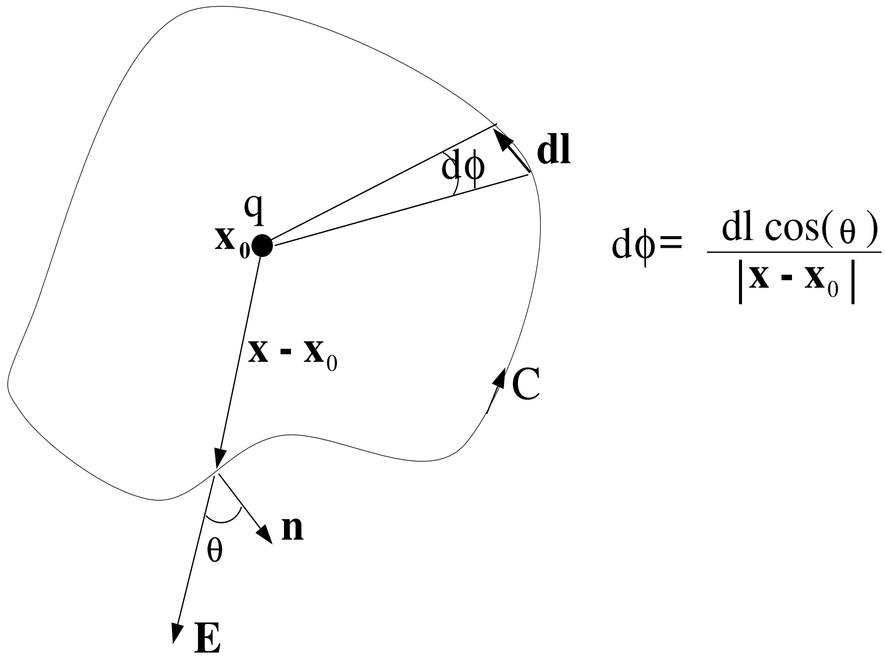
Thus

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= \int d^3x'\, \rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3 \\ &= \sum_{i=1}^{n}\int d^3x'\, q_i\delta(\mathbf{x}'-\mathbf{x}_i)(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3 \\ &= \sum_{i=1}^{n} q_i(\mathbf{x}-\mathbf{x}_i)/|\mathbf{x}-\mathbf{x}_i|^3.\end{aligned} \tag{13}$$

3 Gauss's Law

Although Coulomb's Law is quite sufficient for finding electric fields and forces, the integral form in which we have expressed it is not always the most useful approach to a problem. Another integral form, called Gauss's Law, is often more useful.

Let us look first at a **two-dimensional** version of this law. Consider a point charge q located within a closed path C. In two dimensions, the field produced by this charge is $2q(\mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|^2$. Consider now the integral around C of that component of $\mathbf{E}$ which is normal to the path. This normal component is $\mathbf{E}\cdot\mathbf{n} = q\cos\theta/r$, where r is the distance from the charge to the integration point on the loop. However, $dl\cos\theta/r$ is just the infinitesimal angle $d\phi$ subtended by dl at the charge. Hence we just need to integrate $d\phi$ around the loop.



$$\oint_C dl\mathbf{E}\cdot\mathbf{n} = \oint_C dl\frac{q\cos\theta}{|\mathbf{x}-\mathbf{x}_0|} = q\int d\phi \tag{14}$$

Since the charge is inside, the integral is 2π; if it were outside, the integral would be 0 because over one part of the path, $\cos\theta$ is positive and over another part it is negative with the two parts cancelling one another when the integration is completed. Thus one finds that

$$\int_C dl[\mathbf{n}\cdot\mathbf{E}(\mathbf{x})] = \begin{cases} 2\pi q, & q \text{ inside of C} \\ 0, & q \text{ outside of C} \end{cases} \tag{15}$$

The **three-dimensional** case works out much the same way. The field varies as $1/r^2$ and so one finds that $d^2x\cos\theta/r^2$ is the solid angle element $d\Omega$ subtended by the infinitesimal area element d^2x of S at the position of the charge. Integration over the surface thus reduces to integration over the solid angle subtended by the surface at the charge, and this is 4π if the charge in inside of the surface and 0 otherwise,

$$\int_S d^2x\,[\mathbf{E}(\mathbf{x})\cdot\mathbf{n}] = \begin{cases} 4\pi q, & q \text{ inside of S} \\ 0, & q \text{ outside of S} \end{cases} \tag{16}$$

Next, the superposition principle allows us to add up the fields arising from an arbitrary collection of charges, with Gauss's Law holding for each bit of charge. As a consequence, we may say that

$$\int_S d^2x\,[\mathbf{E}(\mathbf{x})\cdot\mathbf{n}] = 4\pi Q \tag{17}$$

where Q is the total charge contained inside of the surface,

$$Q = \int_V d^3x\,\rho(\mathbf{x}). \tag{18}$$

4 Differential Form of Gauss's Law

A differential form of this law may be found by applying the divergence theorem which states that, for a general vector field $C(\mathbf{x})$,

$$\int_S d^2x\,[C(\mathbf{x})\cdot\mathbf{n}] = \int_V d^3x\,[\nabla\cdot C(\mathbf{x})]. \tag{19}$$

Let us apply this equation to Gauss's Law:

$$4\pi\int_V d^3x\,\rho(\mathbf{x}) = \int_S d^2x\,[\mathbf{E}(\mathbf{x})\cdot\mathbf{n}] = \int_V d^3x\,[\nabla\cdot\mathbf{E}(\mathbf{x})] \tag{20}$$

or

$$\int_V d^3x\,[\nabla\cdot\mathbf{E}(\mathbf{x}) - 4\pi\rho(\mathbf{x})] = 0 \tag{21}$$

Because V is completely arbitrary, we may equate the integrand to zero and find

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \tag{22}$$

which is the differential form of Gauss's Law.

In the process of obtaining this equation from Coulomb's Law, we have lost some of the information contained in it. Merely specifying the divergence of a vector field is not sufficient to determine the field. Hence we need an additional equation to supplement Gauss's Law.

