Chapter 3

Linear Response Theory

3.1 Measurements and Correlation Functions

We will now discuss how to represent *physical measurements*. All local physical measurements of a macroscopic quantum (many-body) system amount in practice to a process in which a localized disturbance is created by an applied external *force*, in the neighborhood of some point $\vec{r}$ at some time t, and the *response* of the system is then measured at some other point $\vec{r}\,'$ at tome *later* time $t' > t$. On the other hand, in many cases we also will be interested in the global response of a system to an uniform perturbation, as in the case of most thermodynamic measurements.

There are many examples of these type of measurements. For example, in a typical optical experiment one has an external electromagnetic wave impinging on a system (for instance, a metal). In the case of an electromagnetic field with sufficiently low energy we can ignore its quantum mechanical nature and treat it as a classical wave. It will interact with the degrees of freedom in a variety of ways. Thus, the scalar potential A_0 couples to the local charge density through a term in the Hamiltonian of the form $\int d^3x A_0(\vec{x},t)\rho(\vec{x},t)$, where $\rho(\vec{x},t)$ is the second quantized charge density operator (in the Heisenberg representation). On the other hand, the vector potential $\vec{A}(\vec{x},t)$ couples to the particle current operator $\vec{j}\,(\vec{x},t)$ (the full form of the coupling is actually more complex, as we shall see below). Thus, at this level, we see that an external classical electromagnetic field can induce density waves and currents in a system.

However, at higher energies the quantum nature of the electromagnetic field cannot be ignored, and in reality what one is considering are the scattering processes of external photons with the internal degrees of freedom of the system of interest. Photons couple to the atoms and through this coupling we can gain information on the phonon degrees of freedom. Thus, Raman scattering of photons provide a direct window to the dynamics of phonons (among many other things).

On the other hand, X-ray photons are more energetic and sufficiently so that they can in fact eject an electron from the system, leaving behind a hole. Thus, by measuring the properties of the ejected *photo-electron* it is possible to infer the spectral properties of the hole. This is an example of a photoemission experiment. If the momentum and energy of the photo-electron are measured, this is an angle-resolved photo-emission (ARPES) experiment, which essentially measures the retarded Green function of the hole.

Other scattering processes couple instead to the spin of the degrees of freedom. For example in *neutron scattering* experiments, the initial state has neutrons with well defined energy, momentum and spin polarization which scatter off the spin degrees of freedom into a final state also with well defined energy momentum and spin polarization. The leading coupling is due to the effectively short-range interaction between the spin of the neutron and the local spin density $\vec{S}(\vec{x}, t)$ of the degrees of freedoms of the system. The interaction between the neutron magnetic moment and the local spin density leads to a term on the Hamiltonian which is linear in $\vec{S}(\vec{x}, t)$. Neutron scattering experiments measure the spin-spin correlation function.

We will now discuss a general theory of these type of measurements. Let H be the full Hamiltonian describing the system in *isolation*. One way to test its properties is to couple the system to a *weak external perturbation* (assuming that the ground state is stable) and to determine how the ground state and the excited states are affected by the perturbation. Let $O(\vec{x}, t)$ be a local observable, such as the local density, the charge current, or the local magnetization. The (total) Hamiltonian H_T for the system weakly coupled to an external perturbation, which we will represent by a Hamiltonian H_{ext}, is

$$H_T = H + H_{\text{ext}} \tag{3.1}$$

We will assume that the perturbation is not only weak, in which case its effects should be describable in perturbation theory, but it is also adiabatically switched on and off. Then as we discussed above, the Heisenberg representation of the *isolated* system, with Hamiltonian H, is the interaction representation of the system coupled to the source. Thus, the physical observables of the coupled system will evolve according to the Hamiltonian of the isolated system but the states follow the external perturbation, which we will refer to as the source. Hence, the expectation value of the observable $O(\vec{x},t)$ in the exact ground state $|G\rangle$ of H, under the action of the weak perturbation H_{ex} is modified as

$$\langle G|O(\vec{x},t)|G\rangle \;\to\; \langle G|U^{-1}(t)O(\vec{x},t)U(t)|G\rangle \tag{3.2}$$

where $U(t)$ is the evolution operator in the interaction representation of H_T. As we have seen, $U(t)$ is given by the time-ordered exponential

$$U(t) = Te^{-\frac{i}{\hbar}\int_{-\infty}^{t} dt' H_{\text{ext}}(t')} \tag{3.3}$$

By expanding $U(t)$ to *linear* order in the perturbation H_{ext}, we see that to lowest (*linear*) order the change in the expectation value $\delta\langle G|O(\vec{x},t)|G\rangle$ is

$$\delta\langle G|O(\vec{x},t)|G\rangle \;=\; \frac{i}{\hbar}\int_{-\infty}^{t} dt' \; \langle G|\left[H_{\text{ext}}(t'),O(\vec{x},t)\right]|G\rangle \tag{3.4}$$

This change represents the *linear response* of the system to the external perturbation. It is given in terms of the ground state expectation value of the *commutator* of the perturbation and the observable. For this reason, this approach is called *Linear Response Theory.* Notice that in Eq.(3.4), there is an ordering of the times t and t': the time t at which the change is observed is always later than the time(s) t' during which the external perturbation acted, $t > t'$. Hence, Eq.(3.4) explicitly obeys *causality.*

In general, if $O(\vec{x},t)$ is a local observable, $H_{\text{ext}}(t)$ represents an external source which couples linearly to the observable,

$$H_{\text{ext}}(t) \;=\; \int d^3x \; O(\vec{x},t)f(\vec{x},t) \tag{3.5}$$

To simplify matters, we will assume that the observable $O(\vec{x},t)$ is *normal ordered* with respect to the exact ground state $|G\rangle$, *i.e.* we will require that $\langle G|O(x,t)|G\rangle = 0$; in other terms, we are considering operators which measure the *fluctuations* of the observable away from the expectation value.

Hence, the linear change (from zero) in the expectation value of the observable induced by the source $f(\vec{x},t)$ is

$$\langle G|O(x,t)|G\rangle_f \;=\; \frac{i}{\hbar}\int_{-\infty}^{t} dt' \int d^3x' \;\langle G|\left[O(\vec{x}',t'),O(\vec{x},t)\right]|G\rangle \; f(\vec{x}\,',t') \quad (3.6)$$

Thus, the *response* is linear in the *force*. The "coefficient" of proportionality between the change in the expectation value $\langle G|O(\vec{x},t)|G\rangle$ and the force $f(\vec{x}\,'t')$ defines a generalized susceptibility $\chi(\vec{x}t,\vec{x}\,'t')$ through the definition

$$\langle G|O(\vec{x},t)|G\rangle \equiv \text{``}\chi\cdot f\text{''} = \int d^3x' \int_{-\infty}^{t} dt' \;\; \chi(\vec{x}t,\vec{x}\,'t')f(x',t') \qquad (3.7)$$

By inspection we see that we can identify the generalized susceptibility $\chi(\vec{x}t;\vec{x}\,'t')$ with the *retarded propagator*, or *correlation function*, of the observable $O(\vec{x},t)$:

$$\chi(\vec{x}t;\vec{x}\,'t') \equiv -\frac{i}{\hbar}\theta(t-t')\langle G|\left[O(\vec{x},t),O(\vec{x}\,',t')\right]|G\rangle \qquad (3.8)$$

That is, the susceptibility is the retarded Green function of the observable.

Let $f(\vec{x},\omega)$ be the time Fourier transform of the force $f(\vec{x},t)$,

$$f(\vec{x},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; e^{i\omega t}\; f(\vec{x},t) \qquad (3.9)$$

The time Fourier transform of Eq.(3.7) is

$$\langle G|O(\vec{x},\omega)|G\rangle = \int d^3\vec{x}'\; \chi(\vec{x},\vec{x}\,';\omega)\; f(\vec{x}\,',\omega) \qquad (3.10)$$

However, since

$$\begin{aligned}
&\langle G|O(\vec{x},t)|G\rangle = \\
&= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\int d^3x' \int_{-\infty}^{t} dt' \left(\frac{-i}{\hbar}\right) e^{i\omega(t'-t)}\langle G|\left[O(\vec{x},t),O(\vec{x}\,',t')\right]|G\rangle f(\vec{x}\,',\omega) \\
&= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\int d^3x' \int_{-\infty}^{0} dt' \left(\frac{-i}{\hbar}\right) e^{i\omega t'}\langle G|\left[O(\vec{x},t),O(\vec{x}\,',t+t')\right]|G\rangle f(\vec{x}\,',\omega)
\end{aligned} \qquad (3.11)$$

we find

$$\langle G|O(\vec{x},\omega)|G\rangle = \int d^3x' \left(\frac{-i}{\hbar}\right) \int_{-\infty}^{0} dt' \langle G|\left[O(\vec{x},t), O(\vec{x}\,',t+t')\right]|G\rangle f(\vec{x}\,',\omega) \tag{3.12}$$

Hence, the time-Fourier transform of the susceptibility becomes

$$\begin{aligned} \chi(\omega;\vec{x},\vec{x}\,') &= -\frac{i}{\hbar} \int_{-\infty}^{0} d\tau e^{i\omega\tau} \langle G|[O(\vec{x},t), O(\vec{x}\,';t+\tau)]|G\rangle \\ &\equiv -\frac{i}{\hbar} \int_{-\infty}^{0} d\tau e^{i\omega\tau} \langle G|[O(\vec{x},0), O(\vec{x}\,',\tau)]|G\rangle \end{aligned} \tag{3.13}$$

Therefore, we find that the susceptibility $\chi(\vec{x},\vec{x}\,';\omega)$ is given by

$$\boxed{\chi(\vec{x},\vec{x}\,';\omega) = -\frac{i}{\hbar} \int_{-\infty}^{0} d\tau\ e^{i\omega\tau}\ \langle G|[O(\vec{x},0), O(\vec{x}\,',\tau)]|G\rangle} \tag{3.14}$$

This result is known as the *Kubo Formula.*

We just showed that the retarded Green function of the observable $O(\vec{x},t)$,

$$\mathcal{D}_O^{\rm ret}(x,x') = -i\theta(t-t')\langle G|[O(x),O(x')]|G\rangle \tag{3.15}$$

is related to the generalized susceptibility,

$$\chi(\omega;\vec{x},\vec{x}') = \frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\ \mathcal{D}_O^{\rm ret}(x,x') \tag{3.16}$$

Hence,

$$\begin{aligned} \langle G|O(x)|G\rangle = & \frac{1}{\hbar} \int_{-\infty}^{\infty} dt' \int d^3x' \mathcal{D}_O^{\rm ret}(x,x') f(\vec{x}\,',t') \\ & \equiv \frac{1}{\hbar} \int d^4x' \mathcal{D}_O^{\rm ret}(x,x') f(x') \end{aligned} \tag{3.17}$$

where $x \equiv (\vec{x},t)$. After a Fourier transform we get

$$\langle G|O(\vec{p},\omega)|G\rangle = \frac{1}{\hbar}\mathcal{D}_O^{\rm ret}(\vec{p},\omega) f(\vec{p},\omega) \tag{3.18}$$

In Fourier space, the generalized susceptibility $\chi(\vec{p},\omega)$ is

$$\chi(\vec{p},\omega) \;=\; \frac{\langle G|O(\vec{p},\omega)|G\rangle}{f(\vec{p},\omega)} \;=\; \frac{1}{\hbar}\mathcal{D}_O^{\text{ret}}(\vec{p},\omega) \qquad \Longleftrightarrow \qquad \chi = \frac{\text{“response”}}{\text{“force”}} \tag{3.19}$$

In conclusion, the response functions of physical interest, and the associated measurable susceptibilities, are given in terms of the retarded Green functions of the physical observables. However, as we saw in the last section, what we can calculate more directly are the time-ordered Green functions, *i.e.* the propagators of the observables. we will show next that the analytic properties of these functions in the complex frequency (or energy) plane are such that there is an analytic continuation procedure relating the time-ordered and the retarded Green functions. In addition, in practice we really are interested in knowing the physical susceptibilities and propagators not only in the ground state but also at non-zero temperature. Fortunately the analytic continuation will enable us to determine all these functions also at non-zero temperature.

