8 Coherent State Path Integral Quantization of Quantum Field Theory

8.1 Coherent states and path integral quantization.

8.1.1 Coherent States

Let us consider a Hilbert space spanned by a complete set of harmonic oscillator states $\{|n\rangle\}$, with $n = 0, \ldots, \infty$. Let $\hat{a}^\dagger$ and $\hat{a}$ be a pair of creation and annihilation operators acting on that Hilbert space, and satisfying the commutation relations

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 \,, \quad \left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0 \,, \quad [\hat{a}, \hat{a}] = 0 \tag{1}$$

These operators generate the harmonic oscillators states $\{|n\rangle\}$ in the usual way,

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle \tag{2}$$

$$\hat{a}|0\rangle = 0 \tag{3}$$

where $|0\rangle$ is the vacuum state of the oscillator.

Let us denote by $|z\rangle$ the *coherent state*

$$|z\rangle = e^{z\hat{a}^\dagger} |0\rangle \tag{4}$$

$$\langle z| = \langle 0| \, e^{\bar{z}\hat{a}} \tag{5}$$

where z is an arbitrary complex number and $\bar{z}$ is the complex conjugate. The coherent state $|z\rangle$ has the defining property of being a wave packet with optimal spread, *i.e.*, the Heisenberg uncertainty inequality is an equality for these coherent states.

How does $\hat{a}$ act on the coherent state $|z\rangle$?

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, \hat{a} \left(\hat{a}^\dagger\right)^n |0\rangle \tag{6}$$

Since

$$\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = n \, \left(\hat{a}^\dagger\right)^{n-1} \tag{7}$$

we get

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, \left(\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] + \left(\hat{a}^\dagger\right)^n \, \hat{a}\right) |0\rangle \tag{8}$$

Thus, we find

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, n \left(\hat{a}^\dagger\right)^{n-1} |0\rangle \equiv z \, |z\rangle \tag{9}$$

Therefore $|z\rangle$ is a right eigenvector of $\hat{a}$ and z is the (right) eigenvalue.

Likewise we get

$$\hat{a}^{\dagger}|z\rangle = \hat{a}^{\dagger} \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\hat{a}^{\dagger}\right)^n |0\rangle$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\hat{a}^{\dagger}\right)^{n+1} |0\rangle$$

$$= \sum_{n=0}^{\infty} (n+1) \frac{z^n}{(n+1)!} \left(\hat{a}^{\dagger}\right)^{n+1} |0\rangle$$

$$= \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} \left(\hat{a}^{\dagger}\right)^n |0\rangle \tag{10}$$

Thus,

$$\hat{a}^{\dagger}|z\rangle = \frac{\partial}{\partial z} |z\rangle \tag{11}$$

Another quantity of interest is the overlap of two coherent states, $\langle z|z'\rangle$,

$$\langle z|z'\rangle = \langle 0|e^{\bar{z}\hat{a}} \; e^{z'\hat{a}^{\dagger}} |0\rangle \tag{12}$$

We will calculate this matrix element using the Baker-Hausdorff formulas

$$e^{\hat{A}} \, e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}\left[\hat{A}, \hat{B}\right]} = e^{\left[\hat{A}, \hat{B}\right]} \, e^{\hat{B}} \, e^{\hat{A}} \tag{13}$$

which holds provided the commutator $\left[\hat{A}, \hat{B}\right]$ is a c-number, *i.e.*, it is proportional to the identity operator. Since $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$, we find

$$\langle z|z'\rangle = e^{\bar{z}z'} \, \langle 0|e^{z'\hat{a}^{\dagger}} \, e^{\bar{z}\hat{a}}|0\rangle \tag{14}$$

But

$$e^{\bar{z}\hat{a}} |0\rangle = |0\rangle \tag{15}$$

and

$$\langle 0| \, e^{z'\hat{a}^{\dagger}} = \langle 0| \tag{16}$$

Hence we get

$$\langle z|z'\rangle = e^{\bar{z}z'} \tag{17}$$

An arbitrary state $|\psi\rangle$ of this Hilbert space can be expanded in the harmonic oscillator basis states $\{|n\rangle\}$,

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!} \left(\hat{a}^{\dagger}\right)^n |0\rangle \tag{18}$$

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The projection of the state $|\psi\rangle$ onto the coherent state $|z\rangle$ is

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!} \langle z| \left(\hat{a}^\dagger\right)^n |0\rangle \tag{19}$$

Since

$$\langle z| \hat{a}^\dagger = \bar{z} \langle z| \tag{20}$$

we find

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!} \bar{z}^n \equiv \psi(\bar{z}) \tag{21}$$

Therefore the projection of $|\psi\rangle$ onto $|z\rangle$ is the *anti-holomorphic (i.e., anti-analytic)* function $\psi(\bar{z})$. In other words, in this representation, the space of states $\{|\psi\rangle\}$ are in one-to-one correspondence with the space of anti-analytic functions.

In summary, the coherent states $\{|z\rangle\}$ satisfy

$$\begin{array}{ll}
\hat{a}|z\rangle = z|z\rangle & \langle z|\hat{a} = \partial_{\bar{z}}\langle z| \\
\hat{a}^\dagger|z\rangle = \partial_z|z\rangle & \langle z|\hat{a}^\dagger = \bar{z}\langle z| \\
\langle z|\psi\rangle = \psi(\bar{z}) & \langle\psi|z\rangle = \bar{\psi}(z)
\end{array} \tag{22}$$

Next we will prove the *resolution of identity*

$$\hat{I} = \int \frac{dz d\bar{z}}{2\pi i} \, e^{-z\bar{z}} |z\rangle\langle z| \tag{23}$$

Let $|\psi\rangle$ and $|\phi\rangle$ be two arbitrary states

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}} |n\rangle$$

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}} |n\rangle$$

$$\langle\phi|\psi\rangle = \sum_{n=0}^{\infty} \frac{\phi_n \psi_n}{n!} \tag{24}$$

Let us compute the matrix element

$$\langle\phi|\hat{I}|\psi\rangle = \sum_{m.n} \frac{\bar{\phi}_n \psi_n}{n!} \langle n|\hat{I}|m\rangle \tag{25}$$

Thus we need to find

$$\langle n|\hat{I}|m\rangle = \int \frac{dz d\bar{z}}{2\pi i} \, e^{-|z|^2} \langle n|z\rangle\langle z|m\rangle \tag{26}$$

Recall that the integration measure is defined to be given by

$$\frac{dz d\bar{z}}{2\pi i} = \frac{d\mathrm{Re}z \, d\mathrm{Im}z}{\pi} \tag{27}$$

where

$$\langle n|z \rangle = \frac{1}{\sqrt{n!}} \langle 0| \left(\hat{a} \right)^n |z \rangle = \frac{z^n}{\sqrt{n!}} \langle 0|z \rangle \tag{28}$$

and

$$\langle z|m \rangle = \frac{1}{\sqrt{m!}} \langle z| \left(\hat{a}^\dagger \right)^m |0 \rangle = \frac{\bar{z}^m}{\sqrt{m!}} \langle z|0 \rangle \tag{29}$$

Now, since $|\langle 0|z \rangle|^2 = 1$, we get

$$\langle n|\hat{I}|m \rangle = \int \frac{dz d\bar{z}}{2\pi i} \frac{e^{-|z|^2}}{\sqrt{n!m!}} z^n \bar{z}^m = \int_0^\infty \rho d\rho \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{e^{-\rho^2}}{\sqrt{n!m!}} \rho^{n+m} e^{i(n-m)\varphi} \tag{30}$$

Thus,

$$\langle n|\hat{I}|m \rangle = \frac{\delta_{n,m}}{n!} \int_0^\infty dx \; x^n e^{-x} = \langle n|m \rangle \tag{31}$$

Hence, we have found that

$$\langle \phi|\hat{I}|\psi \rangle = \langle \phi|\psi \rangle \tag{32}$$

for any pair of states $|\psi \rangle$ and $|\phi \rangle$. Therefore $\hat{I}$ is the identity operator in that space. We conclude that the set of coherent states $\{|z \rangle\}$ is an *over-complete* set of states.