5 An Equation for $\nabla \times \mathbf{E}$; the Scalar Potential

Let us start once again from Coulomb's Law:

$$\mathbf{E}(\mathbf{x}) = \int d^3x'\, \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{23}$$

But one may write part of the integrand as a gradient,

$$\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right), \tag{24}$$

where the gradient is taken with respect to the variable $\mathbf{x}$. Hence

$$\mathbf{E}(\mathbf{x}) = -\nabla \int d^3x'\, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \tag{25}$$

which is to say, $\mathbf{E}$ can be written as the (negative) gradient of a scalar function of $\mathbf{x}$. This function we shall call the scalar potential and denote by $\Phi(\mathbf{x})$:

$$\Phi(\mathbf{x}) \equiv \int d^3x'\, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}; \tag{26}$$

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}). \tag{27}$$

From this statement it follows immediately that $\nabla \times \mathbf{E}(\mathbf{x}) = 0$ because the curl of the gradient of a scalar function is always zero.

To summarize,

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \tag{28}$$

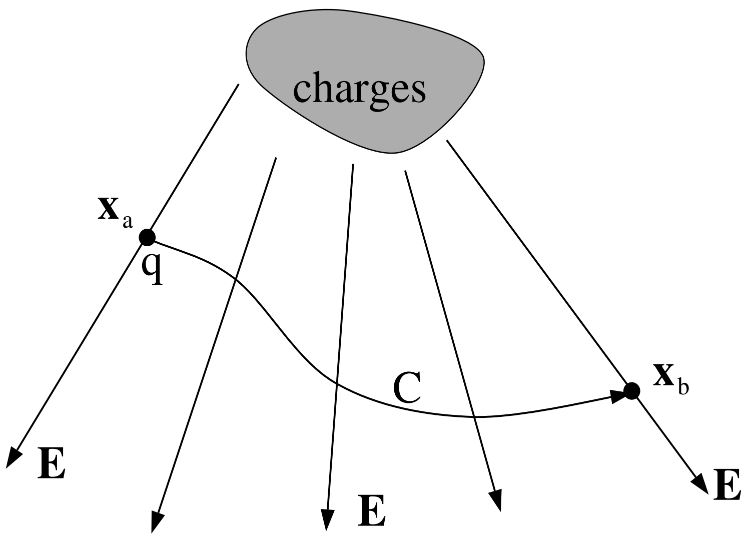
and

$$\nabla \times \mathbf{E}(\mathbf{x}) = 0. \tag{29}$$

5.1 Conservative Potentials

From our derivation of the curl equation, we can see that this simple result follows from the fact that the force (or electric field) is central and depends only on the distance between charges. Such a force is also called conservative, and the potential function is related in a simple way to the energy of a charge in an electric field.

To find this relation, consider that a set of fixed source charges produce a field $\mathbf{E}$ and that a charge q is placed at point $\mathbf{x}_a$. Here it experiences an electric field force $\mathbf{F} = q\mathbf{E}(\mathbf{x}_a)$ and so an equal and opposite force $\mathbf{F}_{ext} = -\mathbf{F} = -q\mathbf{E}(\mathbf{x}_a)$ must be applied by some external agent to keep it in position.



If we now move the charge along a path C from $\mathbf{x}_a$ to $\mathbf{x}_b$, the work done by the external agent is found by integrating the force along the path,

$$W_{a\to b} = -\int_C d\mathbf{l}\cdot\mathbf{F}(\mathbf{x}), \tag{30}$$

or

$$W_{a\to b} = q\int_C d\mathbf{l}\cdot\nabla\Phi(\mathbf{x}) = q\int_C d\Phi = q[\Phi(\mathbf{x}_b)-\Phi(\mathbf{x}_a)] \tag{31}$$

This result shows that $q\Phi(\mathbf{x})$ can be interpreted as the potential energy of charge q in the electrostatic field at point $\mathbf{x}$, aside from a constant defining the zero of potential energy. In going from $\mathbf{x}_a$ to $\mathbf{x}_b$, work $q[\Phi(\mathbf{x}_b)-\Phi(\mathbf{x}_a)]$ is done on the charge, and so the change in the energy of the system (composed of the charge q and the sources of the field) is just this work.

Notice especially that the work does not depend on the path C except through the endpoints. This statement can always be made of conservative systems. In particular, the integral of the work done around a closed path is 0,

$$\int_C d\mathbf{l} \cdot \mathbf{E}(\mathbf{x}) = 0 \tag{32}$$

It is instructive to apply Stokes' Theorem to this relation. His theorem states that, for an arbitrary vector field $\mathbf{A}$, and a closed path C with a surface S "linking" the path (which means that S is an open surface with edges coinciding with C),

$$\int_C d\mathbf{l} \cdot \mathbf{A} = \int_S d^2x \, [\nabla \times \mathbf{A}(\mathbf{x})] \cdot \mathbf{n} \tag{33}$$

where $\mathbf{n}$ is a unit normal to the surface in the right-hand sense relative to the direction in which the path is traversed. As applied to the electric field, we have

$$0 = \int_C d\mathbf{l} \cdot \mathbf{E}(\mathbf{x}) = \int_S d^2x \, [\nabla \times \mathbf{E}(\mathbf{x})] \cdot \mathbf{n}. \tag{34}$$

Because C is arbitrary and can in particular be any infinitesimal closed loop, this relation implies that the integrand is zero, $\nabla \times \mathbf{E}(\mathbf{x}) = 0$. Thus the statement that $\mathbf{E}$ is a conservative field and $\nabla \times \mathbf{E}(\mathbf{x}) = 0$ are equivalent.

6 Poisson's and Laplace's Equations

The differential equations we have determined for $\mathbf{E}$ are sufficient to find it uniquely, given appropriate boundary conditions and the charge density, but they do not necessarily provide the simplest approach to the solution of an electrostatics problem. Often, it is best to solve for Φ from which $\mathbf{E}$ follows easily. Since $\nabla \cdot \mathbf{E} = 4\pi\rho$, and $\mathbf{E} = -\nabla\Phi$, we have

$$\nabla \cdot \nabla\Phi(\mathbf{x}) \equiv \nabla^2\Phi(\mathbf{x}) = -4\pi\rho(\mathbf{x}) \ \ \text{Poisson's Equation}, \tag{35}$$

which is *Poisson's Equation*; the operator ∇^2 is the *Laplacian operator.* In those regions of space where the charge density vanishes, we find the simpler equation,

$$\nabla^2\Phi(\mathbf{x}) = 0 \ \ \text{Laplace's Equation}, \tag{36}$$

which is *Laplace's Equation.*

Consider the effect of operating with ∇^2 on the integral expression for Φ:

$$-4\pi\rho(\mathbf{x}) = \nabla^2\Phi(\mathbf{x}) = \int d^3x'\, \rho(\mathbf{x}')\nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right), \tag{37}$$

or

$$\rho(\mathbf{x}) = -\int d^3x'\, \rho(\mathbf{x}') \left(\frac{1}{4\pi}\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{38}$$

However, we defined $\delta(\mathbf{x}-\mathbf{x}')$ as

$$f(\mathbf{x}) = \int d^3x'\, f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') \tag{39}$$

for general $f(\mathbf{x})$. Since $\rho(\mathbf{x})$ can be quite general, the quantity in large parentheses above satisfies the condition placed on the delta function; hence we conclude that

$$\nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{40}$$

which is only appropriate because $1/|\mathbf{x}-\mathbf{x}'|$ is the potential of a unit point charge, and $\delta(\mathbf{x}-\mathbf{x}')$ is the corresponding charge density. Thus, Eq. (40) expresses Poisson's equation for a unit point charge located at $\mathbf{x}'$.