3.2 Finite Temperature

All physical systems in thermal equilibrium are actually at finite temperature T. We will now show that there is a simple and straightforward way to adapt the $T = 0$ methods to account for the effects of thermal fluctuations. We will assume throughout that the system is in *thermal equilibrium* with a heat bath at temperature T and at a fixed chemical potential μ, *i.e.* we will treat the system in the *Grand Canonical Ensemble*. The *Gibbs density matrix* ρ_G and the Grand partition Function $\mathcal{Z}_G$ are

$$\rho_G \equiv e^{-\beta\,(H-\mu N)}, \qquad \mathcal{Z}_G = \text{Tr}\,\rho_G = e^{-\beta\Omega_G} \tag{3.20}$$

where $\beta = 1/(kT)$, H is the Hamiltonian, N is the particle number operator, which commutes with the Hamiltonian, $[N,H]=0$, $\mathcal{Z}_G$ is the grand partition function and Ω_G is the grand potential (or thermodynamic potential). [1]

The *thermal expectation value*, *i.e.* the expectation value in the the grand

[1]Recall that a chemical potential can *only* be defined for a *conserved* quantity, in this case the particle number operator N.

canonical ensemble, of a physical local observable $A(\vec{x},t)$ is given by

$$\left\langle A(\vec{x},t)\right\rangle = \frac{\text{Tr }\left(A(\vec{x},t)\ \rho_G\ \right)}{\text{Tr }\rho_G} \tag{3.21}$$

Let $\{|\lambda\rangle\}$ be a complete set of eigenstates of the full Hamiltonian H and of the particle number operator N, with eigenvalues $\{E_\lambda\}$ and $\{N_\lambda\}$. (Notice that since we are working in the grand canonical ensemble, necessarily the states $\{|\lambda\rangle\}$ are states in Fock space.) Then, we can write the thermal average as

$$\left\langle A(\vec{x},t)\right\rangle = \frac{\sum_\lambda \langle\lambda|A(\vec{x},t)|\lambda\rangle\ e^{-\beta(E_\lambda\ -\ \mu N_\lambda)}}{\sum_\lambda e^{-\beta(E_\lambda\ -\ \mu N_\lambda)}} \tag{3.22}$$

Thus, in principle in order to compute a thermal average we will have to know first the quantum mechanical expectation value in each state and then compute the thermal average. Although this sounds rather laborious there are very direct ways to compute the thermal average directly. In what follows, as we did before, we will include the chemical potential term in the Hamiltonian, $H \to H - \mu N$, which now depends explicitly on the chemical potential μ.

To calculate thermal averages it useful to define *temperature correlation functions*. Let A be the Schrödinger operator for an observable. We define a *temperature variable* τ, with $0 \leq \tau \leq \beta\hbar$ and the τ-dependent operator $A(\tau)$

$$A(\tau) = e^{H\tau/\hbar}\ A\ e^{-H\tau/\hbar} \tag{3.23}$$

which is formally obtained by the analytic continuation to imaginary time $\tau = it$ of the Heisenberg operator $A(t)$

$$A(t) = e^{iHt/\hbar}\ A\ e^{-iHt/\hbar} \tag{3.24}$$

(Notice that in general $A^\dagger(\tau) \neq A(\tau)^\dagger$.)

Likewise, the evolution operator $U(t)$ of a quantum system in the Schrödinger picture, now becomes

$$U(t) = e^{-iHt/\hbar} \to U_e(\tau) = e^{-H\tau/\hbar} \tag{3.25}$$

In particular the Gibbs density matrix $\rho_G(\beta)$ is

$$\rho_G(\beta) = e^{-\beta H} = U_e(\beta\hbar) \tag{3.26}$$

Thus a quantum system in equilibrium at temperature T can be regarded (formally) as a the imaginary time evolution of a quantum system. In addition since the ensemble average is a *trace* in Fock space, the initial and finite states are the same state. In other words, the *states obey periodic boundary conditions* in the imaginary time coordinate τ. Thus we will restrict the imaginary time variable to the range $0 \leq \tau \leq \beta\hbar$.

Much of what we did at $T = 0$ can be done also at $T \neq 0$. In particular we can similarly define imaginary time ordered products of operators, $T_\tau(A(\tau_1)B(\tau_2))$, by analogy with time ordered products. The temperature Green function for the electron operator is defined as

$$\begin{aligned} G_T^{\sigma\sigma'}(\vec{x},\tau;\vec{x}\,',\tau') &= -\left\langle T_\tau(\psi_\sigma(\vec{x},\tau)\psi^\dagger_{\sigma'}(\vec{x}\,',\tau'))\right\rangle \\ &= -\frac{\text{Tr}\,\left(T_\tau(\psi_\sigma(\vec{x},\tau)\psi^\dagger_{\sigma'}(\vec{x}\,',\tau'))\;\rho_G\right)}{\text{Tr}\;\rho_G} \end{aligned} \tag{3.27}$$

Due to the fact that we are computing a trace it is straightforward to show that, as a consequence of the fermionic statistics (and of the anti-commutation relations of fermionic operators), the imaginary time propagator, the temperature Green function, for *fermionic* operators obeys *anti-periodic* boundary conditions in imaginary time (independently on *both* τ and τ'). Instead, the temperature Green function for *bosonic* operators obey *periodic* boundary conditions in imaginary time. Thus

$$G_T(\vec{x},\tau+\beta\hbar;\vec{x}\,',\tau') = \mp G_T(\vec{x},\tau;\vec{x}\,',\tau') \tag{3.28}$$

where $(-)$ holds for fermions and $(+)$ holds for bosons. These periodic and anti-periodic boundary conditions have important consequences. In particular, since the imaginary time interval is finite, the propagators can be expanded in Fourier *series* as

$$G_T(\vec{x}-\vec{x}\,',\tau-\tau\,') = \sum_{n\in\mathbb{Z}} G_T(\vec{x}-\vec{x}\,',\omega_n)\; e^{i\omega_n(\tau-\tau')} \tag{3.29}$$

Periodic and anti-periodic boundary conditions in τ and in τ' are satisfied if the frequencies $\{\omega_n\}$, where $n \in \mathbb{Z}$, are respectively given by

$$\begin{aligned} \omega_n &= \frac{2\pi}{\beta\hbar}\, n, && \text{for bosons} \\ \omega_n &= \frac{2\pi}{\beta\hbar}\left(n+\frac{1}{2}\right), && \text{for fermions} \end{aligned} \tag{3.30}$$

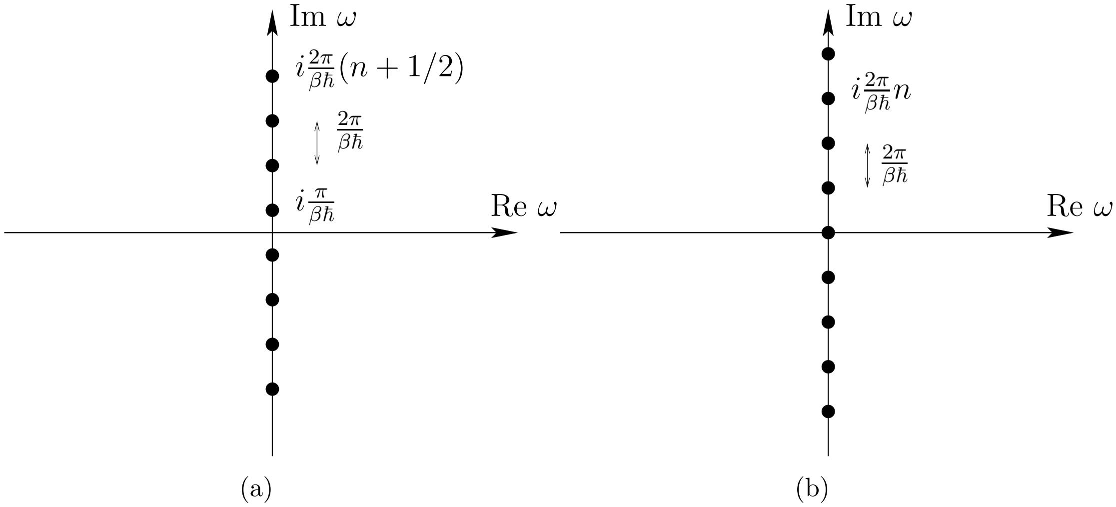

Figure 3.1: Poles on the imaginary frequency axis for (a) fermions and (b) bosons.

These considerations also apply to correlation functions which are always bosonic, even in a theory of fermions as they involve operators which are typically bilinear in fermions.

Since the temperature Green function is formally the analytic continuation of the time-ordered Green function to imaginary time, restricted to the imaginary time interval $0 \leq \tau \leq \beta\hbar$, its imaginary time Fourier transform (or series), $G_F(\vec{x}-\vec{x}\,', \omega_n)$ can also be regarded as the analytic continuation from real frequency ω to the imaginary frequency axis, restricted to the values $i\omega_n$ discussed above for fermions and bosons (See Fig.3.1).

3.3 Green Functions at $T \neq 0$

We will see now that there is a close relationship between the time-ordered, the retarded and the temperature propagators. The same considerations apply for the response and correlation functions. The key connection is the concept of the *spectral function* which describes the *fluctuation spectrum*. In particular, for the case of the response functions we will find a connection between the fluctuations and the dissipation in the system.

We will begin with the connection between the fermion propagator, the fermion retarded Green function and the fermion temperature propagator.

To simplify matters I will drop the spin indices (which are easily restored) and focus on the time dependence. Except for straightforward sign changes the same ideas apply to the boson propagator. Here we will follow closely the discussion in Doniach and Sondheimer.