Furthermore, since

$$\langle z| \left(\hat{a}^\dagger \right)^n \left(\hat{a} \right)^m |z' \rangle = \bar{z}^n z'^m \langle z|z' \rangle = \bar{z}^n z'^m e^{\bar{z}z'} \tag{33}$$

we conclude that the matrix elements of any arbitrary normal ordered operator of the form

$$\hat{A} = \sum_{n,m} A_{n,m} \left(\hat{a}^\dagger \right)^n \left(\hat{a} \right)^m \tag{34}$$

are equal to

$$\langle z|\hat{A}|z' \rangle = \left(\sum_{n,m} A_{n,m} \bar{z}^n z'^m \right) e^{\bar{z}z'} \tag{35}$$

Therefore, if $\hat{A}(\hat{a}, \hat{a}^\dagger)$ is an arbitrary *normal ordered* operator (relative to the state $|0 \rangle$), its matrix elements are given by

$$\langle z|\hat{A}(\hat{a}, \hat{a}^\dagger)|z' \rangle = A(\bar{z}, z') e^{\bar{z}z'} \tag{36}$$

where $A(\bar{z}, z')$ is a function of two complex variables $\bar{z}$ and z', obtained from $\hat{A}$ by the formal replacement

$$\hat{a} \leftrightarrow z' \;, \qquad \hat{a}^\dagger \leftrightarrow \bar{z} \tag{37}$$

For example, the matrix elements of the the operator $\hat{N} = \hat{a}^\dagger \hat{a}$, which measures the number of excitations, is

$$\langle z|\hat{N}|z' \rangle = \langle z|\hat{a}^\dagger \hat{a}|z' \rangle = \bar{z}z' \; e^{\bar{z}z'} \tag{38}$$

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8.1.2   Path Integrals and Coherent States

As usual we will want to compute the matrix elements of the evolution operator $\mathcal{U}$,

$$\mathcal{U} = e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})} \tag{39}$$

where $\hat{H}(\hat{a}^\dagger, \hat{a})$ is a normal ordered operator. Thus, if $|i\rangle$ and $|f\rangle$ denote two arbitrary initial and final states, we can write the matrix element of $\mathcal{U}$ as

$$\langle f| e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})} |i\rangle = \lim_{\epsilon\to 0, N\to\infty} \langle f| \left(1 - -i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)^N |i\rangle \tag{40}$$

However now, instead of inserting a complete set of states at each intermediate time t_j (with $j = 1, \ldots, N$), we will insert an over-complete set $\{|z_j\rangle\}$ at each time t_j through the insertion of the resolution of the identity,

$$\langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)^N |i\rangle =$$

$$= \int \left(\prod_{j=1}^N \frac{dz_j d\bar{z}_j}{2\pi i}\right) e^{-\sum_{j=1}^N |z_j|^2} \langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right) |z_N\rangle$$

$$\times \langle z_N| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right) |z_{N-1}\rangle \ldots \langle z_1| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right) |i\rangle =$$

$$\equiv \int \left(\prod_{j=1}^N \frac{dz_j d\bar{z}_j}{2\pi i}\right) e^{-\sum_{j=1}^N |z_j|^2} \left[\prod_{k=1}^{N-1} \langle z_{k+1}| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right) |z_k\rangle\right]$$

$$\times \langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right) |z_N\rangle\langle z_1| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right) |z_i\rangle \tag{41}$$

In the limit $\epsilon \to 0$ these matrix elements are

$$\langle z_{k+1}| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right) |z_k\rangle = \langle z_{k+1}|z_k\rangle - i\frac{\epsilon}{\hbar}\langle z_{k+1}|\hat{H}(\hat{a}^\dagger, \hat{a})|z_k\rangle$$

$$= \langle z_{k+1}|z_k\rangle \left[1 - i\frac{\epsilon}{\hbar} H(\bar{z}_{k+1}, z_k)\right] \tag{42}$$

where $H(\bar{z}_{k+1}, z_k)$ is a *function* which is obtained from the *normal ordered* Hamiltonian by the substitutions $\hat{a}^\dagger \to \bar{z}_{k+1}$ and $\hat{a} \to z_k$. Hence, we can write

the following expression for the matrix element

$$\langle f|e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})}|i\rangle =$$

$$= \lim_{\epsilon\to 0, N\to\infty} \int \left(\prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i}\right) e^{-\sum_{j=1}^{N}|z_j|^2} e^{\sum_{j=1}^{N-1}\bar{z}_{j+1}z_j} \prod_{j=1}^{N-1}\left[1 - i\frac{\epsilon}{\hbar}H(\bar{z}_{k+1}, z_k)\right]$$

$$\times \langle f|z_N\rangle\langle z_1|i\rangle \left[1 - i\frac{\epsilon}{\hbar}\frac{\langle f|\hat{H}|z_N\rangle}{\langle f|z_N\rangle}\right]\left[1 - i\frac{\epsilon}{\hbar}\frac{\langle z_1|\hat{H}|i\rangle}{\langle z_1|i\rangle}\right]$$

$$(43)$$

By further expanding the initial and final states in coherent states

$$\langle f| = \int \frac{dz_f d\bar{z}_f}{2\pi i} e^{-|z_f|^2} \bar{\psi}_f(z_f)\langle z_f|$$

$$|i\rangle = \int \frac{dz_i d\bar{z}_i}{2\pi i} e^{-|z_i|^2} \psi_i(\bar{z}_i)|z_i\rangle$$

$$(44)$$

we find

$$\langle f|e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})}|i\rangle =$$

$$= \int \mathcal{D}z\mathcal{D}\bar{z}\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\frac{\hbar}{2i}(z\partial_t\bar{z} - \bar{z}\partial_t z) - H(z,\bar{z})\right]} e^{\frac{1}{2}(|z_i|^2 + |z_f|^2)} \bar{\psi}_f(z_f)\psi_i(\bar{z}_i)$$

$$(45)$$

This is the coherent-state form of the path integral. We can identify in this expression the Lagrangian L as the quantity

$$L = \frac{\hbar}{2i}(z\partial_t\bar{z} - \bar{z}\partial_t z) - H(z,\bar{z}) \qquad (46)$$

Notice that the Lagrangian in the coherent-state representation is first order in time derivatives. because of this feature we are not guaranteed that the paths are necessarily differentiable. This property leads to all kinds of subtleties that for the most part we will ignore in what follows.

8.1.3 Path integral for a gas of non-relativistic bosons at finite temperature.

The field theoretic description of a gas of (spinless) non-relativistic bosons is given in terms of the creation and annihilation field operators $\hat{\phi}^\dagger(\vec{x})$ and $\hat{\phi}(\vec{x})$, which satisfy the equal time commutation relations (in d space dimensions)

$$\left[\hat{\phi}(\vec{x}), \hat{\phi}^\dagger(\vec{y})\right] = \delta^d(\vec{x} - \vec{y}) \qquad (47)$$

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Relative to the empty state $|0\rangle$, *i.e.,*

$$\hat{\phi}(\vec{x})|0\rangle = 0 \tag{48}$$

the normal ordered Hamiltonian is

$$\hat{H} = \int d^d x \; \hat{\phi}^\dagger(\vec{x}) \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + \mu + V(\vec{x}) \right] \hat{\phi}(\vec{x})$$
$$+ \frac{1}{2} \int d^d x \int d^d y \; \hat{\phi}^\dagger(\vec{x})\hat{\phi}^\dagger(\vec{y}) U(\vec{x} - \vec{y})\hat{\phi}(\vec{y})\hat{\phi}(\vec{x}) \tag{49}$$

where m is the mass of the bosons, μ is the chemical potential, $V(\vec{x})$ is an external potential and $U(\vec{x} - \vec{y})$ is the interaction potential between pairs of bosons.