As an exercise we may derive this result in a different way. Consider

$$\nabla^2\left(\frac{1}{|\mathbf{x}|}\right)=\nabla^2\left(\frac{1}{r}\right)=\frac{1}{r}\frac{d^2}{dr^2}\left(r\frac{1}{r}\right)=0, \tag{41}$$

except possibly at $r=0$ where r/r is undefined. To determine what happens here, we integrate $\nabla^2(1/r)$ over a small sphere centered on the origin:

$$\int_V d^3x\,\nabla^2(1/r)=\int_V d^3x\,\nabla\cdot\nabla(1/r)=$$
$$\int_S d^2x\,[\nabla(1/r)]\cdot\mathbf{n}=-\int_S d^2x\,(1/r^2)=-\int r^2dr\sin\theta d\theta d\phi(1/r^2)=-4\pi \tag{42}$$

Thus we have shown the following:

$$\begin{array}{lll} (i) & \nabla^2(1/r)=0, & r\neq 0 \\ (ii) & \int_V d^3x\,\nabla^2(1/r)(-1/4\pi)=1, & r=0\in V. \end{array} \tag{43}$$

These results tell us that $\nabla^2(1/r)=-4\pi\delta(\mathbf{x})$.

7 Energy in the Electric Field; Capacitance; Forces

The energy of the static electric field, or of a static charge distribution, is of some importance. Let us start our investigation by constructing the energy of interaction of n point charges q_i located at $\mathbf{x}_i$. As we have seen, the work required to move a charge q from one point to another in an applied electric field is q times the difference in the electric potentials at the end points. If we suppose that this potential is produced by our collection of point charges, then it is given by

$$\Phi(\mathbf{x}) = \sum_{i=1}^{n} [q_i / |\mathbf{x} - \mathbf{x}_i|] \tag{44}$$

and the work done to bring q from infinitely far away, where $\Phi(\mathbf{x}) = 0$, to point $\mathbf{x}$ is

$$W = q \sum_{i=1}^{n} [q_i / |\mathbf{x} - \mathbf{x}_i|]. \tag{45}$$

This is therefore the increase in the total energy of the system of charges when a charge is added to it at some particular point.

We may use this result to calculate the energy of the collection of charges by bringing them in one at a time from points at infinity where they are assumed to be widely separated. The first charge is brought in to $\mathbf{x}_1$, and this costs no energy because $\Phi = 0$ when there are no other charges present. The second charge costs energy

$$W_2 = \frac{q_1 q_2}{|\mathbf{x}_1 - \mathbf{x}_2|}. \tag{46}$$

The third then costs

$$W_3 = q_3 \sum_{j=1}^{2} \frac{q_j}{|\mathbf{x}_3 - x_j|}, \tag{47}$$

and so on. The amount of work which must be done to bring in the i^{th} particle is

$$W_i = q_i \sum_{j=1}^{i-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}. \tag{48}$$

If we add up these energies to find the total work done, it is

$$W = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{49}$$

which is the sum over all pairs, each pair taken once; it may also be written as

$$W = \frac{1}{2} \sum_{i,j}{}' \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|}; \tag{50}$$

the prime on the summation sign means that the terms with $i = j$ are omitted. In this sum, we include each pair i, j with $i \neq j$ twice and so have to multiply by a factor of $1/2$. Given a continuous charge distribution, the same argument can be applied using as the elementary charges infinitesimal charge elements located in infinitesimal volume elements. The result must be

$$W = \frac{1}{2} \int d^3x \, d^3x' \, \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{51}$$

where the integrations are unrestricted and include the points $\mathbf{x} = \mathbf{x}'$ because the interaction energy of an infinitesimal continuously distributed charge element with itself vanishes in the limit of zero extent. However, if the charge distribution contains finite point charges, represented by delta functions in $\rho(\mathbf{x})$, then one has to omit the interaction of each of these charges with itself, as in the original sum, Eq. (50), in order to obtain a finite result.

The expression for W can be cast into a number of other useful forms. Recall that

$$\Phi(\mathbf{x}) = \int d^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}; \tag{52}$$

substitution into the expression for W gives

$$W = \frac{1}{2} \int d^3x\, \rho(\mathbf{x}) \Phi(\mathbf{x}). \tag{53}$$

Further, $\rho(\mathbf{x}) = -\nabla^2 \Phi(\mathbf{x})/4\pi$, so

$$W = -\frac{1}{8\pi} \int d^3x\, \Phi(\mathbf{x}) \nabla^2 \Phi(\mathbf{x}). \tag{54}$$

Let us now do an integration by parts in three dimensions. This operation is easy to achieve by making use of the divergence theorem; for a vector field $f(\mathbf{x})\mathbf{A}(\mathbf{x})$ consider the integral

$$\begin{aligned} \int_V d^3x\, \nabla \cdot [f(\mathbf{x})\mathbf{A}(\mathbf{x})] &= \int_S d^2x\, f(\mathbf{x})[\mathbf{A}(\mathbf{x}) \cdot \mathbf{n}] \\ = \int_V d^3x\, [\nabla f(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x})] &+ \int_V d^3x\, f(\mathbf{x})[\nabla \cdot \mathbf{A}(\mathbf{x})] \end{aligned} \tag{55}$$

or

$$\int_V d^3x\, f(\mathbf{x})[\nabla \cdot \mathbf{A}(\mathbf{x})] = -\int_V d^3x\, [\nabla f(\mathbf{x})] \cdot \mathbf{A}(\mathbf{x}) + \int_S d^2x\, f(\mathbf{x})[\mathbf{A}(\mathbf{x}) \cdot \mathbf{n}], \tag{56}$$

where V and S are related in the usual way. As applied to the integral for W, this useful formula gives, letting V be all space,

$$W = -\frac{1}{8\pi} \int d^3x\, \Phi(\mathbf{x}) \nabla \cdot \nabla \Phi(\mathbf{x}) \tag{57}$$

$$= \frac{1}{8\pi} \int d^3x\, \nabla \Phi(\mathbf{x}) \cdot \nabla \Phi(\mathbf{x}) - \int d^2x\, \Phi(\mathbf{x})[\nabla \Phi(\mathbf{x}) \cdot \mathbf{n}] \tag{58}$$

or

$$W = \frac{1}{8\pi} \int d^3x\, [\mathbf{E}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x})]. \tag{59}$$