3.3.1 Spectral Function

Let us define the fermion correlator $G_>(\vec{x}-\vec{x}\,',t)$

$$G_>(\vec{x}-\vec{x}\,',t) = \left\langle \psi(\vec{x},t)\; \psi^\dagger(\vec{x}\,',0) \right\rangle = \frac{\mathrm{Tr}\; \left(e^{-\beta H} e^{iHt/\hbar} \psi(\vec{x}) e^{-iHt/\hbar} \psi^\dagger(\vec{x}\,')\right)}{\mathrm{Tr}\; e^{-\beta H}} \tag{3.31}$$

Notice that this is a thermal average of a product (not necessarily ordered!) of Heisenberg operators.

Let $\{|\lambda\rangle\}$ be a complete set of eigenstates of the Hamiltonian in Fock space. Completeness means that the identity operator in Fock space can be expanded as

$$1 = \sum_\lambda |\lambda\rangle\langle\lambda| \tag{3.32}$$

Then, we can write the correlator as

$$\begin{aligned} \left\langle \psi(\vec{x},t)\; \psi^\dagger(\vec{x}\,',0) \right\rangle &= \frac{1}{\mathcal{Z}_G} \sum_{\lambda\lambda'} \langle\lambda|\psi^\dagger(\vec{x}\,') e^{-\beta H}|\lambda'\rangle\langle\lambda'| e^{iHt/\hbar}\psi(\vec{x}) e^{-iHt/\hbar}|\lambda\rangle \\ &= \frac{1}{\mathcal{Z}_G} \sum_{\lambda\lambda'} e^{-\beta E_{\lambda'}} \langle\lambda|\psi^\dagger(\vec{x}\,')|\lambda'\rangle\langle\lambda'|\psi(\vec{x})|\lambda\rangle e^{i(E_{\lambda'}-E_\lambda)t/\hbar} \end{aligned} \tag{3.33}$$

where

$$\mathcal{Z}_G = \sum_\lambda e^{-\beta E_\lambda} \tag{3.34}$$

is the (grand) partition function.

Let $J_1(\vec{x}-\vec{x}\,';\omega)$ be the time Fourier transform of $\left\langle \psi(\vec{x},t)\; \psi^\dagger(\vec{x}\,',0) \right\rangle$,

$$\begin{aligned} J_1(\vec{x}-\vec{x}\,';\omega) &= \int_{-\infty}^{\infty} \left\langle \psi(\vec{x},t)\; \psi^\dagger(\vec{x}\,',0) \right\rangle e^{i\omega t} dt \\ &= \frac{1}{\mathcal{Z}_G} \sum_{\lambda\lambda'} e^{-\beta E_{\lambda'}} \langle\lambda|\psi^\dagger(\vec{x}\,')|\lambda'\rangle\langle\lambda'|\psi(\vec{x})|\lambda\rangle\; 2\pi\; \delta\left(\frac{E_{\lambda'}-E_\lambda}{\hbar}+\omega\right) \end{aligned} \tag{3.35}$$

$J_1(\vec{x}-\vec{x}\,';\omega)$ is the *spectral function* of the time correlator $\langle\psi(\vec{x},t)\;\psi^\dagger(\vec{x}\,',0)\rangle$.

Similarly we can also write the correlator $G_<(\vec{x}-\vec{x}\,',t) = \left\langle \psi^\dagger(\vec{x}\,',0)\psi(\vec{x},t)\right\rangle$ and its time Fourier transform $J_2(\vec{x},\vec{x}\,';\omega)$:

$$
\begin{aligned}
J_2(\vec{x}-\vec{x}\,';\omega) &= \int_{-\infty}^{\infty} \left\langle \psi^\dagger(\vec{x}\,',0)\psi(\vec{x},t)\right\rangle e^{i\omega t} dt \\
&= \frac{1}{\mathcal{Z}_G}\sum_{\lambda\lambda'} e^{-\beta E_\lambda}\langle\lambda|\psi^\dagger(\vec{x}\,')|\lambda'\rangle\langle\lambda'|\psi(\vec{x})|\lambda\rangle\; 2\pi\;\delta(\frac{E_{\lambda'}-E_\lambda}{\hbar}+\omega)
\end{aligned}
\tag{3.36}
$$

Using the properties of the Dirac δ-function we get

$$
\begin{aligned}
J_2(\vec{x}-\vec{x}\,';\omega) &= \frac{1}{\mathcal{Z}_G}\sum_{\lambda\lambda'} e^{-\beta(E_{\lambda'}+\hbar\omega)}\langle\lambda|\psi^\dagger(\vec{x}\,')|\lambda'\rangle\langle\lambda'|\psi(\vec{x})|\lambda\rangle\; 2\pi\;\delta(\frac{E_{\lambda'}-E_\lambda}{\hbar}+\omega) \\
&= e^{-\beta\hbar\omega} J_1(\vec{x}-\vec{x}\,';\omega)
\end{aligned}
\tag{3.37}
$$

Notice that at $T=0$, only the state $|\lambda'\rangle = |G\rangle$ (the ground state) survives in sum over λ'. Thus, at $T=0$, $J_1(\vec{x}-\vec{x}\,';\omega)$ *vanishes* for $\omega<0$, while $J_2(\vec{x}-\vec{x}\,';\omega)$ vanishes (at $T=0$) for $\omega>0$.

3.3.2 The Retarded Green Function

Let us consider now the *retarded fermion Green function* $G^R(\vec{x}-\vec{x}\,',t)$ (at finite temperature),

$$
\begin{aligned}
G^R(\vec{x}-\vec{x}\,',t) &= -i\theta(t)\left\langle\;\left\{\psi(\vec{x},t),\psi^\dagger(\vec{x}\,',0)\right\}\;\right\rangle \\
&= -i\theta(t)\left(\langle\psi(\vec{x},t)\;\psi^\dagger(\vec{x}\,',0)\rangle + \langle\psi^\dagger(\vec{x}\,',0)\psi(\vec{x},t)\;\rangle\right) \\
&= -i\theta(t)\left(G_>(\vec{x}-\vec{x}\,',t)+G_<(\vec{x}-\vec{x}\,',t)\right)
\end{aligned}
\tag{3.38}
$$

where we have used *anti-commutators* because we are dealing with fermion operators. For a bosonic operator we must replace the anticommutator by a commutator.

For a translationally invariant system we can clearly express the retarded Green function in terms of the spectral functions $J_1(\vec{k},\omega)$ and $J_2(\vec{k},\omega)$, the space Fourier transforms of the functions $J_1(\vec{x},\vec{x}\,';\omega)$ and $J_2(\vec{x},\vec{x}\,';\omega)$ (where

$\vec{k}$ is the wave vector) defined above. To do that we will note that for the case of translationally invariant systems the Hamiltonian also commutes with the total linear momentum operator $\vec{P}$, $[\vec{P}, H] = 0$, and hence the states $\{|\lambda\rangle\}$ can also be chosen to be eigenstates of the linear momentum $\vec{P}$, with eigenvalues $\{\vec{P}_\lambda\}$. If we now recall that since $\vec{P}$ is the generator of infinitesimal translations in space, we can also write

$$\psi(\vec{x}) = e^{i\vec{P}\cdot\vec{x}/\hbar}\psi(0)e^{-i\vec{P}\cdot\vec{x}/\hbar} \tag{3.39}$$

Hence, in Fourier space, the spectral function $J_1(\vec{k}, \omega)$ becomes

$$J_1(\vec{k}, \omega) = \frac{1}{\mathcal{Z}_G}\sum_{\lambda\lambda'} e^{-\beta E_{\lambda'}}|\langle\lambda'|\psi(0)|\lambda\rangle|^2 \, 2\pi \, \delta(\frac{E_{\lambda'} - E_\lambda}{\hbar}+\omega) \, (2\pi)^3 \, \delta^{(3)}(\frac{\vec{P}_{\lambda'} - \vec{P}_\lambda}{\hbar}+\vec{k}) \tag{3.40}$$

where $\vec{P}_\lambda$ and $\vec{P}_{\lambda'}$ are the total linear momentum of the states $|\lambda\rangle$ and $|\lambda'\rangle$. Notice that $J_1(\vec{k}, \omega)$ is a *real function*.

By the same argument used above we also get

$$J_2(\vec{k}, \omega) = e^{-\beta\hbar\omega} J_1(\vec{k}, \omega) \tag{3.41}$$

Then, the retarded *fermion* Green function at $T > 0$ is

$$\begin{aligned} G^R(\vec{x}, t; \vec{x}\,', 0) &= -i\theta(t)\int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left(J_1(\vec{k}, \omega') + J_2(\vec{k}, \omega')\right) \, e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}\,')-\omega' t\right)} \\ &= -i\theta(t)\int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left(1 + e^{-\beta\hbar\omega'}\right) J_1(\omega') \, e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}\,')-\omega' t\right)} \end{aligned} \tag{3.42}$$

Its Fourier transform $G^R(\vec{k}, \omega)$ is

$$G^R(\vec{k}, \omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left(1 + e^{-\beta\hbar\omega'}\right) \frac{J_1(\vec{k}, \omega')}{\omega - \omega' + i\eta} \tag{3.43}$$

(where $\eta \to 0^+$). This function has poles in the lower half plane $\text{Im}\omega < 0$.

We will now use the fact that $J_1(\vec{k}, \omega)$ is a real function and the property

$$\lim_{\eta\to 0^+} \frac{1}{x + i\eta} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{3.44}$$

where x is real and

$$\mathcal{P}\frac{1}{x} = \lim_{\eta \to 0^+} \frac{x}{x^2 + \eta^2} \tag{3.45}$$

is the principal value of $1/x$, to show that

$$\begin{aligned} \text{Im } G^R(\vec{k}, \omega) &= -\pi \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left(1 + e^{-\beta\hbar\omega'}\right) J_1(\vec{k}, \omega') \, \delta(\omega - \omega') \\ &= -\frac{1}{2} \left(1 + e^{-\beta\hbar\omega}\right) J_1(\vec{k}, \omega) \end{aligned} \tag{3.46}$$

Hence we obtain

$$J_1(\vec{k}, \omega) = -\left(\frac{2}{1 + e^{-\beta\hbar\omega}}\right) \text{ Im } G^R(\vec{k}, \omega) \tag{3.47}$$

For a bosonic operator, and in particular this will apply to the generalized susceptibilities, the corresponding relation between the spectral function $J_1(\vec{k}, \omega)$ and the imaginary part of the Green function is

$$J_1(\vec{k}, \omega) = -\left(\frac{2}{1 - e^{-\beta\hbar\omega}}\right) \text{ Im } G^R(\vec{k}, \omega) \tag{3.48}$$

3.3.3 The Time-Ordered (Feynman) Fermion Propagator

Let us consider now the Feynman fermion propagator (also at $T > 0$)

$$G_F(\vec{x} - \vec{x}\,', t) = -i\left\langle T(\psi(\vec{x}, t)\psi^\dagger(\vec{x}\,', 0)\right\rangle \tag{3.49}$$

where we use a fermion time-ordered product. It can also be expressed in terms of the spectral function $J_1(\vec{x} - \vec{x}\,'; \omega)$ and of its Fourier transform (in space) $J_1(\vec{k}, \omega)$:

$$\begin{aligned} G_F(\vec{x} - \vec{x}\,', t) &= -i\theta(t) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} J_1(\vec{x} - \vec{x}\,', \omega') e^{-i\omega' t} \\ &\quad + i\theta(-t) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} J_2(\vec{x} - \vec{x}\,', \omega') e^{-i\omega' t} \end{aligned} \tag{3.50}$$

and its time (and space) Fourier transform

$$\begin{aligned} G_F(\vec{k},\omega) &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\{ \frac{J_1(\vec{k},\omega')}{\omega-\omega'+i\eta} + \frac{J_2(\vec{k},\omega')}{\omega-\omega'-i\eta} \right\} \\ &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, J_1(\vec{k},\omega') \, \left\{ \frac{1}{\omega-\omega'+i\eta} + \frac{e^{-\beta\hbar\omega'}}{\omega-\omega'-i\eta} \right\} \end{aligned} \tag{3.51}$$

which has poles both in the upper and in the lower half of the complex ω plane. Thus, contrary to the retarded Green function, the Feynman propagator is not analytic on either half plane.