Following our discussion of the coherent state path integral we see that it is immediate to write down a path integral for a thermodynamic system of bosons. The boson coherent states are now labelled by a *complex field* $\phi(\vec{x})$ and its *complex conjugate* $\bar{\phi}(\vec{x})$.

$$|\{\phi(\vec{x})\}\rangle = e^{\int d\vec{x} \; \phi(\vec{x})\hat{\phi}(\vec{x})}|0\rangle \tag{50}$$

which has the coherent state property

$$\hat{\phi}(\vec{x})|\{\phi\}\rangle = \phi(\vec{x})|\{\phi\}\rangle \tag{51}$$

as well as the resolution of the identity

$$\mathcal{I} = \int \mathcal{D}\phi\mathcal{D}\phi^* e^{-\int d\vec{x} \; |\phi(\vec{x})|^2} |\{\phi\}\rangle\langle\{\phi\}| \tag{52}$$

The matrix element between an initial state $|i\rangle$ and a final state $|f\rangle$ separated by a time span $T = t_f - t_i$ (not to be confused with the temperature!) now takes the form

$$\langle f|e^{-\frac{i}{\hbar}\hat{H}T}|i\rangle =$$
$$\int \mathcal{D}\phi\mathcal{D}\bar{\phi} \; \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\int d^d x \; \frac{\hbar}{i} [\phi(\vec{x},t)\partial_t\bar{\phi}(\vec{x},t) - \bar{\phi}(\vec{x},t)\partial_t\phi(\vec{x},t)] - H[\phi,\bar{\phi}] \right) \right\}$$
$$\times \bar{\Psi}_f(\phi(\vec{x},t_f))\Psi_i(\bar{\phi}(\vec{x},t_i)) \; e^{\frac{1}{2}\int d\vec{x}(|\phi(\vec{x},t_f)|^2 + |\phi(\vec{x},t_i)|^2)} \tag{53}$$

where $H[\phi,\bar{\phi}]$ is

$$H[\phi,\bar{\phi}] = \int d^d x \; \bar{\phi}(\vec{x}) \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + \mu + V(\vec{x}) \right] \phi(\vec{x}) + \frac{1}{2} \int d^d x \int d^d y \; |\phi(\vec{x})|^2 |\phi(\vec{y})|^2 U(\vec{x} - \vec{y}) \tag{54}$$

It is also possible to write the action S in the less symmetric but simpler form $(dx \equiv dt d\vec{x})$

$$
\begin{aligned}
S = \quad & \int dx \, \bar{\phi}(x) \left(i\hbar \partial_t + \frac{\hbar^2}{2m} \vec{\nabla}^2 - \mu - V(\vec{x}) \right) \phi(x) \\
& - \frac{1}{2} \int d^d x \int d^d y \, |\phi(x)|^2 |\phi(y)|^2 U(x-y)
\end{aligned}
$$

$$(55)$$

where $U(x-y) = U(\vec{x} - \vec{y}) \delta(t_x - t_y)$.

This formulation is useful to study superfluid Helium and similar problems. Suppose for instance that we want to compute the partition function Z for this system of bosons at finite temperature T,

$$
Z = \text{tr} e^{-\beta \hat{H}} \tag{56}
$$

where $\beta = 1/T$ (in units where $k_B = 1$). The coherent-state path integral representation of the partition function is obtained by

1. restricting the initial and final states to be the same $|i\rangle = |f\rangle$ and arbitrary

2. Summing over all possible states

3. and finally a Wick rotation to imaginary time $t \to -i\tau$, with the *time-span* $T \to -i\beta\hbar$ (*i.e.*, Periodic Boundary Conditions in imaginary time)

The result is the (imaginary time) path integral

$$
Z = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \, e^{-S_E(\phi, \bar{\phi})} \tag{57}
$$

where S_E is the Euclidean action

$$
\begin{aligned}
S_E(\phi, \bar{\phi}) = \quad & \frac{1}{\hbar} \int_0^\beta d\tau \int d\vec{x} \, \bar{\phi} \left[-\hbar \partial_\tau + \mu - \frac{\hbar^2}{2m} \nabla^2 - V(x) \right] \phi \\
& + \frac{1}{2\hbar} \int_0^\beta d\tau \int d\vec{x} \int d\vec{y} \, U(x-y) \, |\phi(x)|^2 |\phi(y)|^2
\end{aligned}
$$

$$(58)$$

The fields $\phi(x) = \phi(\vec{x}, \tau)$ satisfy Periodic Boundary Conditions (PBC's) is imaginary time

$$
\phi(\vec{x}, \tau) = \phi(\vec{x}, \tau + \beta\hbar) \tag{59}
$$

This requirement suggests an expansion of the field $\phi(x)$ in Fourier modes of the form

$$
\phi(\vec{x}, \tau) = \sum_{n=-\infty}^{\infty} e^{i\omega_n \tau} \phi(\vec{x}, \omega_n) \tag{60}
$$

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where the frequencies ω_n (the *Matsubara* frequencies) must be chosen so that ϕ obeys the required PBCs. We find

$$\omega_n = \frac{2\pi}{\beta\hbar}n = \frac{2\pi T}{\hbar}n , \qquad n \in \mathbb{Z} \tag{61}$$

where n is an arbitrary integer.

8.2 Fermion Coherent States

In this section we will develop a formalism for *fermions* which follows closely what we have done for bosons while accounting for the anti-commuting nature of fermionic operators, *i. e.* the Pauli Principle.

Let $\{c_i^\dagger\}$ be a set of *fermion* creation operators, with $i = 1, \ldots, N$, and $\{c_i\}$ the set of their N adjoint operators, *i. e.* the associated annihilation operators. The number operator for the i-th fermion is $n_i = c_i^\dagger c_i$. Let us define the kets $|0_i\rangle$ and $|1_i\rangle$, which obey the obvious definitions:

$$
\begin{array}{ll}
c_i|0_i\rangle = 0 & c_i^\dagger|0_i\rangle = |1_i\rangle \\
c_i^\dagger c_i|0_i\rangle = 0 & c_i^\dagger c_i|1_i\rangle = |1_i\rangle
\end{array}
\tag{62}
$$

For N fermions the Hilbert space is spanned by the anti-symmetrized states $|n_1, \ldots, n_N\rangle$. Let

$$|0\rangle \equiv |0_1, \ldots, 0_N\rangle \tag{63}$$

and

$$|n_1, \ldots, n_N\rangle = c_1^\dagger \ldots c_N^\dagger |0\rangle \tag{64}$$

As we saw before, the wave function $\langle n_1, \ldots, n_N|\Psi\rangle$ is a Slater determinant.

8.2.1 Definition of Fermion Coherent States

We now define fermion coherent states. Let $\{\bar{\xi}_i, \xi_i\}$, with $i = 1, \ldots, N$, be a set of $2N$ *Grassmann variables* (also known as the *generators* of a Grassmann algebra.) These variables satisfy, by definition, the following properties

$$\{\xi_i, \xi_j\} = \{\bar{\xi}_i, \bar{\xi}_j\} = \{\xi_i, \bar{\xi}_j\} = \xi_i^2 = \bar{\xi}_i^2 = 0 \tag{65}$$

We will also require that the Grassmann variables anti-commute with the fermion operators:

$$\{\xi_i, c_j\} = \left\{\bar{\xi}_i, c_j^\dagger\right\} = \{\bar{\xi}_i, c_j\} = \left\{\bar{\xi}_i, c_j^\dagger\right\} = 0 \tag{66}$$

Let us define the *fermion coherent states*

$$|\xi\rangle \equiv e^{-\xi c^\dagger}|0\rangle \tag{67}$$

$$\langle \xi| \equiv \langle 0| e^{\bar{\xi} c} \tag{68}$$

As a consequence of these definitions we have:

$$e^{-\xi c^\dagger} = 1 - \xi c^\dagger \tag{69}$$

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Similarly, if ψ is a Grassmann variable, we have

$$\langle\xi|\psi\rangle = \langle 0|e^{\bar{\xi}c}e^{-\psi c^{\dagger}}|0\rangle = 1 + \bar{\xi}\psi = e^{\bar{\xi}\psi} \tag{70}$$

For N fermions we have,

$$|\xi\rangle \equiv |\xi_1,\ldots,\xi_N\rangle = \Pi_{i=1}^{N}e^{-\xi_i c_i^{\dagger}}|0\rangle \equiv e^{-\sum_{i=1}^{N}\xi_i c_i^{\dagger}}|0\rangle \tag{71}$$

since the following commutator vanishes,

$$\left[\xi_i c_i^{\dagger}, \xi_j c_j^{\dagger}\right] = 0 \tag{72}$$

8.2.2 Analytic functions of Grassmann variables

We will define $\psi(\xi)$ to be an *analytic* function of the Grassmann variable if it has a power series expansion in ξ,

$$\psi(\xi) = \psi_0 + \psi_1\xi + \psi_2\xi^2 + \ldots \tag{73}$$

where $\psi_n \in \mathbb{C}$. Since

$$\xi^n = 0, \qquad \forall n \geq 0 \tag{74}$$

then, all analytic functions of a Grassmann variable reduce to a first degree polynomial,

$$\psi(\xi) \equiv \psi_0 + \psi_1\xi \tag{75}$$

Similarly, we define complex conjugation by

$$\overline{\psi(\xi)} \equiv \bar{\psi}_0 + \bar{\psi}_1\bar{\xi} \tag{76}$$

where $\bar{\psi}_0$ and $\bar{\psi}_1$ are the complex conjugates of ψ_0 and ψ_1 respectively.