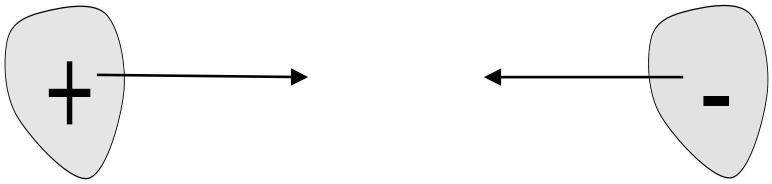
The surface integral has vanished because Φ falls off at distances r which are large compared to the extent of the charge distribution at least as fast as $1/r$. Hence the integrand in the surface integral falls off at least as fast as $1/r^3$ while the area of the surface at distance r varies as r^2. This integral therefore falls off at least as fast as $1/r$ and so vanishes when the surface is at infinity.

An interesting and plausible interpretation of the final expression is that the integrand is the energy density $u(\mathbf{x})$ of the electric field,

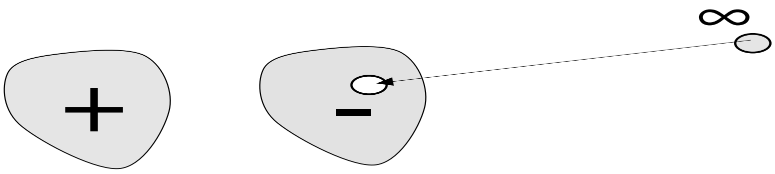
$$u(\mathbf{x}) = \frac{1}{8\pi}\mathbf{E}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}). \tag{60}$$

This is only an interpretation, however. All we really know is that the total energy is the integral of this quantity over all space. The idea is plausible because $u(\mathbf{x})$ so defined is everywhere positive or zero (a negative energy density would be disturbing). Note, too, that our other expression for the energy as an integral over a single position variable has an integrand that can be both positive and negative which makes it unreasonable to interpret that integrand, $\rho(\mathbf{x})\Phi(\mathbf{x})/2$, as the energy density.

Given Eq. (59) for W, we can see that the energy will be positive definite. Yet the energy of, *e.g.,* a pair of point charges q and $-q$ at $\mathbf{x}$ and $\mathbf{x}'$ is negative, $-q^2/|\mathbf{x} - \mathbf{x}'|$. The reason is that the expression we have for the energy of a set of point charges does not include the (infinite) energy required to assemble each of the point charges in the first place, but Eq. (59) would include this (positive) energy. A more concrete example involves two oppositely charged masses. The energy required to bring them together from infinity is negative,



whereas the energy required to assemble the entire charge distribution



at its final location is positive.

7.1 Conductors

Consider now the special case that our electrostatic system consists of a collection of n electrically isolated conductors; for our present purposes, a conductor may be defined as an object which cannot support an electric field (because it contains "free" charges which move under the influence of a field until there is no field). Thus the interior of a conductor is an equipotential. Using Eq. (53), we see that for such a system,

$$W = \frac{1}{2}\int d^3x\, \rho(\mathbf{x})\Phi(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i \tag{61}$$

where Q_i and V_i are, respectively, the charge and potential on the i^{th} conductor. Now, because the potential is a linear function of charge (superposition theorem), it is true that

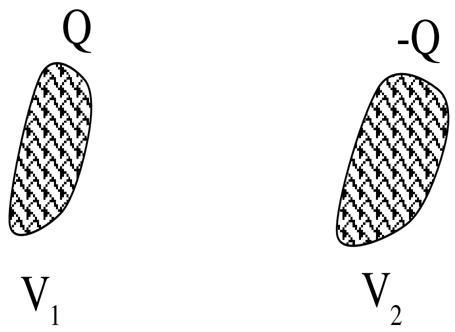
$$V_i = \sum_{j=1}^{n} p_{ij} Q_j. \tag{62}$$

The coefficients p_{ij} are independent of the charges; they depend only on the distribution and shapes of the conductors and are called the *coefficients of potential*. To see that this relation is valid, one need only think of the potentials produced on each conductor given charge Q_j on the j^{th} conductor and zero charge on all others; then superpose the solutions to each of the problems of this kind. Inversion of Eq. (62) yields the charges Q_i as linear combinations of the potentials V_j,

$$Q_i = \sum_{i=1}^{n} C_{ij} V_j. \tag{63}$$

The coefficients C_{ij} are called *coefficients of capacitance*; the diagonal elements C_{ii} are more commonly referred to simply as *capacitances* while the off-diagonal ones C_{ij} are called *coefficients of electrostatic induction* and are not to be confused with the inductances introduced in connection with Faraday's Law. The capacitance of a single conductor, C_{ii}, is thus the total charge on that conductor when it is maintained at unit potential while all other conductors are held at zero potential.

As an example the capacitance of a pair of conductors with equal and opposite charge is defined as the ratio of the charge on one conductor to the potential difference between them when all other conductors are maintained at zero potential.



$$\begin{pmatrix} Q \\ -Q \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{64}$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{\begin{pmatrix} C_{22} & -C_{21} \\ -C_{12} & C_{11} \end{pmatrix} \begin{pmatrix} Q \\ -Q \end{pmatrix}}{C_{11}C_{22} - C_{12}C_{21}} \tag{65}$$

The capacitance $C(1,2) = Q/|V_1 - V_2|$ turns out to be

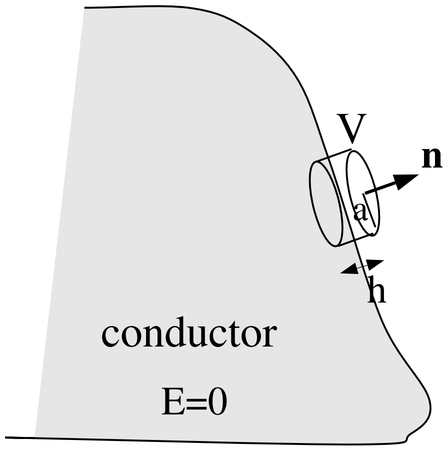
$$C(1,2) = (C_{11}C_{22} - C_{12}^2)/(C_{11} + C_{22} + 2C_{12}). \tag{66}$$