Since $J_1(\vec{k},\omega)$ is real, we get

$$\text{Re}\; G^R(\vec{k},\omega) = \text{Re}\; G_F(\vec{k},\omega) = \mathcal{P}\int_{-\infty}^{\infty} \left(1+e^{-\beta\hbar\omega'}\right)\, \frac{J_1(\vec{k},\omega')}{\omega-\omega'}\, \frac{d\omega'}{2\pi} \tag{3.52}$$

and

$$\begin{aligned} \text{Im}\; G^R(\vec{k},\omega) &= -\frac{1}{2}\left(1+e^{-\beta\hbar\omega}\right)\, J_1(\vec{k},\omega) \\ \text{Im}\; G_F(\vec{k},\omega) &= -\frac{1}{2}\left(1-e^{-\beta\hbar\omega}\right)\, J_1(\vec{k},\omega) \end{aligned} \tag{3.53}$$

which imply that the real and imaginary parts of the retarded and time ordered functions are related by

$$\begin{aligned} \text{Re}\; G^R(\vec{k},\omega) &= \text{Re}\; G_F(\vec{k},\omega) \\ \text{Im}\; G^R(\vec{k},\omega) &= \coth(\frac{\beta\hbar\omega}{2})\; \text{Im}\; G_F(\vec{k},\omega) \end{aligned} \tag{3.54}$$

At $T=0$,

$$\text{Im}\; G^R(\vec{k},\omega) = \text{sign}(\omega)\; \text{Im}\; G_F(\vec{k},\omega) \tag{3.55}$$

Finally the real and imaginary parts of the retarded (and time-ordered) Green function are related by the Kramers-Kronig relation

$$\text{Re}\; G^R(\vec{k},\omega) = -\mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im}\; G^R(\vec{k},\omega')}{\omega-\omega'} \tag{3.56}$$

3.3.4 The temperature Green Function

The temperature Green function $G_T(\vec{x}-\vec{x}\,', \tau)$ is the time-ordered propagator in imaginary time τ,

$$G_T(\vec{x}-\vec{x}\,', \tau) = -\left\langle T_\tau(\psi(\vec{x},\tau)\psi^\dagger(\vec{x}\,',0))\right\rangle \tag{3.57}$$

where T_τ is the imaginary time ordering operator, and $0 \leq \tau \leq \beta\hbar$. Hence

$$G_T(\vec{x}-\vec{x}\,', \tau) = -\theta(\tau)\langle\psi(\vec{x},\tau)\psi^\dagger(\vec{x}\,',0)\rangle + \theta(-\tau)\langle\psi^\dagger(\vec{x}\,',0)\psi(\vec{x},\tau)\rangle \tag{3.58}$$

Once again we expand the thermal expectation values in terms of complete sets of states $\{|\lambda\rangle\}$, to obtain

$$\begin{aligned}
\left\langle\psi(\vec{x},\tau)\psi^\dagger(\vec{x}\,',0)\right\rangle &= \frac{1}{\mathcal{Z}_G}\sum_{\lambda\lambda'} e^{-\beta E_{\lambda'}}\langle\lambda|\psi^\dagger(\vec{x}\,')|\lambda'\rangle\langle\lambda'|\psi(\vec{x})|\lambda\rangle e^{\tau(E_{\lambda'}-E_\lambda)/\hbar} \\
\left\langle\psi^\dagger(\vec{x}\,',0)\psi(\vec{x},\tau)\right\rangle &= \frac{1}{\mathcal{Z}_G}\sum_{\lambda\lambda'} e^{-\beta E_{\lambda}}\langle\lambda|\psi^\dagger(\vec{x}\,')|\lambda'\rangle\langle\lambda'|\psi(\vec{x})|\lambda\rangle e^{\tau(E_{\lambda'}-E_\lambda)/\hbar}
\end{aligned} \tag{3.59}$$

Since the temperature (or imaginary time) fermion Green function is anti-periodic in time, $G_T(\vec{x}-\vec{x}\,', \tau) = -G_T(\vec{x}-\vec{x}\,', \tau+\beta\hbar)$, it can be expanded in Fourier series with coefficients

$$\begin{aligned}
G_T(\vec{x}-\vec{x}\,', \omega_n) &= \frac{1}{\beta\hbar}\int_0^{\beta\hbar} d\tau\; e^{-i\omega_n\tau}\; G_T(\vec{x}-\vec{x}\,', \tau) \\
&= \frac{1}{\beta\hbar}\int_0^{\beta\hbar} d\tau\; e^{-i\omega_n\tau}\; \langle\psi(\vec{x},\tau)\psi^\dagger\vec{x}\,')\rangle \\
&= -\frac{1}{\beta\hbar\mathcal{Z}_G}\sum_{\lambda\lambda'}\langle\lambda|\psi^\dagger(\vec{x}\,')|\lambda'\rangle\langle\lambda'|\psi(\vec{x})|\lambda\rangle\left\{\frac{e^{-\beta E_\lambda}+e^{-\beta E_{\lambda'}}}{E_{\lambda'}-E_\lambda-i\hbar\omega_n}\right\}
\end{aligned} \tag{3.60}$$

where $\omega_n = \frac{2\pi}{\beta\hbar}(n+1/2)$.

In Fourier space we find

$$\begin{aligned}
G_T(\vec{k},\omega_n) &= -\frac{1}{\beta\hbar}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left(\frac{J_1(\vec{k},\omega')+J_2(\vec{k},\omega')}{-i\omega_n-\omega'}\right) \\
&= -\frac{1}{\beta\hbar}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left(1+e^{-\beta\hbar\omega'}\right)\frac{J_1(\vec{k},\omega')}{-i\omega_n-\omega'}
\end{aligned} \tag{3.61}$$

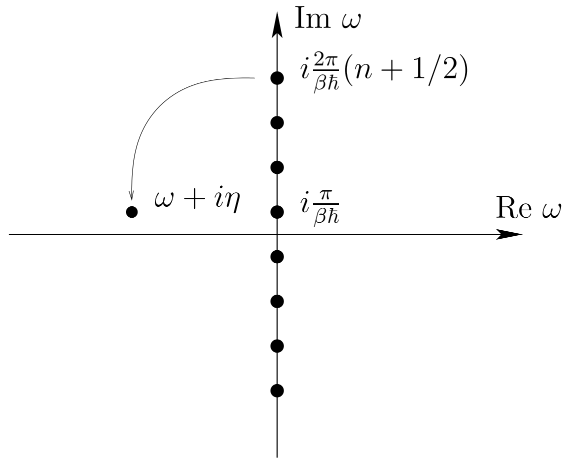

Figure 3.2: Analytic continuation to real frequency at finite temperature.

Thus, the temperature Green function $G_T(\vec{k}, \omega_n)$ can also be determined from the spectral function $J_1(\vec{k}, \omega)$.

Furthermore, if we now compare this result with the analogous expression for the retarded Green function $G^R(\vec{k}, \omega)$ we see that

$$G^R(\vec{k}, -i\omega_n) = -\beta\hbar G_T(\vec{k}, \omega_n) \tag{3.62}$$

Hence, $G_R(\vec{k}, \omega)$ is the analytic continuation of $-\beta\hbar G_T(\vec{k}, \omega_n)$ to a patch of the complex plane, that includes the imaginary axis:

$$\boxed{-i\omega_n \to \omega + i\eta \;\Rightarrow\; -\beta\hbar G_T(\vec{k}, \omega_n) \to G^R(\vec{k}, \omega)} \tag{3.63}$$

Since $G_T(\vec{k}, \omega_n)$ is known only for a discrete set of frequencies on the imaginary axis, $\{i\omega_n\}$, this procedure amounts to an analytic continuation from the imaginary frequency axis to the whole complex frequency plane, and thus to the real axis. (See Fig.3.2)

Thus we have now a direct way to compute these functions

1. We first compute the temperature Green function $G_T(\vec{k}, \omega_n)$
2. Next, by analytic continuation we find the retarded Green function $G^R(\vec{k}, \omega)$
3. The spectral function $J_1(\vec{k}, \omega)$ is obtained from the imaginary part of the retarded Green function.

4. The time-ordered and the retarded Green functions are both determined in terms on the spectral function

3.4 Dissipation and Response

We will now extend these ideas to response functions. We will find along the way a very important result known as the *fluctuation-dissipation theorem*.