We can also define functions of two Grassmann variables ξ and $\bar{\xi}$,

$$A(\bar{\xi}, \xi) = a_0 + a_1\xi + \bar{a}_1\bar{\xi} + a_{12}\bar{\xi}\xi \tag{77}$$

where a_1, $\bar{a}_1$ and a_{12} are complex numbers; a_1 and $\bar{a}_1$ are *not* necessarily complex conjugates of each other.

1. *Differentiation of Grassmann variables*:
   Since analytic functions of Grassmann variables have such a simple structure, differentiation is just as simple. Indeed, we define the *derivative* as the coefficient of the linear term

   $$\partial_{\xi}\psi(\xi) \equiv \psi_1 \tag{78}$$

   Likewise we also have

   $$\partial_{\bar{\xi}}\overline{\psi(\xi)} \equiv \bar{\psi}_1 \tag{79}$$

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Clearly, using this rule we can write

$$\partial_\xi(\bar{\xi}\xi) = -\partial_\xi(\xi\bar{\xi}) = -\bar{\xi} \tag{80}$$

A similar argument shows that

$$\partial_\xi A(\bar{\xi},\xi) = a_1 - a_{12}\,\bar{\xi} \tag{81}$$

$$\partial_{\bar{\xi}} A(\bar{\xi},\xi) = \bar{a}_1 + a_{12}\,\xi \tag{82}$$

$$\partial_{\bar{\xi}}\partial_\xi A(\bar{\xi},\xi) = -a_{12} = -\partial_\xi\partial_{\bar{\xi}} A(\bar{\xi},\xi) \tag{83}$$

from where we conclude that ∂_ξ and $\partial_{\bar{\xi}}$ *anti-commute*,

$$\{\partial_{\bar{\xi}}, \partial_\xi\} = 0, \qquad \text{and} \qquad \partial_\xi\partial_\xi = \partial_{\bar{\xi}}\partial_{\bar{\xi}} = 0 \tag{84}$$

2. *Integration over Grassmann variables*
   The basic differentiation rule of Eq. (78) implies that

$$1 = \partial_\xi\xi \tag{85}$$

which suggests the following *definitions*:

$$\int d\xi\, 1 = 0 \tag{86}$$

$$\int d\xi\, \partial_\xi\xi = 0 \qquad (\text{``exact differential''}) \tag{87}$$

$$\int d\xi\, \xi = 1 \tag{88}$$

Analogous rules also apply for the conjugate variables $\bar{\xi}$.

It is instructive to compare the differentiation and integration rules:

$$\begin{array}{ccc}
\int d\xi\, 1 = 0 & \leftrightarrow & \partial_\xi 1 = 0 \\
\int d\xi\, \xi = 1 & \leftrightarrow & \partial_\xi\xi = 1
\end{array} \tag{89}$$

Thus, for Grassmann variables differentiation and integration are exactly equivalent

$$\partial_\xi \iff \int d\xi \tag{90}$$

These rules imply that the integral of an analytic function $f(\xi)$ is

$$\int d\xi\, f(\xi) = \int d\xi\, (f_0 + f_1\xi) = f_1 \tag{91}$$

and

$$\int d\xi A(\bar{\xi},\xi) = \int d\xi \left(a_0 + a_1\xi + \bar{a}_1\bar{\xi} + a_{12}\bar{\xi}\xi\right) = a_1 - a_{12}\bar{\xi}$$

$$\int d\bar{\xi} A(\bar{\xi},\xi) = \int d\xi \left(a_0 + a_1\xi + \bar{a}_1\bar{\xi} + a_{12}\bar{\xi}\xi\right) = \bar{a}_1 + a_{12}\xi$$

$$\int d\bar{\xi}d\xi A(\bar{\xi},\xi) = -\int d\xi d\bar{\xi} A(\bar{\xi},\xi) = -a_{12}$$

$$\tag{92}$$

It is straightforward to show that with these definitions, the following expression is a consistent definition of a delta-function:

$$\delta(\xi', \xi) = \int d\eta \; e^{-\eta(\xi - \xi')} \tag{93}$$

where ξ, ξ' and η are Grassmann variables.

Finally, given that we have a vector space of analytic functions we can define an *inner product* as follows:

$$\langle f | g \rangle = \int d\bar{\xi} d\xi \; e^{-\bar{\xi}\xi} \; \bar{f}(\xi) \, g(\bar{\xi}) = \bar{f}_0 g_0 + \bar{f}_1 g_1 \tag{94}$$

as expected.

8.2.3 Properties of Fermion Coherent States

We defined above the fermion bra and ket coherent states

$$|\{\xi_j\}\rangle = e^{-\sum_j \xi_j c_j^\dagger} |0\rangle, \qquad \langle\{\xi_j\}| = \langle 0| \, e^{\sum_j \bar{\xi}_j c_j} \tag{95}$$

After a little algebra, using the rules defined above, it is easy to see that the following identities hold:

$$c_i |\{\xi_j\}\rangle = \xi_i |\{\xi_j\}\rangle \tag{96}$$
$$c_i^\dagger |\{\xi_j\}\rangle = -\partial_{\xi_i} |\{\xi_j\}\rangle \tag{97}$$
$$\langle\{\xi_j\}| \, c_i = \partial_{\bar{\xi}_i} \langle\{\xi_j\}| \tag{98}$$
$$\langle\{\xi_j\}| \, c_i^\dagger = \bar{\xi}_i \langle\{\xi_j\}| \tag{99}$$

The inner product of two coherent states $|\{\xi_j\}\rangle$ and $|\{\xi'_j\}\rangle$ is

$$\langle\{\xi_j\}|\{\xi'_j\}\rangle = e^{\sum_j \bar{\xi}_j \xi_j} \tag{100}$$

Similarly, we also have the *Resolution of the Identity* (which is easy to prove)

$$I = \int \left(\Pi_{i=1}^N d\bar{\xi}_i d\xi_i\right) \; e^{-\sum_{i=1}^N \bar{\xi}_i \xi_i} \; |\{\xi_i\}\rangle\langle\{\xi_i\}| \tag{101}$$

Let $|\psi\rangle$ be some state. Then, we can use Eq. (101) to expand the state $|\psi\rangle$ in fermion coherent states $|\xi\rangle$,

$$|\psi\rangle = \int \left(\Pi_{i=1}^N d\bar{\xi}_i d\xi_i\right) \; e^{-\sum_{i=1}^N \bar{\xi}_i \xi_i} \; \psi(\xi)|\{\xi_i\}\rangle \tag{102}$$

where
$$\psi(\bar{\xi}) \equiv \psi(\bar{\xi}_1, \ldots, \bar{\xi}_N) \tag{103}$$

We can use the rules derived above to compute the following matrix elements

$$\langle \xi | c_j | \psi \rangle = \partial_{\bar{\xi}_j} \psi(\bar{\xi}) \tag{104}$$
$$\langle \xi | c_j^\dagger | \psi \rangle = \bar{\xi}_j \, \psi(\bar{\xi}) \tag{105}$$

which is consistent with what we concluded above.