The energy of the system of conductors may be written in terms of potentials and the coefficients C_{ij} as

$$W = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i = \frac{1}{2}\sum_{i,j=1}^{n} C_{ij} V_i V_j. \tag{67}$$

7.2 Forces on Charged Conductors

Another useful application of the expressions for the energy is in the calculation of forces on charged conductors. Consider the surface of a conductor. The field at the surface can be inferred from $\nabla \cdot \mathbf{E}(\mathbf{x}) = 4\pi\rho(\mathbf{x})$ and $\nabla \times \mathbf{E}(\mathbf{x}) = 0$. Consider an integral of the first of these equations over a "pillbox" or short right circular cylinder oriented with the "drumhead" parallel to the surface of a conductor and situated half inside and half outside of the conductor.



$$\int_V d^3x\, \nabla \cdot \mathbf{E} = \int_S d^2x\, \mathbf{E} \cdot \mathbf{n} = \mathbf{E}_n(\mathbf{x})\pi\, a^2 = 4\pi \int_V d^3x\, \rho(\mathbf{x}) = \pi a^2 4\pi\sigma(\mathbf{x}) \tag{68}$$

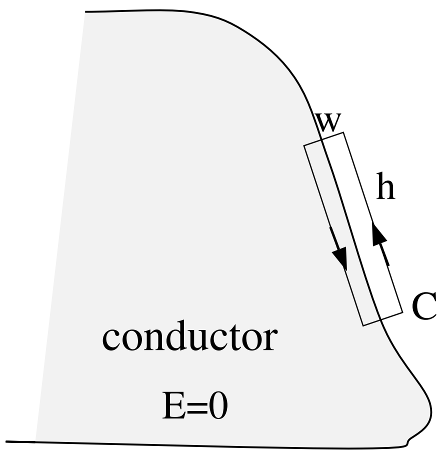
Using the divergence theorem, we may convert to a surface integral. Given that the height h of the cylinder is much smaller than its radius, $h << a$, the only important contribution to the surface integral must come from the drumheads. But $\mathbf{E}(\mathbf{x}) = 0$ on the one inside of the conductor, so we pick up only the contribution from the component of $\mathbf{E}$ normal to the surface of the conductor on the outside. Given that a is much smaller than distances over which the field varies appreciably, we get simply $\pi a^2 \mathbf{E}_n(\mathbf{x})$ where $\mathbf{x}$ is a point just outside of the surface of the conductor and the subscript n designates the outward (from the conductor) normal component of the field. The volume integral of $\rho(\mathbf{x})$, on the other hand, yields the total charge within the pillbox. Because $h << a$, we will get a vanishingly small contribution from any finite volume density of charge. However, we get an important contribution from a *surface* charge density; we shall denote such a thing by $\sigma(\mathbf{x})$. It can be written as a volume charge density by using a δ function:

$$\rho(\mathbf{x}) = \sigma(\mathbf{x})\delta(\xi(\mathbf{x})) \tag{69}$$

where ξ is the normal distance of the point $\mathbf{x}$ from the surface of the conductor. When this surface charge density is integrated over the volume of the pillbox, it reduces to a surface integral of the surface charge density over a disc on the conductor surface and having the area of the crosssection of the pillbox, *i.e.,* πa^2. Hence one finds $\pi a^2(4\pi\sigma(\mathbf{x}))$. Putting the two sides of the equation together, we find the following relation between the charge density on the surface of a conductor and the normal component of the electric field just outside of the conductor:

$$\mathbf{E}_n(\mathbf{x}) = 4\pi\sigma(\mathbf{x}) \tag{70}$$

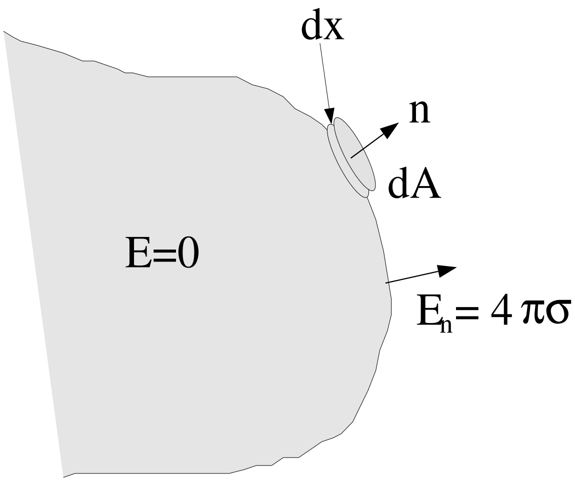
We may also find an equation for the **tangential component** of the electric field at the surface of a conductor. Consider the line integral of $d\mathbf{l} \cdot \mathbf{E}(\mathbf{x})$ around a rectangle which straddles the conductor's surface.



$$\int_S d^2x\,(\nabla\times\mathbf{E})\cdot\mathbf{n} = \oint_C d\mathbf{l}\cdot\mathbf{E} = \mathbf{E}_t h = 0 \implies \mathbf{E}_t = 0 \tag{71}$$

The width w of the rectangle, which is its size in the direction normal to the interface, is much smaller than its height h, which is its size parallel to the interface. The dominant contribution to the line integral, which is zero, comes from the two sides parallel to the interface. On the inside, $\mathbf{E}(\mathbf{x})$ is zero, so we have only the integral along the side which is exterior to the conductor. Since the whole integral is zero, the integral along this single side must be zero, and hence we can conclude that the tangential component of $\mathbf{E}(\mathbf{x})$, or $\mathbf{E}_t(\mathbf{x})$, just outside of a conductor must vanish,

$$\mathbf{E}_t(\mathbf{x}) = 0 \qquad \text{at the surface of a conductor.} \tag{72}$$



Now we are in a position to consider **the force on the surface of a conductor**. We use the method of virtual work. Imagine moving a small element dA of the conductor's surface, along with the charge on it, a distance dx from its initial position in the direction normal to the surface. It will sweep out a volume $dAdx$. In this volume, to a first approximation [4], there will no longer be any electric field (since the field is zero within a conductor) while the electric field elsewhere will be unchanged. Hence there will be a change in the field energy of

$$dW = -dAdx\frac{1}{8\pi}E_n^2 = -2\pi\sigma^2 dAdx + \mathcal{O}(dAd^2x, d^2Adx) \qquad (73)$$

where Eq. (70) has been used for the normal component of the electric field. Energy conservation demands that the amount of work done on the system in making this displacement is equal to dW. It is also dx times the negative of the electric force acting on the area element dA.