We will begin by considering first a very simple problem. Imagine that we have a classical linear harmonic oscillator, represented by a coordinate $x(t)$. The oscillator has mass m and natural frequency ω_0. The oscillator is in thermal equilibrium with a bath represented by a set of random time-dependent "internal forces" $F_{\text{int}}(t)$, which represent the collisions between the degrees of freedom of the bath with the oscillator. Thus in equilibrium the classical equation of motion of the oscillator is

$$m\frac{d^2x(t)}{dt^2} + m\omega_0^2 x(t) = F_{\text{int}}(t) \tag{3.64}$$

We will now imagine that we act with an external force $F_{\text{ext}}(t)$ on the oscillator. [2] The *average non-equilibrium* displacement (averaged with respect to the random internal forces) which we will denote by $\langle x(t)\rangle_{\text{n.e.}}$, is linearly related to the external force by an expression of the form

$$\langle x(t)\rangle_{\text{n.e.}} = \int_{-\infty}^{\infty} \bar{\chi}(t,t')F_{\text{ext}}(t')dt' \tag{3.65}$$

The equation of motion in the presence of the external force is

$$m\frac{d^2x(t)}{dt^2} + m\omega_0^2 x(t) = F_{\text{int}}(t) + F_{\text{ext}}(t) \tag{3.66}$$

so that on average we have

$$m\left\langle \frac{d^2x(t)}{dt^2} \right\rangle_{\text{n.e.}} + m\omega_0^2 \langle x(t)\rangle_{\text{n.e.}} = \langle F_{\text{int}}(t)\rangle_{\text{n.e.}} + F_{\text{ext}}(t) \tag{3.67}$$

where we have used the fact that, in the presence of the external force, the average effect of the internal forces, which we define as *friction*, does not

[2]Notice that in the presence of the external force, the system (the oscillator) is no longer in equilibrium.

vanish, $\langle F_{\rm int}(t)\rangle_{\rm n.e.} \neq 0$. In simple phenomenological models we normally use the "constitutive relation" which asserts that the friction force is a linear function of the velocity of the oscillator

$$\langle F_{\rm int}(t)\rangle_{\rm n.e.} = -m\gamma\left\langle \frac{dx(t)}{dt}\right\rangle_{\rm n.e.} \tag{3.68}$$

where γ is the friction constant. The equation of motion now is

$$m\left\langle \frac{d^2x(t)}{dt^2}\right\rangle_{\rm n.e.} + m\gamma\left\langle \frac{dx(t)}{dt}\right\rangle_{\rm n.e.} + m\omega_0^2\langle x(t)\rangle_{\rm n.e.} = F_{\rm ext}(t) \tag{3.69}$$

Hence, $\bar{\chi}(t,t')$ satisfies

$$m\frac{d^2\bar{\chi}(t,t')}{dt^2} + m\gamma\frac{d\bar{\chi}(t,t')}{dt} + m\omega_0^2\bar{\chi}(t,t') = \delta(t-t') \tag{3.70}$$

which is to say that $\bar{\chi}(t,t') = \bar{\chi}(t-t')$ is the retarded Green function for this non-equilibrium system.

It is straightforward to show that $\bar{\chi}(t-t')$ has the integral representation

$$\bar{\chi}(t-t') = -\frac{1}{m}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega^2-\omega_0^2+i\gamma\omega} \tag{3.71}$$

where the integration runs along the real axis on any path in the upper half plane. Thus, the real and imaginary parts of its Fourier transform, $\chi'(\omega)$ and $\chi''(\omega)$, in this simple problem are, respectively, given by

$$\chi'(\omega) = \frac{\omega_0^2-\omega^2}{m\left[\left(\omega^2-\omega_0^2\right)^2+\left(\gamma\omega\right)^2\right]} \tag{3.72}$$

$$\chi''(\omega) = \frac{\omega\gamma}{m\left[\left(\omega^2-\omega_0^2\right)^2+\left(\gamma\omega\right)^2\right]} \tag{3.73}$$

As usual, $\chi'(\omega)$ and $\chi''(\omega)$ obey the Kramers-Kronig relation.

Using Cauchy's Theorem we can define a function $\chi(z)$ where z is a complex number,

$$\chi(z) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\chi(\omega)}{\omega-z} = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\chi''(\omega)}{\omega-z} \tag{3.74}$$

where z is in the upper half plane, $\text{Im}z > 0$. By explicit calculation we find

$$\chi(z) = -\frac{1}{m} \frac{1}{z^2 - \omega_0^2 - i\gamma z} \tag{3.75}$$

which does not have a pole on either half plane but instead a *branch cut along the real axis*, with discontinuity $2\chi''(\omega)$. Notice that the branch cut arises only in the presence of a finite damping coefficient γ. In other words, it is due to the existence of friction forces.

We will now show that $\chi''(\omega)$ represents the *dissipation*. Let us compute the work done by the external force, per unit time,

$$\begin{aligned} -\frac{dW}{dt} &= F_{\text{ext}} \left\langle \frac{dx(t)}{dt} \right\rangle_{\text{n.e.}} = F_{\text{ext}}(t) \int_{-\infty}^{\infty} dt' \, \frac{d\chi}{dt}(t-t') \, F_{\text{ext}}(t') \\ &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \, e^{-i\omega t} \, F_{\text{ext}}(t) \, (-i\omega')\chi(\omega') \int_{-\infty}^{\infty} dt' \, e^{i\omega' t'} F_{\text{ext}}(t') \end{aligned} \tag{3.76}$$

For a "monochromatic" force $F_{\text{ext}}(t) = \text{Re}\ (\mathcal{F}_{\text{ext}} e^{i\omega t})$, the power dissipated, averaged over one cycle, is

$$-\overline{\frac{dW}{dt}} = \frac{\omega}{2} \, \chi''(\omega) \, \overline{F_{\text{ext}}^2(t)} \tag{3.77}$$

where we used that $\chi''(\omega) = -\chi''(-\omega)$, which holds for this simple model but it is more generally true. Hence, we see that the dissipated power is proportional to $\chi''(\omega)$, *i.e.* to the discontinuity across the branch cut. In particular this simple model suggests that there is a direct connection between dissipation and susceptibilities. This result is actually quite general and we will identify the imaginary part of the response functions with the dissipative response.

3.5 The Fluctuation-Dissipation Theorem

Let us now return to the discussion of the *response functions*. As we saw above, even in Fermi systems the response functions are related to correlators of bosonic operators (*e.g.* bilinears of Fermi operators). In particular, for a general bosonic operator $O(\vec{x}, t)$, the generalized susceptibility $\chi(\vec{x}, t; \vec{x}\,', t')$ is simply related to the retarded Green function $\mathcal{D}^{\rm ret}(\vec{x},t;\vec{x}\,',t')$ for the operator $O(\vec{x},t)$:

$$\chi(\vec{x},t;\vec{x}\,',t') = \frac{1}{\hbar}\mathcal{D}^{\rm ret}(\vec{x},t;\vec{x}\,',t') \tag{3.78}$$

This relation is also correct at finite temperature T. Thus, just as in the case of the fermion correlators, we will need the connection between the retarded, time-ordered and temperature correlation functions. These relations are very similar expect for minor changes due to Bose statistics. We will not repeat all the arguments and we will instead just write down the important results. Just as in the fermionic case, the key concept is the *spectral function*.

Thus, for a general bosonic operator $O(\vec{x},t)$ we define the correlators $\mathcal{D}_>(\vec{x}-\vec{x}\,',t)$, $\mathcal{D}_<(\vec{x}-\vec{x}\,',t)$, $\mathcal{D}^{\rm ret}(\vec{x}-\vec{x}\,',t)$, $\mathcal{D}_F(\vec{x}-\vec{x}\,',t)$ and $\mathcal{D}_T(\vec{x}-\vec{x}\,',\tau)$, by the thermal averages

$$\begin{aligned}
\mathcal{D}_>(\vec{x}-\vec{x}\,',t) &= \Big\langle O(\vec{x},t)O(\vec{x}\,',0)\Big\rangle \\
\mathcal{D}_<(\vec{x}-\vec{x}\,',t) &= \Big\langle O(\vec{x}\,',0)\; O(\vec{x},t)\Big\rangle \\
\mathcal{D}^{\rm ret}(\vec{x}-\vec{x}\,',t) &= -i\theta(t)\Big\langle\; [O(\vec{x}\,',t),O(\vec{x},0)]\;\Big\rangle \\
\mathcal{D}_F(\vec{x}-\vec{x}\,',t) &= -i\Big\langle T(O(\vec{x}\,',t),O(\vec{x},0))\Big\rangle \\
\mathcal{D}_T(\vec{x}-\vec{x}\,',\tau) &= \Big\langle T_\tau(O(\vec{x}\,',\tau),O(\vec{x},0))\Big\rangle
\end{aligned} \tag{3.79}$$

The spectral functions $J_1(\vec{k},\omega)$ and $J_2(\vec{k},\omega)$ are once again the space and time Fourier transforms of the correlators $D_>(\vec{x}-\vec{x}\,',t)$ and $\mathcal{D}_<(\vec{x}-\vec{x}\,',t)$ respectively:

$$\begin{aligned}
J_1(\vec{k},\omega) &= \mathcal{D}_>(\vec{k},\omega) \\
J_2(\vec{k},\omega) &= \mathcal{D}_<(\vec{k},\omega)
\end{aligned} \tag{3.80}$$

which obey

$$J_2(\vec{k},\omega) = e^{-\beta\hbar\omega}\; J_1(\vec{k},\omega) \tag{3.81}$$

However, since the operators $O(\vec{x},t)$ are bosonic (that is, they obey commutation relations), the temperature Green function obeys *periodic* boundary conditions in imaginary time, $\mathcal{D}_T(\vec{x}-\vec{x}\,',\tau+\beta\hbar) = \mathcal{D}_T(\vec{x}-\vec{x}\,',\tau)$. This condition, which implies that the Matusbara frequencies now take the values $\omega_n = \frac{2\pi}{\beta\hbar}n$, where $n\in\mathbb{Z}$, has important consequences. In particular, paraphrasing what we did for the Fermi case, we now find that the spectral function $J_1(\vec{k},\omega)$ and the imaginary part of the Fourier transform of the retarded Green function Im $\mathcal{D}^{\rm ret}(\vec{k},\omega)$ are related by

$$J_1(\vec{k},\omega) = -\frac{2}{1-e^{-\beta\hbar\omega}} \text{Im } \mathcal{D}^{\rm ret}(\vec{k},\omega) \tag{3.82}$$

In particular, this relation implies that the dissipative component of the dynamical susceptibility $\chi''(\vec{k},\omega)$ is proportional to the spectral function $J_1(\vec{k},\omega)$,

$$\chi''(\vec{k},\omega) = -\frac{\hbar}{2}\left(1-e^{-\beta\hbar\omega}\right) J_1(\vec{k},\omega) \tag{3.83}$$

This relation is known as the *Fluctuation-Dissipation Theorem*.

On the other hand the retarded and time ordered Green functions are now related by

$$\begin{aligned} \text{Re}D^{\rm ret}(\vec{k},\omega) &= \text{Re}\mathcal{D}_F(\vec{k},\omega) & (3.84)\\ \text{Im}D^{\rm ret}(\vec{k},\omega) &= \tanh(\frac{\beta\hbar\omega}{2})\text{Im}\mathcal{D}_F(\vec{k},\omega) & (3.85)\end{aligned}$$

Also, just as in the fermionic case, the real and imaginary parts of the retarded correlation function are related by the usual Kramers-Kronig relation

$$\text{Re } \mathcal{D}^{\rm ret}(\vec{k},\omega) = -\mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\, \frac{\text{Im } \mathcal{D}^{\rm ret}(\vec{k},\omega')}{\omega-\omega'} \tag{3.86}$$

Finally, the temperature Green function $\mathcal{D}_T(\vec{k},\omega_n)$, where $\omega_n = \frac{2\pi}{\beta\hbar}n$, is related to the spectral function by the integral transform

$$\mathcal{D}_T(\vec{k},\omega_n) = -\frac{1}{\beta\hbar}\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left(1-e^{-\beta\hbar\omega'}\right) \frac{J_1(\vec{k},\omega')}{-i\omega_n-\omega'} \tag{3.87}$$

Thus, here too, all the correlation functions of interest can be found by the *same* analytic continuation procedure we already discussed for fermions. The bosonic nature of the correlation functions follows form (a) the physically correct spectral functions and (b) the temperature-dependent prefactors in the integrand of this equation.