Let $|0\rangle$ be the "empty state" (we will not call it the "vacuum" since it is not in the sector of the ground state). Let $A\left(\{c_j^\dagger\}, \{c_j\}\right)$ be a *normal ordered operator* (with respect to the state $|0\rangle$). By using the formalism worked out above one can show without difficulty that its matrix elements in the coherent states $|\xi\rangle$ and $|\xi'\rangle$ are

$$\langle \xi | A\left(\{c_j^\dagger\}, \{c_j\}\right) | \xi' \rangle = e^{\sum_i \bar{\xi}_i \xi_i'} A\left(\{\bar{\xi}_j\}, \{\xi_j'\}\right) \tag{106}$$

Thus, for example, the expectation value of the fermion number operator $\hat{N}$,

$$\hat{N} = \sum_j c_j^\dagger c_j \tag{107}$$

in the coherent state $|\xi\rangle$ is

$$\frac{\langle \xi | \hat{N} | \xi \rangle}{\langle \xi | \xi \rangle} = \sum_j \bar{\xi}_j \xi_j \tag{108}$$

8.2.4   Grassmann Gaussian Integrals

Let us consider a Gaussian integral over Grassmann variables of the form

$$\mathcal{Z}[\bar{\zeta}, \zeta] = \int \left(\prod_{i=1}^N d\bar{\xi}_i d\xi_i \right) e^{-\sum_{i,j} \bar{\xi}_i M_{ij} \xi_j + \bar{\xi}_i \zeta_i + \bar{\zeta}_i \xi_i} \tag{109}$$

where $\{\zeta_i\}$ and $\{\bar{\zeta}_i\}$ are a set of $2N$ Grassmann variables, and the matrix M_{ij} is a complex Hermitian matrix. We will now show that

$$\mathcal{Z}[\bar{\zeta}, \zeta] = (\det M) \, e^{\sum_{ij} \bar{\zeta}_i \left(M^{-1}\right)_{ij} \zeta_j} \tag{110}$$

Before showing that Eq. (110) is correct let us make a few observations:

1. Eq. (110) looks like the familiar expression for Gaussian integrals for bosons except that instead of a factor of $(\det M)^{-1/2} = \mathrm{Pf}(M)$ (here $\mathrm{PF}(M)$ denotes the pfaffian of M) we get a factor of $\det M$. This is the main effect of the statistics!.

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2. If we had considered a system of N Grassmann variables (instead of $2N$) we would have obtained instead a factor of $\sqrt{\det M}$ where M would now be an $N \times N$ *real anti-symmetric* matrix.

To prove that Eq. (110) is correct we will consider only the case $\zeta_i = \bar{\zeta}_i = 0$, since the contribution from these sources is identical to the bosonic case. Using the Grassmann identities we can write the exponential factor as

$$e^{-\sum_{i,j} \bar{\xi}_i M_{ij} \xi_j} = \prod_{ij} \left(1 - \bar{\xi}_i M_{ij} \xi_j\right) \tag{111}$$

The integral that we need to do is

$$\mathcal{Z}[0,0] = \int \left(\prod_{i=1}^{N} d\bar{\xi}_i \xi_i\right) \prod_{ij} \left(1 - \bar{\xi}_i M_{ij} \xi_j\right) \tag{112}$$

From the integration rules, we can easily see that the only non-vanishing terms in this expression are those that have the just one ξ_i and one $\bar{\xi}_i$ (for each i). Hence we can write

$$
\begin{aligned}
\mathcal{Z}[0,0] &= (-1)^N \int \left(\prod_{i=1}^{N} d\bar{\xi}_i \, d\xi_i\right) \bar{\xi}_1 M_{12} \xi_2 \bar{\xi}_2 M_{23} \xi_3 \ldots + \text{permutations} \\
&= (-1)^N M_{12} M_{23} M_{34} \ldots \int \left(\prod_{i=1}^{N} d\bar{\xi}_i \, d\xi_i\right) \bar{\xi}_1 \xi_2 \bar{\xi}_2 \xi_3 \bar{\xi}_3 \ldots \bar{\xi}_N \xi_N + \text{permutations} \\
&= (-1)^{2N} M_{12} M_{23} M_{34} \ldots M_{N-1,N} + \text{permutations} \tag{113}
\end{aligned}
$$

What is the contribution of the terms labeled "permutations"? It is easy to see that if we permute any pair of labels, say 2 and 3, we will get a contribution of the form

$$(-1)^{2N}(-1) M_{13} M_{32} M_{24} \ldots \tag{114}$$

Hence we conclude that the Gaussian Grassmann integral is just the determinant of the matrix M,

$$\mathcal{Z}[0,0] = \int \left(\prod_{i=1}^{N} d\bar{\xi}_i \xi_i\right) e^{-\sum_{ij} \bar{\xi}_i M_{ij} \xi_j} = \det M \tag{115}$$

(Alternatively, we can diagonalize the quadratic form and notice that the Jacobian is "upside-down").

8.2.5 Grassmann Path Integrals for Fermions

We are now ready to give a prescription for the construction of a fermion path integral in a general system. Let H the a *normal-ordered* Hamiltonian (with

respect to *some* reference state $|0\rangle\rangle$ of a system of *fermions*. Let $|\Psi_i\rangle$ be the ket at the initial time t_i and $|\Psi_f\rangle$ be the final state at time t_f. The matrix element of the evolution operator can be written as a Grassmann Path Integral

$$
\begin{aligned}
\langle \Psi_f, t_f | \Psi_i, t_i \rangle &= \langle \Psi_f | e^{-\frac{i}{\hbar} H(t_f - t_i)} | \Psi_i \rangle \\
&\equiv \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{\frac{i}{\hbar} S(\bar{\psi}, \psi)} \times \text{projection operators} \quad (116)
\end{aligned}
$$

where we have not written down the explicit form of the projection operators onto the initial and final states. The action $S(\bar{\psi}, \psi)$ is

$$
S(\bar{\psi}, \psi) = \int_{t_i}^{t_f} dt \, \left[i\hbar\bar{\psi}\partial_t\psi - H(\bar{\psi}, \psi) \right] \quad (117)
$$

This expression of the fermion path integral holds for any theory of fermions, relativistic or not. Notice that it has *the same form* as the bosonic path integral. The only change is that for fermions the determinant appears in the numerator while for bosons it is in the denominator!

8.3 Path integral quantization of Dirac fermions.

We will now apply the methods we just developed to the case of the Dirac Theory.

Let us define the Dirac field $\psi_\alpha(x)$, with $\alpha = 1, \ldots, 4$. It satisfies the Dirac Equation as an equation of motion,

$$
\left(i\slashed{\partial} - m \right) \psi = 0 \quad (118)
$$

where ψ is a 4-spinor and $\slashed{\partial} = \gamma^\mu \partial_\mu$. Recall that the Dirac γ-matrices satisfy the algebra

$$
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (119)
$$

where $g^{\mu\nu}$ is the Minkowski space metric tensor (in the Bjorken-Drell form).

We saw before that in the quantum field theory description of the Dirac theory, ψ is an operator acting on the Fock space of (fermionic) states. We also saw that the Dirac equation can be regarded as the classical equation of motion of the Lagrangian density

$$
\mathcal{L} = \bar{\psi} \left(i\slashed{\partial} - m \right) \psi \quad (120)
$$

where $\bar{\psi} = \psi^\dagger \gamma^0$. We also noted that the momentum canonically conjugate to the field ψ is $i\psi^\dagger$, from where the standard fermionic equal time anti-commutation relations follow

$$
\left\{ \psi_\alpha(\vec{x}, x_0), \psi_\beta^\dagger(\vec{y}, x_0) \right\} = \delta_{\alpha\beta}\delta^3(\vec{x} - \vec{y}) \quad (121)
$$

The Lagrangian density $\mathcal{L}$ for a Dirac fermion coupled to sources η_α and $\bar{\eta}_\alpha$ is

$$
\mathcal{L} = \bar{\psi} \left(i\slashed{\partial} - m \right) \psi + \bar{\psi}\eta + \bar{\eta}\psi \quad (122)
$$

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and the path-integral is found to be

$$
\begin{aligned}
\mathcal{Z}[\bar{\eta}, \eta] &= \frac{1}{\langle 0|0 \rangle} \langle 0|T\, e^{\,i \int d^4 x \left(\bar{\psi}\eta + \bar{\eta}\psi \right)} |0 \rangle \\
&\equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{\,iS}
\end{aligned}
\tag{123}
$$

where $S = \int d^4 x \mathcal{L}$.