[4] This approximation improves as the ratio of dx to $\sqrt{A}$ goes to zero

Thus,

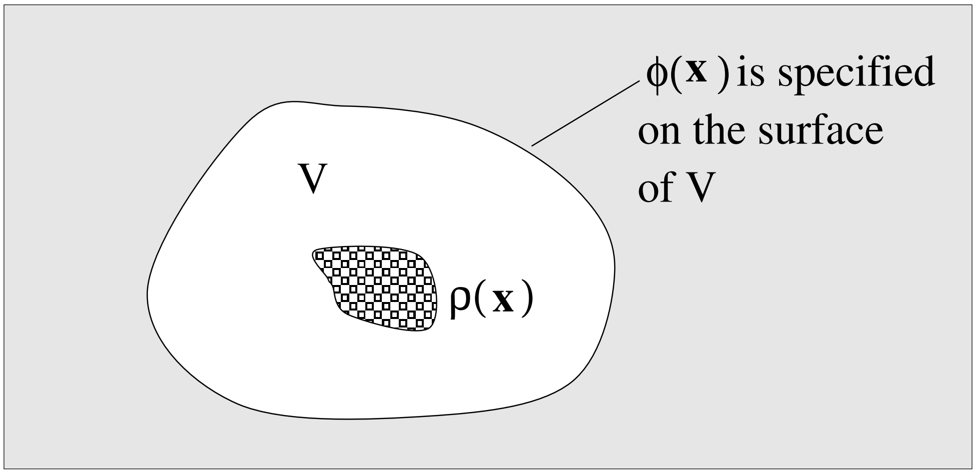
$$-F dx = dW \text{ or } F = 2\pi\sigma^2 \, dA. \tag{74}$$

Hence the force per unit area on the surface of the conductor is $2\pi\sigma^2$; it is directed normally outward from the conductor ("negative pressure").

A second way of looking at this problem is to calculate the force directly. It must be the charge on dA, or σdA, times the electric field which acts on this charge, *i.e.,* that part of the electric field at the surface which is produced by charges other than those on dA. This field is just $2\pi\sigma$ (Why?), so the force comes out at before.

8 Green's Theorem

In everything we have discussed thus far, we have assumed that $\rho(\mathbf{x})$ is known, and that there is a simple boundary condition on $\phi(\mathbf{x})$ at infinity (that it must at least as fast as $1/r$). *This is not generally true*! Usually, we only know $\rho(\mathbf{x})$ within some finite volume V, and the value of $\phi(\mathbf{x})$ on the corresponding surface S.



For these problems Green's theorem and functions are useful. The simplest greens function is that for free space

$$G_{free}(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}\,, \tag{75}$$

and the corresponding potential is

$$\phi(\mathbf{x}) = \int_V d^3x'\, G_{free}(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\,. \tag{76}$$

Note that $G_{free}(\mathbf{x}, \mathbf{x}')$ gives the response at $\mathbf{x}$, due to a unit point charge at $\mathbf{x}'$. For the response due to a collection of charges $\rho(\mathbf{x})$, superposition yields the integral above.

For the more general problem of a collection of charges and boundary conditions on a surface, we might expect that this integral relation will become

$$\phi(\mathbf{x}) = \int_V d^3x'\, G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') + \text{ surface term} \tag{77}$$

Before we can proceed further, we must develop some formalism.

8.1 Green's Theorem

To this end, let us develop Green's Theorem, starting from the divergence theorem as applied to a vector field $\mathbf{A}$:

$$\int_V d^3x\,[\nabla \cdot \mathbf{A}(\mathbf{x})] = \int_S d^2x\,[\mathbf{A}(\mathbf{x}) \cdot \mathbf{n}] \tag{78}$$

where S is the surface bounding the domain V. Consider the special case $\mathbf{A}(\mathbf{x}) = \phi(\mathbf{x})\nabla\psi(\mathbf{x})$; then

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = \nabla\phi(\mathbf{x}) \cdot \nabla\psi(\mathbf{x}) + \phi(\mathbf{x})\nabla^2\psi(\mathbf{x}) \tag{79}$$

and

$$\mathbf{n} \cdot \mathbf{A}(\mathbf{x}) = \phi(\mathbf{x})[\mathbf{n} \cdot \nabla\psi(\mathbf{x})] = \phi(\mathbf{x})\frac{\partial\psi(\mathbf{x})}{\partial n} \tag{80}$$

where $\frac{\partial\psi}{\partial n}$ is the outward normal derivative of ψ at the surface. Substitution into the divergence theorem produces

$$\int_V d^3x\,[\nabla\phi(\mathbf{x}) \cdot \nabla\psi(\mathbf{x}) + \phi(\mathbf{x})\nabla^2\psi(\mathbf{x})] = \int_S d^2x\,\phi(\mathbf{x})\frac{\partial\psi(\mathbf{x})}{\partial n}, \tag{81}$$

a result known as *Green's first identity.* We may also start from a vector field $\mathbf{A}(\mathbf{x}) = \psi(\mathbf{x})\nabla\phi(\mathbf{x})$ and wind up with

$$\int_V d^3x\,[\nabla\psi(\mathbf{x}) \cdot \nabla\phi(\mathbf{x}) + \psi(\mathbf{x})\nabla^2\phi(\mathbf{x})] = \int_S d^2x\,\psi(\mathbf{x})\frac{\partial\phi(\mathbf{x})}{\partial n}. \tag{82}$$

Subtract the second expression from the first and obtain

$$\int_V d^3x\,[\phi(\mathbf{x})\nabla^2\psi(\mathbf{x}) - \psi(\mathbf{x})\nabla^2\phi(\mathbf{x})] = \int_S d^2x\,\left[\phi(\mathbf{x})\frac{\partial\psi(\mathbf{x})}{\partial n} - \psi(\mathbf{x})\frac{\partial\phi(\mathbf{x})}{\partial n}\right], \tag{83}$$

which is *Green's second identity*, also known as *Green's Theorem.*

8.2 Applying Green's Theorem 1

We next make a particular choice of the scalar functions and also change the integration variable to $\mathbf{x}'$:

$$\phi(\mathbf{x}') = \Phi(\mathbf{x}') \text{ and } \psi(\mathbf{x}') = 1/|\mathbf{x} - \mathbf{x}'| \equiv 1/R. \tag{84}$$