3.6 Perturbation Theory for $T > 0$

We will now discuss briefly perturbation theory at finite temperature T. The approach is very similar to what we did at $T = 0$.

Let is consider a system with Hamiltonian $H = H_0 + H_1$. We will be interested in the fermion temperature Green function $G_T(\vec{x} - \vec{x}\,', \tau)$

$$G_T(\vec{x} - \vec{x}\,', \tau - \tau') = \left\langle T_\tau(\psi(\vec{x}, \tau)\psi^\dagger(\vec{x}\,', \tau'))\right\rangle \tag{3.88}$$

for a system described by the Hamiltonian H.

We now introduce a representation analogous to the interaction representation but for $T > 0$. Let $U(\tau, \tau')$ be the operator

$$U(\tau, \tau') = e^{\tau H_0/\hbar} e^{-(\tau-\tau')(H_0+H_1)/\hbar} e^{-\tau' H_0/\hbar} \tag{3.89}$$

Notice that although this operator is not unitary, it satisfies the group property

$$U(\tau, \tau')U(\tau', \tau'') = U(\tau, \tau'') \tag{3.90}$$

and it obeys the (imaginary time) equation of motion

$$-\hbar \frac{\partial U}{\partial \tau}(\tau, \tau') = H_1(\tau)U(\tau, \tau') \tag{3.91}$$

where

$$H_1(\tau) = e^{\tau H_0/\hbar} H_1 e^{-\tau H_0/\hbar} \tag{3.92}$$

In particular the full density matrix is given by

$$e^{-\beta H} = e^{-\beta H_0} U(\beta\hbar, 0) \tag{3.93}$$

It is straightforward to show that the temperature Green function of the full system is given by the following expression in terms of thermal averages of the unperturbed system:

$$G_T(\vec{x} - \vec{x}\,', \tau - \tau') = \frac{\left\langle T_\tau(U(\beta\hbar, 0)\psi(\vec{x}, \tau)\psi^\dagger(\vec{x}\,', \tau'))\right\rangle_0}{\left\langle U(\beta\hbar, 0)\right\rangle_0} \tag{3.94}$$

Here we used the notation

$$\langle A\rangle_0 = \frac{\text{Tr }(e^{-\beta H_0} A)}{\text{Tr } e^{-\beta H_0}} \tag{3.95}$$

where A is an arbitrary operator.

Notice that the operator $U(\beta\hbar, 0))$ plays here a role analogous to the S-matrix. Hence, using the same line of reasoning that we used for the S-matrix, we can write

$$\begin{aligned} U(\beta\hbar, 0) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{\hbar^n} \int_0^{\beta\hbar} d\tau_1 \ldots \int_0^{\beta\hbar} d\tau_n \; T_\tau(H_1(\tau_1) \ldots H_1(\tau_n)) \\ &\equiv T_\tau \; e^{-\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau H_1(\tau)} \end{aligned} \tag{3.96}$$

Thus we can carry out perturbation theory much in the same way as we did at $T = 0$. The only missing ingredient is a generalization of Wick's Theorem for $T > 0$. It turns out that this works exactly in the same way provided that the contraction of an arbitrary pair of operators $A(\tau_1)$ and $B(\tau_2)$ is identified with *imaginary time ordered* thermal expectation values of the operators with respect to H_0:

$$\overbrace{A(\tau_1) B}(\tau_2) = \langle T_\tau (A(\tau_1) B(\tau_2)) \rangle_0 \tag{3.97}$$

which is a *thermal contraction*. Notice that we are using the same notation for the thermal contraction that we used at $T = 0$ although it now has a different meaning.

Wick's Theorem simply states that the thermal average of the product of any number of operators is equal to the sum of the products of all possible pair-wise thermal contractions. Consequently, the structure of the perturbation series is the same as the one we found at $T = 0$. In particular we will also find that disconnected diagrams, which at $T = 0$ are called the "vacuum diagrams", also cancel out exactly at $T > 0$. They only survive in the computation of the grand partition function. Thus, all of the Feynman rules used at $T = 0$ carry over to $T > 0$ with the only change that the factor of $-i/\hbar$ now simply becomes $-1/\hbar$. All other rules remain intact, except that the propagators are now temperature Green functions, which obey specific periodic and anti-periodic boundary conditions in imaginary time.

3.7 The Electrical Conductivity of a Metal

We will now consider the response of an electron gas to weak external electromagnetic fields $A_\mu(x)$. The formalism can be generalized easily to other systems and responses. In particular, we will consider the electrical conductivity of a metal.

There are three effects (and couplings) that we need to take into consideration: a) electrostatic, b) diamagnetic (or orbital) and c) paramagnetic. The electrostatic coupling is simply the coupling to an external potential with H_{ext} given by

$$H_{\text{ext}} = \sum_{\sigma=\uparrow,\downarrow} \int d^3x \; e\phi(x,t) \; \psi_\sigma^\dagger(x)\psi_\sigma(x) \equiv \int d^3x \; J_0(\vec{x})A_0(\vec{x}) \tag{3.98}$$

where $\phi \equiv A_0$ is the scalar potential (or time component of the vector potential A_μ). The diamagnetic coupling (or orbital) follows from the minimal coupling to the external vector potential $\vec{A}$. The kinetic energy term H_{kin} is modified following the minimal coupling prescription to become

$$H_{\text{kin}}(A) = \int d^3x \frac{\hbar^2}{2m} \left(\vec{\bigtriangledown} + \frac{ie}{\hbar c}\vec{A}(x) \right) \psi^\dagger(x) \cdot \left(\vec{\bigtriangledown} - \frac{ie}{\hbar c}\vec{A}(\vec{x}) \right) \psi(x) \tag{3.99}$$

which can be written as a sum of two terms

$$H_{\text{kin}}(A) = H_{\text{kin}}(0) + H_{\text{ext}}(A) \tag{3.100}$$

where $H_{\text{kin}}(0)$ is the kinetic energy term of the Hamiltonian in the absence of the field, and $H_{\text{ext}}(A)$ is the total perturbation, *i.e.*,

$$\begin{aligned} H_{\text{ext}}(A) &= \int d^3x \left[J_0(\vec{x})A_0(\vec{x}) - \vec{J}(\vec{x}) \cdot \vec{A}(\vec{x}) - \frac{e}{2mc^2}\vec{A}^2(\vec{x})J_0(x) \right] \\ &= \int d^3x \left[J_\mu(x)A^\mu(x) - \frac{e}{2mc^2}\vec{A}^{\,2}(x)J_0(x) \right] \end{aligned} \tag{3.101}$$

Here $J_0(x) = e\rho(x) = e\sum_\sigma \psi_\sigma^\dagger \psi_\sigma$ is the local charge density, and $\vec{J}(\vec{x})$ is the gauge-invariant charge current defined as

$$\begin{aligned} \vec{J}(\vec{x}) &= \frac{ie\hbar}{2mc} \sum_\sigma [\psi_\sigma^\dagger(\vec{x})\vec{\bigtriangledown}\psi_\sigma(\vec{x}) - \vec{\bigtriangledown}\psi^\dagger(\vec{x})\psi(\vec{x})] - \frac{e^2}{mc^2}\vec{A}(\vec{x}) \sum_\sigma \psi_\sigma^\dagger(\vec{x})\psi_\sigma(\vec{x}) \\ &\equiv \frac{ie\hbar}{2mc} \sum_\sigma [\psi_\sigma^\dagger(\vec{x})\vec{D}\psi_\sigma(\vec{x}) - \left(\vec{D}\psi_\sigma(\vec{x})\right)^\dagger \psi_\sigma(\vec{x})] \end{aligned} \tag{3.102}$$

where $\vec{D} = \vec{\nabla} - i\frac{e}{\hbar c}\vec{A}$ is the (spatial) covariant derivative. We are using here a 4-vector notation $J_\mu = (J_0, \vec{J})$ and $J_\mu A^\mu = J_0 A_0 - \vec{J} \cdot \vec{A}$.

Clearly $\vec{J}(\vec{x})$ is the sum of the two terms, one which represents the mass current and the diamagnetic term, $\frac{e^2}{mc^2}\vec{A}\psi^\dagger\psi$. We can write the total perturbation, including the scalar potential A_0, if we write

$$H_{\text{ext}} = \int d^3x [J_\mu(x) A^\mu(x) - \frac{e}{2mc^2}\vec{A}^2 J_0(x)] \tag{3.103}$$

Finally, we can also consider a paramagnetic coupling to the spin degrees of freedom which has the Zeeman form

$$H_{\text{ext}}^{\text{Zeeman}} = \int d^3x \; g\vec{B}(\vec{x}) \cdot \psi_\sigma^\dagger(\vec{x})\vec{S}_{\sigma\sigma'}\psi_{\sigma'}(\vec{x}) \tag{3.104}$$

where g is typically of the order of the Bohr magneton μ_B and $\vec{S} = \frac{\hbar}{2}$ for spin one-half systems. We will discuss the Zeeman term later in the context of the magnetically ordered states. For now will focus on the purely electromagnetic response of a general interacting Fermi system.

A straightforward application of the Linear Response formulas derived above yields an expression for the current $\langle J_\mu \rangle'$ in the presence of the perturbation.

$$\langle J_\mu(x)\rangle' = \langle J_\mu(x)\rangle_G - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \int d^3x' \langle G|[J_\nu(x'), J_\mu(x)]|G\rangle A_\nu(x') + \ldots \tag{3.105}$$

This formula suggests that we should define the *retarded current correlation function* $\mathcal{D}_{\mu\nu}^{\text{ret}}(x, x')$

$$\mathcal{D}_{\mu\nu}^{\text{ret}}(x, x') = -i\Theta(x_0 - x_0')\langle G|[J_\mu(x), J_\nu(x')]|G\rangle \tag{3.106}$$

The *induced current* $\langle J_k \rangle_{\text{ind}}$

$$\langle J_k \rangle_{\text{ind}} = \langle J_k \rangle' - \langle j_k \rangle_G \tag{3.107}$$

(where j_k is the "paramagnetic" component of the current) has a very simple form in terms of $\mathcal{D}_{\mu\nu}^{\text{ret}}(x, x')$, namely

$$\begin{aligned} \Big\langle J_k(x) \Big\rangle_{\text{ind}} &= \frac{1}{\hbar}\int d^4x' \frac{1}{\hbar}\left\{\mathcal{D}_{k\ell}^R(x, x')A_\ell(x') - \mathcal{D}_{k0}^R(x, x')A_0(x')\right\} \\ &\quad - \frac{e}{mc^2} J_0(x) A_k(x) + \ldots \end{aligned} \tag{3.108}$$

3.7.1 The dielectric tensor and the conductivity tensor

The induced current has two important properties: a) it is conserved, namely the the induced current and the induced density satisfy a continuity equation

$$\frac{\partial \rho_{\text{ind}}}{\partial t} + \vec{\nabla} \cdot \vec{J}_{\text{ind}} = 0 \tag{3.109}$$

and b) it is *gauge invariant*. This means that we should be able express the induced current directly in terms of gauge invariant quantities such as the electric and magnetic fields.