From this result it follows that the Dirac propagator is

$$
\begin{aligned}
iS_{\alpha\beta}(x-y) &= \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0 \rangle \\
&= \frac{(-i)^2}{\mathcal{Z}[0,0]} \left. \frac{\delta^2 \mathcal{Z}[\bar{\eta},\eta]}{\delta \bar{\eta}_\alpha(x) \delta \eta_\beta(y)} \right|_{\bar{\eta}=\eta=0} \\
&= \langle x, \alpha | \frac{1}{i\slashed{\partial} - m} | y, \beta \rangle
\end{aligned}
\tag{124}
$$

8.4 Functional determinants.

We will now discuss more generally how to compute functional determinants. We have discussed before how to do that for path-integrals with a few degrees of freedom (*i. e.* in Quantum Mechanics). We will now generalize these ideas to Quantum Field Theory. We will begin by discussing some simple determinants that show up in systems of fermions and bosons at finite temperature and density.

8.4.1 Functional determinants for Coherent States

Consider a system of fermions (or bosons) with one-body Hamiltonian $\hat{h}$ at non-zero temperature T and chemical potential μ. The partition function

$$
Z = \operatorname{tr}\, e^{-\beta \left(\hat{H} - \mu \hat{N} \right)}
\tag{125}
$$

where $\beta = 1/k_B T$,

$$
\hat{H} = \int dx\, \hat{\psi}(x)^\dagger\, \hat{h}\, \hat{\psi}(x)
\tag{126}
$$

and

$$
\hat{N} = \int dx\, \hat{\psi}(x)^\dagger\, \hat{\psi}(x)
\tag{127}
$$

is the number operator. Here x denotes both spacial and internal (spin) labels.

The functional (or path) integral expression for the partition function is

$$
Z = \int \mathcal{D}\psi^* \mathcal{D}\psi\, e^{\frac{i}{\hbar} \int d\tau\, \psi^* \left(i\hbar \partial_\tau + \hat{h} + \mu \right) \psi}
\tag{128}
$$

In imaginary time we set $t \to -i\tau$, with $0 \le \tau \le \beta\hbar$.

The fields $\psi(\tau)$ can represent either be *bosons*, in which case they are just complex functions of x and τ, or *fermions*, in which case they are complex Grassmann functions of x and τ. The only subtlety resides in the choice of boundary conditions

1. *Bosons*:
   Since the partition function is a trace, in this case the fields (be complex or real) must obey the usual *periodic boundary conditions* in imaginary time, *i. e.*,

   $$\psi(\tau) = \psi(\tau + \beta\hbar) \tag{129}$$

2. *Fermions*:
   In the case of fermions the fields are complex Grassmann variables. However, if we want to compute a trace it turns out that, due to the anti-commutation rules, it is necessary to require the fields to obey *anti-periodic boundary conditions*, *i. e.*

   $$\psi(\tau) = -\psi(\tau + \beta\hbar) \tag{130}$$

Let $\{|\lambda\rangle\}$ be a complete set of eigenstates of the one-body Hamiltonian $\hat{h}$, $\{\varepsilon_\lambda\}$ be its eigenvalue spectrum with λ a spectral parameter (*i. e.* a suitable set of quantum numbers spanning the spectrum of $\hat{h}$), and $\{\phi_\lambda(\tau)\}$ be the associated complete set of eigenfunctions. We now expand the field configurations in the basis of eigenfunctions of $\hat{h}$,

$$\psi(\tau) = \sum_\lambda \psi_\lambda \, \phi_\lambda(\tau) \tag{131}$$

The eigenfunctions of $\hat{h}$ are complete and orthonormal.

Thus if we expand the fields, the path-integral of Eq. (128) becomes (with $\hbar = 1$)

$$Z = \int \left(\prod_\lambda d\psi_\lambda^* d\psi_\lambda \right) e^{-\int d\tau \sum_\lambda \psi_\lambda^* \left(-\partial_\tau - \varepsilon_\lambda + \mu\right) \psi_\lambda} \tag{132}$$

which becomes (after absorbing all uninteresting constant factors in the integration measure)

$$Z = \prod_\lambda \left[\text{Det} \left(-\partial_\tau - \varepsilon_\lambda + \mu\right) \right]^\sigma \tag{133}$$

where $\sigma = +1$ is the results for *fermions* and $\sigma = -1$ for *bosons*.

Let $\psi_n^\lambda(\tau)$ be the solution of eigenvector equation

$$\left(-\partial_\tau - \varepsilon_\lambda + \mu\right) \psi_n^\lambda(\tau) = \alpha_n \psi_n^\lambda(\tau) \tag{134}$$

where α_n is the (generally complex) eigenvalue. The eigenfunctions $\psi_n^\lambda(\tau)$ will be required to satisfy either periodic or anti-periodic boundary conditions,

$$\psi_n^\lambda(\tau) = -\sigma \psi_n^\lambda(\tau + \beta) \tag{135}$$

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where, once again, $\sigma = \pm 1$.

The eigenvalue condition, Eq. (134) is solved by

$$\psi_n(\tau) = \psi_n \, e^{i\alpha_n \tau} \tag{136}$$

provided α_n satisfies

$$\alpha_n = -i\omega_n + \mu - \varepsilon_\lambda \tag{137}$$

where the Matsubara frequencies are given by (with $k_B = 1$)

$$\omega_n = \begin{cases} 2\pi T \left(n + \frac{1}{2}\right), & \text{for fermions} \\ 2\pi T n, & \text{for bosons} \end{cases} \tag{138}$$

Let us consider now the function $\varphi_\alpha(\tau)$ which is an eigenfunction of $-\partial_\tau - \varepsilon_\lambda + \mu$,

$$(-\partial_\tau - \varepsilon_\lambda + \mu) \, \varphi_\alpha(\tau) = \alpha \, \varphi_\alpha(\tau) \tag{139}$$

which satisfies only an *initial condition* for $\varphi_\alpha^\lambda(0)$, such as

$$\varphi_\alpha^\lambda(0) = 1 \tag{140}$$

Notice that since the operator is linear in ∂_τ we cannot impose additional conditions on the derivative of φ_α.

The solution of

$$\partial_\tau \ln \varphi_\alpha^\lambda(\tau) = \mu - \varepsilon - \alpha \tag{141}$$

is

$$\varphi_\alpha^\lambda(\tau) = \varphi_\alpha^\lambda(0) \, e^{\left(\mu - \varepsilon_\lambda - \alpha\right) \tau} \tag{142}$$

After imposing the initial condition of Eq. (140), we find

$$\varphi_\alpha^\lambda(\tau) = e^{-\left(\alpha + \varepsilon_\lambda - \mu\right) \tau} \tag{143}$$

But, although this function $\varphi_\alpha^\lambda(\tau)$ satisfies all the requirements, it does not have the same zeros as the determinant $\mathrm{Det}(-\partial_\tau + \mu - \varepsilon_\lambda - \alpha)$. However, the function

$$_\sigma F_\alpha^\lambda(\tau) = 1 + \sigma \, \varphi_\alpha^\lambda(\tau) \tag{144}$$

does satisfies all the properties. Indeed,

$$_\sigma F_\alpha^\lambda(\beta) = 1 + \sigma \, e^{-\left(\alpha + \varepsilon_\lambda - \mu\right) \beta} \tag{145}$$

which vanishes for $\alpha = \alpha_n$. Then, a version of Coleman's argument tells us that

$$\frac{\mathrm{Det}\left(-\partial_\tau + \mu - \varepsilon_\lambda - \alpha\right)}{_\sigma F_\alpha^\lambda(\beta)} = \text{constant} \tag{146}$$

where the right hand side is a constant in the sense that it does not depend on the choice of the eigenvalues $\{\varepsilon_\lambda\}$.