In the latter function, $\mathbf{x}$ is to be regarded as a parameter which will eventually become the point at which we evaluate the potential. Substitution into Green's Theorem gives

$$\int_V d^3x' \left[\Phi(\mathbf{x}')\nabla'^2(1/R) - (1/R)\nabla'^2\Phi(\mathbf{x}')\right] \tag{85}$$

$$= \int_S d^2x' \left[\Phi(\mathbf{x}')\frac{\partial(1/R)}{\partial n'} - \frac{1}{R}\frac{\partial\Phi(\mathbf{x}')}{\partial n'}\right], \tag{86}$$

or

$$-4\pi\Phi(\mathbf{x}) + 4\pi\int_V d^3x' \,\frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \int_S d^2x' \left[\Phi(\mathbf{x}')\frac{\partial(1/R)}{\partial n'} - \frac{1}{R}\frac{\partial\Phi(\mathbf{x}')}{\partial n'}\right] \tag{87}$$

where we have assumed $\mathbf{x}$ is inside of V. With a little rearrangement, the final equation can be written as

$$\Phi(\mathbf{x}) = \int_V d^3x' \,\frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{1}{4\pi}\int_S d^2x' \left[\frac{1}{R}\frac{\partial\Phi(\mathbf{x}')}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial(1/R)}{\partial n'}\right] \tag{88}$$

The first term on the right is the familiar volume integral over the charge density, but notice that it no longer is over all space. The charge outside of V does, of course, contribute to $\Phi(\mathbf{x})$; its contribution is now taken into account by the integral over the surface surrounding V. Note, too, that if the point $\mathbf{x}$ is outside of V, then the left-hand side (LHS) of the equation is zero. Further, if V is all space so that S is at infinity, then the surface integral will vanish and we recover the volume integral over all space. Note also that if $\rho(\mathbf{x}') = 0$ for all $\mathbf{x}'$ in V, then the potential is found simply from the surface integral of Φ and its normal derivative.

8.3 Applying Green's Theorem 2

This equation is not the best one of its kind. It is in fact (as we shall see) possible to find $\Phi(\mathbf{x})$ from the charge density in V and from **either** $\Phi(\mathbf{x}')$ **or** $\partial\Phi(\mathbf{x}')/\partial n'$ on the surface S; that is, it is not necessary to know both of these things on the surface. When Φ is specified, that is called *Dirichlet boundary conditions*; when the normal derivative of Φ is given, that is called *Neumann boundary conditions.* Various combinations are also possible, such as Dirichlet conditions on part of S and Neumann conditions on the remainder, a case known as *mixed boundary conditions.*

specification	Boundary Condition
$\phi(\mathbf{x}')$ specified on S	Dirichlet
$\partial\Phi(\mathbf{x}')/\partial n'$ specified on S	Neumann
$\phi(\mathbf{x}')$ specified on part of S $\partial\Phi(\mathbf{x}')/\partial n'$ specified on remainder	Mixed

Let us demonstrate that it is sufficient to know either $\Phi(\mathbf{x}')$ or its normal derivative on S in order to obtain a unique solution to

$$\nabla^2\Phi(\mathbf{x}) = -4\pi\rho(\mathbf{x}) \tag{89}$$

with some volume V bounded by S. Start by supposing that either type of boundary condition is given and that there are two distinct solutions Φ_1 and Φ_2. Define $U = \Phi_1 - \Phi_2$. This function is such that

$$\nabla^2 U(\mathbf{x}) = 0 \text{ inside of V} \tag{90}$$

and

$$\begin{array}{ll} \text{either} & U(\mathbf{x}) = 0, \;\; \text{for } \mathbf{x} \text{ on S (Dirichlet)} \\ \text{or} & \frac{\partial U(\mathbf{x})}{\partial n} = 0, \;\; \text{for } \mathbf{x} \text{ on S (Neumann).} \end{array} \tag{91}$$

In Green's first identity Eq. (81), let $\psi = \phi = U$:

$$\int_V d^3x \, [U(\mathbf{x})\nabla^2 U(\mathbf{x}) + \nabla U(\mathbf{x}) \cdot \nabla U(\mathbf{x})] = \int_S d^2x \, U(\mathbf{x}) \frac{\partial U(\mathbf{x})}{\partial n}. \tag{92}$$

Now, since U satisfies the Laplace equation, the first term in brackets on the left vanishes. From the boundary condition for either the Neumann or the Dirichlet problem, the surface integral also vanishes. Hence we have just

$$\int_V d^3x \, |\nabla U(\mathbf{x})|^2 = 0 \tag{93}$$

from which it follows that $\nabla U(\mathbf{x}) = 0$ in V. Therefore $U(\mathbf{x})$ is a constant in V and so the two solutions Φ_1 and Φ_2 are the same up to a constant. For Dirichlet conditions, this constant is zero since the two functions are the same on the boundary. For Neumann conditions it is arbitrary and amounts only to a choice of the zero of potential.

The preceding proof is also valid for the case of mixed boundary conditions because the surface integral vanishes in this case also. Finally, one cannot in general specify *both* $\Phi(\mathbf{x})$ and $\frac{\partial \Phi}{\partial n}$ everywhere on S (*Cauchy* boundary conditions); either one alone is sufficient to determine a unique solution and the two solutions so determined are not necessarily the same. However, if Cauchy boundary conditions are given on just an appropriate part of S, that can be sufficient to give a unique solution for the potential.