Since $\langle J_\mu(x)\rangle_{\text{ind}}$ is gauge invariant, we can compute its form in any gauge. In the gauge $A_0 = 0$ the spacial components of $\langle J_\mu(x)\rangle_{\text{ind}}$ are

$$\langle J_k(x)\rangle_{\text{ind}} = -\frac{e^2\rho}{mc^2} A_k(x) + \frac{1}{\hbar}\int d^4x' \mathcal{D}^{\text{ret}}_{k\ell}(x-x') A_\ell(x') + \ldots \tag{3.110}$$

In this gauge, the external electric field $\vec{E}_{\text{ext}}$ and magnetic field $\vec{H}$ are

$$\vec{E}_{\text{ext}} = -\partial_0 \vec{A} \qquad \vec{H} = \vec{\nabla}\times\vec{A} \tag{3.111}$$

Now, in Fourier space, we can write

$$\begin{aligned} \langle J_k(\vec{p},\omega)\rangle_{\text{ind}} &= -\frac{e^2\rho}{mc^2} A_k(\vec{p},\omega) + \frac{1}{\hbar}\mathcal{D}^{\text{ret}}_{k\ell}(\vec{p},\omega) A_\ell(\vec{p},\omega) \\ &\equiv \left(\frac{1}{\hbar}\mathcal{D}^{\text{ret}}_{k\ell}(\vec{p},\omega) - \frac{e^2\rho}{mc^2}\delta_{k\ell}\right) \frac{E^{\text{ext}}_\ell}{i\omega}(p,\omega) \end{aligned} \tag{3.112}$$

This expression is *almost* the conductivity. It is not quite that it since the conductivity is a relation between the *total current* $\vec{J} = \vec{J}_{\text{ind}} + \vec{J}_{\text{ext}}$ and the *total electric field* $\vec{E}$, instead of one between the *induced current* and the *external electric field*. In order to take these electromagnetic effects into account, we must use Maxwell's equations in a medium which involve the fields $\vec{E}, \vec{D}, \vec{B}$ and $\vec{H}$

$$\begin{array}{ll} \vec{\nabla}\cdot\vec{D} = \rho & \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \end{array} \tag{3.113}$$

where

$$\vec{B} = \vec{H} + \vec{M} \qquad \vec{E} = \vec{E}^{\text{ext}} + \vec{E}^{\text{ind}} \tag{3.114}$$

Here $\vec{M}$ and $\vec{E}^{\rm ind}$ are the magnetic and electric polarization vectors. In particular

$$\vec{J}^{\rm ind} = \partial_t \vec{E}^{\rm ind} \tag{3.115}$$

and

$$\partial_t \vec{D} = \partial_t \vec{E} + \vec{J}^{\rm ind} \tag{3.116}$$

Linear Response theory is the statement that $\vec{D}$ must be proportional to $\vec{E}$

$$D_j = \varepsilon_{jk} E_k \tag{3.117}$$

where ε_{jk} is the dielectric tensor. Since $\vec{E}$ and $\vec{E}^{\rm ext}$ satisfy similar equations

$$\begin{aligned} -\vec{\bigtriangledown} \times \vec{\bigtriangledown} \times \vec{E} \quad &= \partial_t^2 \vec{E} + \partial_t \vec{J} \\ -\vec{\bigtriangledown} \times \vec{\bigtriangledown} \times \vec{E}^{\rm ext} \quad &= \partial_t^2 \vec{E}^{\rm ext} + \partial_t \vec{J}^{\rm ext} \end{aligned} \tag{3.118}$$

and $\vec{\bigtriangledown} \times \vec{\bigtriangledown} \times \vec{E} = \vec{\bigtriangledown}(\vec{\bigtriangledown} \cdot \vec{E}) - \bigtriangledown^2 \vec{E}$, we can write, for the Fourier transforms, the equations

$$\begin{aligned} p_i p_j E_j(\vec{p},\omega) - \vec{p}^2 E_i(\vec{p},\omega) \quad &= \quad -\omega^2 E_j(\vec{p},\omega) - i\omega J_i(\vec{p},\omega) \\ p_i p_j E_j^{\rm ext}(\vec{p},\omega) - \vec{p}^2 E_i^{\rm ext}(\vec{p},\omega) \quad &= \quad -\omega^2 E_i^{\rm ext}(\vec{p},\omega) - i\omega J_i^{\rm ext}(\vec{p},\omega) \end{aligned} \tag{3.119}$$

Thus, we get

$$\begin{aligned} p_i p_j E_j(\vec{p},\omega) - \vec{p}^2 E_i(\vec{p},\omega) \quad &+ \quad \omega^2 E_i(\vec{p},\omega) = -i\omega J_i^{\rm ind}(\vec{p},\omega) \\ &+ \quad p_i p_j E_j^{\rm ext}(\vec{p},\omega) - \vec{p}^2 E_i^{\rm ext}(\vec{p},\omega) + \omega^2 E_i^{\rm ext}(\vec{p},\omega) \end{aligned} \tag{3.120}$$

Since we showed above that

$$-i\omega J_i^{\rm ind}(\vec{p},\omega) = \left(\delta_{ij} \frac{e^2 \rho}{mc^2} - \frac{1}{\hbar} \mathcal{D}_{ij}^{\rm ret}(\vec{p},\omega) \right) E_j^{\rm ext}(\vec{p},\omega) \tag{3.121}$$

we conclude that

$$\begin{aligned} (p_i p_j - \vec{p}^2 \delta_{ij} \quad &+ \quad \omega^2 \delta_{ij}) \; E_j(\vec{p},\omega) = \\ & (\delta_{ij} \frac{e^2 \rho}{mc^2} - \frac{1}{\hbar} \mathcal{D}_{ij}^{R}(\vec{p},\omega) + p_i p_j - \vec{p}^2 \delta_{ij} + \omega^2 \delta_{ij}) \; E_j^{\rm ext}(\vec{p},\omega) \end{aligned} \tag{3.122}$$

In matrix form, these equations have the simpler form

$$(p \otimes p - \vec{p}^2 I + \omega^2 I)\vec{E} = \left(\frac{e^2 \rho}{mc^2} I - \frac{1}{\hbar} \mathcal{D}^R + p \otimes p - \vec{p}^2 I + \omega^2 I \right) \vec{E}_{\text{ext}} \quad (3.123)$$

This equation allows us to write $\vec{E}_{\text{ext}}$ in terms of $\vec{E}$.

Hence we find that the induced current is related to the total field by

$$\begin{aligned} i\omega \vec{J}_{\text{ind}} \quad &= \quad \left(\mathcal{D}^{\text{ret}} - \frac{e^2 \rho}{mc^2} 1 \right) [\frac{e^2 \rho}{mc^2} I - \frac{1}{\hbar} \mathcal{D}^{\text{ret}} + p \otimes p - \vec{p}^2 I + \omega^2 1]^{-1} \\ & \quad \left(p \otimes p - \vec{p}^2 I + \omega^2 I \right) \vec{E} \end{aligned} \quad (3.124)$$

It follows that the *conductivity tensor* σ is

$$\begin{aligned} i\omega\sigma(\vec{p}, \omega) \quad &= \quad \left(\frac{1}{\hbar} \mathcal{D}^{\text{ret}}(\vec{p}, \omega) - \frac{e^2 \rho}{mc^2} I \right) + \left(\frac{1}{\hbar} \mathcal{D}^{\text{ret}}(\vec{p}, \omega) - \frac{e^2 \rho}{mc^2} I \right) \\ & \quad [\frac{e^2 \rho}{mc^2} I - \frac{1}{\hbar} \mathcal{D}^{\text{ret}}(\vec{p}, \omega) + p \otimes p - \vec{p}^2 I + \omega^2 I]^{-1} \\ & \quad \left(\frac{1}{\hbar} \mathcal{D}^{\text{ret}}(\vec{p}, \omega) - \frac{e^2 \rho}{mc^2} I \right) \end{aligned} \quad (3.125)$$

Also, since $\vec{D} = \varepsilon \vec{E}$, the dielectric tensor ε is

$$\varepsilon = I + \frac{i}{\omega} \sigma \quad (3.126)$$

We conclude that both the conductivity tensor and the dielectric tensor can be determined from the retarded current and density correlation functions.

3.7.2 Correlation Functions and Conservation Laws

In the problem discussed in the previous section, we saw that we had to consider a correlation function of currents. Since the currents are conserved, *i.e.*, $\partial_\mu J^\mu = 0$, we expect that the correlation function $\mathcal{D}_{\mu\nu}(x, x')$ should obey a similar equation. Let us compute the divergence of the retarded correlation function, $\partial_\mu^x \mathcal{D}_{\text{ret}}^{\mu\nu}(x, x')$,

$$\partial_\mu^x \mathcal{D}_{\text{ret}}^{\mu\nu}(x, x') = \partial_\mu^x [-i\Theta(x_0 - x_0') \langle G | [J^\mu(x), J^\nu(x')] | G \rangle \quad (3.127)$$

Except for the contribution coming from the step function, we see that we can operate with the derivative inside the expectation value to get

$$\begin{aligned}\partial^x_\mu \mathcal{D}^{\mu\nu}_{\rm ret}(x,x') &= -i\left(\partial^x_\mu \Theta(x_0-x'_0)\right)\langle G|[J^\mu(x),J^\nu(x')]|G\rangle \\ &\quad -i\Theta(x_0-x'_0)\langle G|[\partial^x_\mu J^\mu(x),J^\nu(x')]|G\rangle\end{aligned} \tag{3.128}$$

The second term vanishes since $J^\mu(x)$ is a conserved current and the first term is non zero only if $\mu=0$. Hence we find

$$\partial^x_\mu \mathcal{D}^{\mu\nu}_{\rm ret}(x,x') = -i\delta(x_0-x'_0)\langle G|[J^0(x),J^\nu(x')]|G\rangle \tag{3.129}$$

which is the v.e.v. of an equal-time commutator. These commutators are given by

$$\begin{aligned}\langle G|[J^0(\vec{x},x_0),J^0(\vec{x}',x_0)]|G\rangle &= 0 \\ \langle G|[J^0(\vec{x},x_0),J^i(\vec{x}',x_0)]G\rangle &= \frac{ie^2}{mc^2}\partial^x_k[\delta(\vec{x}-\vec{x}')\langle\rho(\vec{x})\rangle]\end{aligned} \tag{3.130}$$