Hence,

$$\mathrm{Det}\,(-\partial_\tau + \mu - \varepsilon_\lambda) = \mathrm{const.}\ {}_\sigma F_0^\lambda(\beta) \tag{147}$$

The partition function is

$$Z = e^{-\beta F} = \prod_\lambda [\mathrm{Det}\,(-\partial_\tau + \mu - \varepsilon_\lambda)]^\sigma \tag{148}$$

where F is the free energy, which we find it is given by

$$\begin{aligned} F &= -\sigma T \sum_\lambda \ln \mathrm{Det}\,(-\partial_\tau + \mu - \varepsilon_\lambda - \alpha) \\ &= -\sigma T \sum_\lambda \ln\left(1 + \sigma e^{\beta(\mu - \varepsilon_\lambda)}\right) + f(\beta\mu) \end{aligned} \tag{149}$$

which is the correct result for non-interacting fermions and bosons. Here, we have set

$$f(\beta\mu) = \begin{cases} 0 & \text{fermions} \\ -2TN\,\ln\left(1 - e^{\beta\mu}\right) & \text{bosons} \end{cases} \tag{150}$$

where N is the number of states in the spectrum $\{\lambda\}$.

In some cases the spectrum has the symmetry $\varepsilon_\lambda = -\varepsilon_{-\lambda}$, *e. g.* the Dirac theory whose spectrum is $\varepsilon_\pm = \pm\sqrt{p^2 + m^2}$, and these expressions cam be simplified further,

$$\begin{aligned} \prod_\lambda \mathrm{Det}\,(-\partial_t + i\varepsilon_\lambda) &= \prod_{\lambda>0} [\mathrm{Det}\,(-\partial_t + i\varepsilon_\lambda)\,\mathrm{Det}\,(-\partial_t - i\varepsilon_\lambda)] \tag{151} \\ &= \prod_{\lambda>0} \mathrm{Det}\,(\partial_t^2 + \varepsilon_\lambda^2) \\ \text{after a Wick rotation} \longrightarrow &\ \prod_{\lambda>0} \mathrm{Det}\,(-\partial_\tau^2 + \varepsilon_\lambda^2) \tag{152} \end{aligned}$$

This last expression we have encountered before. The result is

$$\prod_{\lambda>0} \mathrm{Det}\,(-\partial_\tau^2 + \varepsilon_\lambda^2) = \mathrm{const.}\ \psi_0(\beta) \tag{153}$$

where $\psi_0(\tau)$ is the solution of the differential equation

$$\left(-\partial_\tau^2 + \varepsilon_\lambda^2\right)\psi_0(\tau) = 0 \tag{154}$$

which satisfies the initial conditions

$$\psi_0(0) = 0 \qquad \partial_\tau \psi_0(0) = 1 \tag{155}$$

The solution is

$$\psi_0(\tau) = \frac{\sinh(|\varepsilon_\lambda|\tau)}{|\varepsilon_\lambda|} \tag{156}$$

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Hence

$$
\begin{aligned}
\psi_0(\beta) &= \frac{\sinh(|\varepsilon_\lambda|\beta)}{|\varepsilon_\lambda|} \\
&\longrightarrow \frac{e^{|\varepsilon_\lambda|\beta}}{2|\varepsilon_\lambda|} \qquad \text{as } \beta \to \infty,
\end{aligned} \tag{157}
$$

In particular, since

$$
\prod_{\lambda>0} \frac{e^{|\varepsilon_\lambda|\beta}}{2|\varepsilon_\lambda|} = e^{\beta \sum_{\lambda>0}|\varepsilon_\lambda|} = e^{-\beta \sum_{\lambda<0}\varepsilon_\lambda} \tag{158}
$$

we get that the ground state energy E_G is the sum of the single particle energies of the occupied (negative energy) states:

$$
E_G = \sum_{\lambda<0} \varepsilon_\lambda \tag{159}
$$

8.5 Summary of the general strategy

We will now summarize the general strategy to compute path integrals of any type.

We begin with the path integral for the vacuum persistence amplitude (or "partition function") which we write in general as

$$
\mathcal{Z} \int \mathcal{D}\,(\text{fields})\ e^{i\int d^D x\,\mathcal{L}(\text{fields})} \tag{160}
$$

and consider a field configuration $\phi_c(x)$ which is a solution of the Classical Equation of Motion

$$
\frac{\delta\mathcal{L}}{\delta\phi} - \partial^\mu \frac{\delta\mathcal{L}}{\partial^\mu \phi} = 0 \tag{161}
$$

we then expand about this classical solution

$$
\phi(x) = \phi_c(x) + \varphi(x) \tag{162}
$$

the action $S[\phi] = \int d^D x\,\mathcal{L}[\phi]$,

$$
\begin{aligned}
S[\phi_c + \varphi] &= S[\phi_c] + \int d^D x\,\delta\phi(x) \left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial^\mu \frac{\delta\mathcal{L}}{\partial^\mu \phi}\right]_{\phi_c} \\
&\quad + \int d^D x \int d^D y\,\frac{1}{2}\frac{\delta^2\mathcal{L}}{\delta\phi(x)\delta\phi(y)}\phi(x)\phi(y) + O(\phi^3)
\end{aligned} \tag{163}
$$

Thus, for example, for the case of a relativistic scalar field with Lagrangian density

$$
\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - V(\phi) \tag{164}
$$

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we write
$$\mathcal{L} = \mathcal{L}[\phi_c] + \frac{1}{2}\varphi(x)\left[-\partial^2 - V''[\phi_c]\right]\varphi(x) + \dots \tag{165}$$
The partition function now takes the form
$$\mathcal{Z} = \int \mathcal{D}\phi\, e^{iS[\phi]} = \text{const.}\; \left[\text{Det}\left(-\partial^2 - V''[\phi_c]\right)\right]^{-1/2}\, e^{iS[\phi_c]}\, [1 + \dots] \tag{166}$$

For a fermionic system the main difference is that the determinant factor gets flipped around
$$\left[\text{Det}\,(\text{differential operator})\right]^{-n/2} \to \left[\text{Det}\,(\text{differential operator})\right]^{n/2} \tag{167}$$
where n is the number of real field components.

Thus, if we wished to compute the ground state energy E_G, we would use the fact that
$$S[\phi_c] = -TE_0 \tag{168}$$
where T is the total time-span and E_0 is the ground state energy for the free field, and that the partition function for large T becomes
$$\mathcal{Z} = e^{-iE_G T}, \qquad \text{for large real time } T \tag{169}$$

Hence,
$$E_G = \lim_{T \to \infty} \frac{i}{T}\ln \mathcal{Z} = -\frac{1}{T}S[\phi_c] + \frac{i}{2}\ln \text{Det}\left(-\partial^2 - V''[\phi_c]\right) + \dots \tag{170}$$

Notice that for fermions the sign of the term with the determinant would be the opposite.

In imaginary time we have
$$\mathcal{Z} = e^{-\beta E_G} = e^{-\beta E_0}\left[\text{Det}\left(-\nabla^2 + V''[\phi_c]\right)\right]^{-1/2} \times [1 + \dots] \tag{171}$$
and
$$E_G = E_0 + \frac{1}{2\beta}\ln \text{Det}\left(-\nabla^2 + V''[\phi_c]\right) + \dots \tag{172}$$

The correction terms can be computed using perturbation theory; we will do this later on. Let us note for now that the way will do this will require the we introduce a set of sources J which couple linearly to the field ϕ, in the form of an extra term to the action of the form $\int d^D x J(x)\phi(x)$. Upon shifting the fields and expanding we get

$$
\begin{aligned}
\mathcal{Z}[J] \quad = \quad & e^{\displaystyle iS[\phi_c] + i\int d^D x J(x)\phi_c(x)} \\[2ex]
\times \quad & \left[\text{Det}\left(-\partial^2 - V''[\phi_c]\right)\right]^{-1/2}\, e^{\displaystyle \frac{i}{2}\int_x \int_y J(x)G(x,y)J(y)}
\end{aligned} \tag{173}
$$

where $G(x,y)$ is the Green's function,
$$G(x,y) = \langle x|\frac{1}{-\partial^2 - V''[\phi_c]}|y\rangle \tag{174}$$

8.6 How to Compute Functional Determinants: Heat kernels and ζ-function regularization.

We have seen before that the evaluation of the effects of quantum fluctuations involves the calculation of the determinant of a differential operator. In the case of non-relativistic single particle Quantum Mechanics we discussed in detail how to calculate a functional determinant of the form $\text{Det}\left[-\partial_t^2 + W(t)\right]$. However the method we used for that purpose becomes unmanageably cumbersome if applied to the calculation of determinants of partial differential operators of the form $\text{Det}\left[-D^2 + W(x)\right]$, where $x \equiv x_\mu$. Fortunately there are better and more efficient ways of doing such calculations.