In the light of what we have learned, it is evident that our integral expression for $\Phi(\mathbf{x})$, Eq. (88), which involves surface integrals of both the potential and its normal derivative, is not a very effective way to solve an electrostatic boundary value problem; it requires more input information than is actually needed to determine a solution and so is an integral equation as opposed to a solution in the form of an integral. If we had made a better choice of $\psi(\mathbf{x}')$ at the outset, we could have come up with a better result. Let's try again, choosing for $\psi(\mathbf{x}')$ a function we shall call $G(\mathbf{x}, \mathbf{x}')$; it is given by $1/|\mathbf{x} - \mathbf{x}'|$ plus an as yet undetermined function $F(\mathbf{x}, \mathbf{x}')$ which is to be a solution of the Laplace equation in V,

$$\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0 \tag{94}$$

for $\mathbf{x}$ and $\mathbf{x}'$ in V. Since

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}'), \tag{95}$$

it is the case that

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{96}$$

for $\mathbf{x}$ and $\mathbf{x}'$ in V. Physically, the function G, viewed as a function of $\mathbf{x}'$, is a solution of Poisson's equation in V given a unit point charge at $\mathbf{x}$. The function $F(\mathbf{x},\mathbf{x}')$ is a solution of Laplace's equation; we shall presently determine its properties further by requiring that it satisfy certain conditions on S. Using $G(\mathbf{x},\mathbf{x}')$ for $\psi(\mathbf{x}')$ and $\Phi(\mathbf{x})$ for $\phi(\mathbf{x})$ in Green's Theorem, we can easily show that

$$\Phi(\mathbf{x}) = \int_V d^3x'\, G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') + \frac{1}{4\pi}\int_S d^2x' \left[G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi(\mathbf{x}')}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right]. \tag{97}$$

8.3.1 Greens Theorem with Dirichlet B.C.

Now consider in turn two different sets of boundary conditions for G. First, require that $G(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}'$ on S and denote this function by G_D. Then the preceding equation becomes

$$\Phi(\mathbf{x}) = \int_V d^3x'\, G_D(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi}\int_S d^2x' \,\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}\Phi(\mathbf{x}'). \tag{98}$$

This is a useful equation when we have a Dirichlet problem with $\Phi(\mathbf{x}')$ specified for $\mathbf{x}'$ on S. Then we have in principle the information we need to complete the integration and so find $\Phi(\mathbf{x})$ in which case Eq. (98) is an integral solution for $\Phi(\mathbf{x})$ as opposed to an integral equation for the potential.

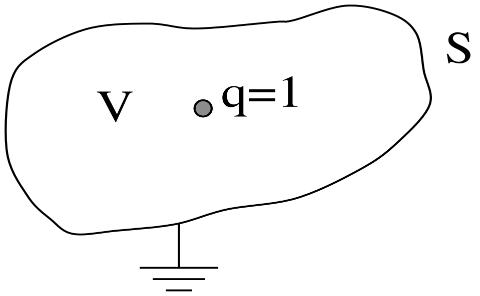
The question naturally arises, does the function G_D exist? That is, is it possible to find the Dirichlet Green's function G_D which is specified by the conditions

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}');\ \ \mathbf{x},\mathbf{x}' \text{ in V} \tag{99}$$

and

$$G_D(\mathbf{x},\mathbf{x}')|_{\mathbf{x}' \text{ on S}} = 0? \tag{100}$$

The answer is that this function does exist; further, it is unique. The preceding two conditions are sufficient to determine it completely. We know this without resorting to a mathematical proof because we can see that $G_D(\mathbf{x},\mathbf{x}')$ is just the scalar potential at $\mathbf{x}'$ given a unit point charge at $\mathbf{x}$ inside of a cavity with conducting walls coincident with S and held at zero potential.



This is the physical interpretation of the Dirichlet Green's function. Notice in particular that this is a strongly geometry-dependent function (it depends on S very much) but it is not dependent on any other properties of the system. In other words, we can solve any Dirichlet problem for a given geometry if we can solve the "point charge with grounded conducting surfaces" problem for the same geometry in the sense that we can reduce the solution to a quadrature, *i.e.,* to an integral.

An important property of the Dirichlet Green's function is that it is invariant under interchange of $\mathbf{x}$ and $\mathbf{x}'$, $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$. To demonstrate this property, let $\phi(\mathbf{y}) = G(\mathbf{x},\mathbf{y})$ and $\psi(\mathbf{y}) = G(\mathbf{x}',\mathbf{y})$. Then insert these functions into Green's Theorem (with $\mathbf{y}$ as the integration variable)

$$\begin{aligned}
&\int_V d^3y\,[G(\mathbf{x},\mathbf{y})\nabla^2 G(\mathbf{x}',\mathbf{y}) - G(\mathbf{x}',\mathbf{y})\nabla^2 G(\mathbf{x},\mathbf{y})] = \\
&\int_S d^2y \left[G(\mathbf{x},\mathbf{y})\frac{\partial G(\mathbf{x}',\mathbf{y})}{\partial n} - G(\mathbf{x}',\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n}\right], \qquad (101)
\end{aligned}$$

and make use of the properties of the Dirichlet Green's function that $\nabla^2_{\mathbf{y}} G(\mathbf{x},\mathbf{y}) = -4\pi\delta(\mathbf{x}-\mathbf{y})$ and $G(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{y}$ on S. The result is that $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$.

8.3.2 Greens Theorem with Neumann B.C.

The second case of boundary conditions we consider on G is $\partial G(\mathbf{x},\mathbf{x}')/\partial n' = 0$ for $\mathbf{x}'$ on S. Then application of Green's Theorem (Eq. (97)) leads to

$$\Phi(\mathbf{x}) = \int_V d^3x'\,G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') + \frac{1}{4\pi}\int_S d^2x'\,\frac{\partial\Phi(\mathbf{x}')}{\partial n'}G(\mathbf{x},\mathbf{x}'). \qquad (102)$$

Unfortunately, this G does not exist as we may show by applying Gauss's Law,

$$0 = \int_S d^2x' \, \frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'} = \int_S d^2x' \, [\mathbf{n}' \cdot \nabla' G(\mathbf{x},\mathbf{x}')] = \int_V d^3x' \, \nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi. \tag{103}$$

Clearly, we cannot have a G with zero normal derivative everywhere on S. The next simplest possibility is that

$$\frac{\partial G_N(\mathbf{x},\mathbf{x}')}{\partial n'}|_{\mathbf{x}' \text{ on S}} = -\frac{4\pi}{S}, \tag{104}$$

where S is the area of the surface. Given such a function, we can use it in Green's Theorem and will be led to the following integral expression for the scalar potential:

$$\Phi(\mathbf{x}) =< \Phi >_S + \int_V d^3x' \, G_N(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') + \frac{1}{4\pi} \int_S d^2x' \, \frac{\partial \Phi(\mathbf{x}')}{\partial n'} G_N(\mathbf{x},\mathbf{x}') \tag{105}$$

where $< \Phi >_S$ is the average of the potential over the surface S,

$$< \Phi >_S \equiv \frac{1}{S} \int_S d^2x' \, \Phi(\mathbf{x}') \tag{106}$$

One can understand the necessity of the presence of this term from the fact that the Neumann boundary condition problem can only be solved up to an arbitrary constant.

The Dirichlet Green's function is the one that we shall use most often as one more commonly specifies the potential on the boundary than the normal component of the electric field.