Hence, the divergence of $\mathcal{D}^{\rm ret}_{\mu\nu}$ is

$$\partial^\mu_x \mathcal{D}^{\rm ret}_{\mu k}(x,x') = \frac{e^2}{mc^2}\partial^x_k[\delta^4(x-x')\langle\rho(x)\rangle] \quad ; \quad \partial^\mu_{x'}\mathcal{D}^{\rm ret}_{0\mu}(x,x')=0 \tag{3.131}$$

and

$$\partial^\nu_{x'} \mathcal{D}^{\rm ret}_{k\nu}(x,x') = -\frac{e^2}{mc^2}\partial^x_k[\delta^4(x-x')\langle\rho(x')\rangle] \quad ; \quad \partial^\mu_{x'}\mathcal{D}^{\rm ret}_{0\mu}(x,x')=0 \tag{3.132}$$

Notice that the time-ordered functions also satisfy these identities. These identities can be used to prove that $\langle\vec{J}^{\rm ind}\rangle$ is indeed gauge-invariant and conserved. Furthermore, in momentum and frequency space, the identities become

$$\begin{aligned}-i\omega\mathcal{D}^{\rm ret}_{00}(\vec{p},\omega)-ip_k\mathcal{D}^{\rm ret}_{k0}(\vec{p},\omega) &= 0 \\ -i\omega\mathcal{D}^{\rm ret}_{0k}(\vec{p},\omega)-ip_\ell\mathcal{D}^{\rm ret}_{\ell k}(\vec{p},\omega) &= -\frac{e^2\rho}{mc^2}ip_k \\ -i\omega\mathcal{D}^{\rm ret}_{00}(p,\omega)-ip_k\mathcal{D}^{\rm ret}_{0k}(\vec{p},\omega) &= 0 \\ -i\omega\mathcal{D}^{\rm ret}_{k0}(\vec{p},\omega)-ip_\ell\mathcal{D}^{\rm ret}_{k\ell}(\vec{p},\omega) &= -\frac{e^2\rho}{mc^2}ip_k\end{aligned} \tag{3.133}$$

We can combine these identities to get

$$\omega^2 \mathcal{D}^{\text{ret}}_{00}(\vec{p},\omega) - p_\ell p_k \mathcal{D}^{\text{ret}}_{\ell k}(\vec{p},\omega) = -\frac{e^2\rho}{mc^2}\vec{p}^{\,2} \tag{3.134}$$

Hence, the density-density and the current-current correlation functions are not independent. A number of interesting identities follow from this equation. In particular if we take the static limit $\omega \to 0$ at fixed momentum $\vec{p}$, we get

$$\lim_{\omega\to 0} p_\ell p_k \mathcal{D}^{\text{ret}}_{\ell k}(\vec{p},\omega) = \frac{e^2\rho}{mc^2}\vec{p}^{\,2} \tag{3.135}$$

provided that $\lim_{\omega\to 0}\mathcal{D}^{\text{ret}}_{00}(\vec{p},\omega)$ is not singular for $\vec{p}\neq 0$. Also from the equal-time commutator

$$\langle G|[J_k(\vec{x},x_0),J_0(\vec{x}',x_0)]|G\rangle = \frac{ie^2}{mc^2}\partial^x_k\left(\delta(\vec{x}-\vec{x}')\langle\rho(x)\rangle\right) \tag{3.136}$$

we get

$$\lim_{x'_0\to x_0}\partial^x_k \mathcal{D}^{ret}_{k0}(x,x') = \frac{e^2}{mc^2}\,\bigtriangledown^2_x\left(\delta(x-x')\langle\rho(x)\rangle\right) \tag{3.137}$$

If the system is uniform, $\langle\rho(x)\rangle=\rho$, we can Fourier transform this identity to get

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi} ip_k \mathcal{D}^{\text{ret}}_{k0}(\vec{p},\omega) = -\frac{e^2\rho}{mc^2}\vec{p}^{\,2} \tag{3.138}$$

The conservation laws yield the alternative expression

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi} i\omega \mathcal{D}^{\text{ret}}_{00}(\vec{p},\omega) = \frac{e^2\rho}{mc^2}\vec{p}^{\,2} \tag{3.139}$$

This identity is known as the *f-sum rule*.

If the system is isotropic, these relations can be used to yield a simpler form for the conductivity tensor. Indeed, if the system is isotropic, $\mathcal{D}^{\text{ret}}_{k\ell}(\vec{p},\omega)$ is a sum of a longitudinal part $\mathcal{D}^{\text{ret}}_{\parallel}$ and a transverse part $\mathcal{D}^{\text{ret}}_{\perp}$

$$\mathcal{D}^{\text{ret}}_{\ell k}(\vec{p},\omega) = \mathcal{D}^{\text{ret}}_{\parallel}(\vec{p},\omega)\frac{p_\ell p_k}{\vec{p}^{\,2}} + \mathcal{D}^{\text{ret}}_{\perp}(\vec{p},\omega)\left(\frac{p_\ell p_k}{\vec{p}^{\,2}} - \delta_{\ell k}\right) \tag{3.140}$$

Thus, we get a relation between $\mathcal{D}^{\text{ret}}_{00}$ and the longitudinal part $\mathcal{D}^{\text{ret}}_{\parallel}$

$$\omega^2\mathcal{D}^{\text{ret}}_{00}(p,\omega) - \vec{p}^{\,2}\mathcal{D}^{\text{ret}}_{\parallel}(\vec{p},\omega) = -\frac{e^2\rho}{mc^2}\vec{p}^{\,2} \tag{3.141}$$

Hence

$$\mathcal{D}_{00}^{\rm ret}(\vec{p},\omega) = \frac{\vec{p}^{\,2}}{\omega^2}\left(\mathcal{D}_{\parallel}^{\rm ret}(\vec{p},\omega) - \frac{e^2\rho}{mc^2}\right) \tag{3.142}$$

and

$$\lim_{\omega\to 0} \mathcal{D}_{\parallel}^{\rm ret}(\vec{p},\omega) = \frac{e^2\rho}{mc^2} \tag{3.143}$$

for all $\vec{p}$.

The conductivity tensor can similarly be split into longitudinal $\sigma_\parallel$ and transverse $\sigma_\perp$ components

$$\sigma_{ij} = \sigma_\parallel \frac{p_i p_j}{\vec{p}^{\,2}} + \sigma_\perp\left(\frac{p_i p_j}{\vec{p}^{\,2}} - \delta_{ij}\right) \tag{3.144}$$

We find

$$\sigma_\parallel = \frac{1}{i\omega}\left[\frac{\mathcal{D}_{\parallel}^{\rm ret} - \frac{e^2\rho}{mc^2}}{-\mathcal{D}_{\parallel}^{\rm ret} + \frac{e^2\rho}{mc^2} + \omega^2}\right] \tag{3.145}$$

and

$$\sigma_\perp = \frac{1}{i\omega}\left(\mathcal{D}_{\perp}^{\rm ret} - \frac{e^2\rho}{mc^2}\right)\left[1 + \frac{\mathcal{D}_{\perp}^{\rm ret} - \frac{e^2\rho}{mc^2}}{\frac{e^2\rho}{mc^2} - \mathcal{D}_{\perp}^{\rm ret} + \omega^2 - \vec{p}^{\,2}}\right] \tag{3.146}$$

These relations tell us that the real part of $\sigma_\parallel$ is determined by the imaginary part of $\mathcal{D}_{\parallel}^{\rm ret}$. Thus, the *resistive part* of $\sigma_\parallel$ (which is responsible for dissipation in the system) is determined by the imaginary part of a response function. This is generally the case as it follows from the fluctuation-dissipation theorem.

3.7.3 Consequences of the f-sum rule

We will now assume that there is a stable collective mode, *i.e.,* a *plasmon* branch. Therefore the retarded density-density correlation function $\mathcal{D}_{00}^R$ must have a pole at the plasmon dispersion, $\hbar\omega_{\rm pl}(\vec{p})$:

$$\mathcal{D}_{00}^R(p,\omega) = \frac{A(p,\omega)}{\omega^2 - \omega_{\rm pl}^2(p)} + \left[\mathcal{D}_{00}^R\right]_{\rm reg} \tag{3.147}$$

where $A(\vec{p},\omega)$ is the residue, and $\left[\mathcal{D}_{00}^R\right]_{\rm reg}$ is regular (analytic) near the plasma branch. In order to enforce the condition that $\mathcal{D}_{00}^R$ is the retarded correlation function we will perform the analytic continuation $\omega \to \omega + i\varepsilon$. This analytic structure of the correlation function holds for $\omega \approx |\omega_{p\ell}(p)|$.

We will now see that the sum rule determines the residue $A(p,\omega)$. However, since $\left[\mathcal{D}_{00}^R\right]_{\rm reg}$ is non-singular, we have

$$\int_{-\infty}^{+\infty} d\omega \; \omega \; \left[\mathcal{D}_{00}^R\right]_{\rm reg}(\omega, p) = 0 \tag{3.148}$$

since we can close the contour on the upper half plane without enclosing any singularities. Thus, in order to satisfy the f-sum rule we must demand that

$$\int_{-\infty}^{+\infty} d\omega \; i\omega \frac{A(p,\omega)}{(\omega + i\varepsilon)^2 - \omega_{\rm p\ell}^2(p)} = \frac{e^2 \rho}{mc^2} \; p^2 \tag{3.149}$$

We will now close the contour on the pole and compute the residue. The pole is at $\omega = \pm|\omega_{\rm p\ell}(p)| + i\epsilon$, and the residue is

$$\text{residue}: \quad = -\pi A(p, \pm|\omega_{\rm p\ell}(p)|) \tag{3.150}$$

which leads to the a condition for the amplitude $A(p,\omega)$:

$$-\pi (A(p, +|\omega_{\rm p\ell}(p)|) + A(p, -|\omega_{\rm p\ell}(p)|)) = \frac{e^2 \rho}{mc^2} p^2 \tag{3.151}$$

Hence,

$$A = -\frac{1}{2\pi} \, \frac{e^2 \rho}{mc^2} \; p^2 \tag{3.152}$$

(typically $A(\omega) = A^*(-\omega)$).

Thus we conclude that for frequencies close to the plasma branch the retarded density-density correlation function must have the form

$$\mathcal{D}_{00}^R(p,\omega) = -\frac{1}{2\pi} \, \frac{e^2 \rho}{mc^2} \, \frac{p^2}{\omega^2 - \omega_{\rm p\ell}^2(p)} + \left[\mathcal{D}_{00}^R\right]_{\rm reg}(p,\omega) \tag{3.153}$$

Notice that the residue is determined by the *exact density* and by the *bare mass*. This is a consequence of Galilean Invariance.