Let $\hat{A}$ be an operator, and $\{f_n(x)\}$ be a complete set of eigenstates of $\hat{A}$, with the eigenvalue spectrum $S(A) = \{a_n\}$,

$$\hat{A} f_n(x) = a_n f_n(x) \tag{175}$$

We will assume that $\hat{A}$ has a discrete spectrum of real positive eigenvalues. For the case of a continuous spectrum we will put the system in a finite box, which makes the spectrum discrete, and take limits at the end of the calculation.

The function $\zeta(s)$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for } s > 0 \tag{176}$$

is the well known Riemann ζ-function. We will now use the eigenvalue spectrum of the operator $\hat{A}$ to define the *generalized ζ-function*

$$\zeta_A(s) = \sum_n \frac{1}{a_n^s} \tag{177}$$

where the sum runs over the labels of the spectrum of the operator $\widehat{A}$. Then, upon differentiation we find

$$\begin{aligned}
\frac{d\zeta_A}{ds} &= \sum_n \frac{d}{ds} e^{-s \ln a_n} \\
&= -\sum_n \frac{\ln a_n}{a_n^s}
\end{aligned} \tag{178}$$

Then, in the limit $s \to 0^+$ we find,

$$\lim_{s \to 0^+} \frac{d\zeta_A}{ds} = -\sum_n \ln a_n = -\ln \prod_n a_n \equiv -\ln \text{Det} A \tag{179}$$

Hence,

$$\left. \frac{d\zeta_A}{ds} \right|_{s \to 0^+} = -\ln \text{Det} A \tag{180}$$

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Let us define now the generalized *heat kernel*,

$$G_A(x, y; \tau) = \sum_n e^{-a_n \tau} f_n(x) f_n^*(y) \tag{181}$$

where $\tau > 0$. The heat kernel $G_A(x, y; \tau)$ clearly obeys

$$-\partial_\tau G_A(x, y; \tau) = \hat{A} G_A(x, y; \tau) \tag{182}$$

which can be regarded as a Heat equation. Indeed, for $\hat{A} = -D \, \nabla^2$, this is the regular Heat Equation (where D is the diffusion constant); in this case τ represents time. In general we will refer to τ as *proper time.*

The heat kernel $G_A(x, y; \tau)$ satisfies the initial condition

$$\lim_{\tau \to 0^+} G_A(x, y; \tau) = \sum_n f_n(x) f_n^*(y) = \delta(x - y) \tag{183}$$

where we have used the completeness relation of the eigenfunctions $\{f_n(x)\}$. Hence, $G_A(x, y; \tau)$ is the solution of a generalized Heat Equation with kernel $\hat{A}$. It defines a generalized random walk or Markov process.

We will now show that $G_A(x, y; \tau)$ is related to the function $\zeta_A(s)$. Indeed, let us consider the heat kernel $G_A(x, y; \tau)$ at *short distances*, $y \to x$, and compute the integral (below D is the dimensionality of space-time)

$$\int d^D x \, \lim_{y \to x} G_A(x, y; \tau) = \sum_n e^{-a_n \tau} \int d^D x \, f_n(x) f_n^*(x)$$

$$= \sum_n e^{-a_n \tau} \equiv \left(\text{tr} \, e^{-\tau \hat{A}} \right) \tag{184}$$

where we assumed that the eigenfunctions are normalized to unity

$$\int d^D x \, |f_n(x)|^2 = 1 \tag{185}$$

i. e. normalized inside a box.

We will now use that, for $s > 0$,

$$\int_0^\infty d\tau \, \tau^{s-1} \, e^{-a_n \tau} = \frac{\Gamma(s)}{a_n^s} \tag{186}$$

where $\Gamma(s)$ is the Gamma function:

$$\Gamma(s) = \int_0^\infty d\tau \, \tau^{s-1} \, e^{-\tau} \tag{187}$$

Then,

$$\int_0^\infty d\tau \, \tau^{s-1} \int d^D x \, \lim_{y \to x} G_A(x, y; \tau) = \sum_n \frac{\Gamma(s)}{a_n^s} \tag{188}$$

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Therefore, we find that the generalized ζ-function, $\zeta_A(s)$, can be written obtained from the generalized heat kernel $G_A(x, y; \tau)$:

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \tau^{s-1} \int d^D x \, \lim_{y \to x} G_A(x, y; \tau) \tag{189}$$

This result suggests the following strategy for the computation of determinants:

1. Given the Hermitian operator $\hat{A}$, we solve the Generalized Heat Equation

$$\hat{A} \, G_A = -\partial_\tau \, G_A \tag{190}$$

   subject to the initial condition

$$\lim_{\tau \to 0^+} G_A(x, y; \tau) = \delta^D(x - y) \tag{191}$$

2. Next we evaluate find the associated ζ-function, $\zeta_A(s)$, using the expression

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \tau^{s-1} \underbrace{\int d^D x \, \lim_{y \to x} G_A(x, y; \tau)}_{\text{tr } e^{-\tau \hat{A}}} \tag{192}$$

3. We next take the limit $s \to 0^+$ to relate the ζ-function to the determinant:

$$\lim_{s \to 0^+} \frac{d\zeta_A(s)}{ds} = -\ln \det \hat{A} \tag{193}$$

   In practice we will have to exercise some care in this step since we will find singularities as we take this limit. Most often we will keep the points x and y apart by a small but finite distance a, which we will eventually take to zero. Hence, we will need to understand in detail the short distance behavior of the heat kernel.

Furthermore, the propagator

$$S_A(x, y) = \langle x| \, \hat{A}^{-1} \, |y\rangle \tag{194}$$

can also be related to the heat kernel. Indeed by expending Eq.(194) in the eigenstates of $\hat{A}$, we find

$$S(x, y) = \sum_n \frac{\langle x|n\rangle \, \langle n|y\rangle}{a_n} = \sum_n \frac{f_n(x) f_n^*(y)}{a_n} \tag{195}$$

We can now write the following integral of the heat kernel as

$$\int_0^\infty d\tau \, G_A(x, y; \tau) = \sum_n f_n(x) f_n^*(y) \int_0^\infty d\tau \, e^{-a_n \tau}$$

$$= \sum_n \frac{f_n(x) f_n^*(y)}{a_n} = S(x, y) \tag{196}$$

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Hence, the propagator $S_A(x, y)$ is an integral over the heat kernel:

$$S_A(x, y) = \int_0^\infty d\tau \; G_A(x, y; \tau) \tag{197}$$

Equivalently, we can say that since G_A satisfies the Heat Equation

$$\hat{A}G_A = -\partial_\tau G_A \qquad \Longrightarrow \qquad G_A(x, y; \tau) = \langle x | \, e^{-\tau \hat{A}} \, | y \rangle \tag{198}$$

Thus,

$$S_A(x, y) = \int_0^\infty d\tau \; \langle x | \, e^{-\tau \hat{A}} \, | y \rangle = \langle x | \, \hat{A}^{-1} \, | y \rangle \tag{199}$$

and that it is indeed the Green's function of $\hat{A}$,

$$\hat{A}_x S(x, y) = \delta(x - y) \tag{200}$$

It is worth to note that the heat kernel $G_A(x, y; \tau)$, as can be seen from Eq. (198), is also the *density matrix* of the bounded Hermitian operator $\hat{A}$. As such it has an imaginary time (τ!) path-integral representation.(To actually insure convergence we must also require that the spectrum of $\hat{A}$ be positive.) In that picture we view $G_A(x, y; \tau)$ as the amplitude for the imaginary-time (proper time) evolution from the initial state $|y\rangle$ to the final state $|x\rangle$. In other words, we picture $S_A(x, y)$ as the amplitude to go from y to x in an arbitrary time.

8.6.1   The determinant of the Euclidean Klein-Gordon Operator

As an example of the use of the heat kernel method we will use it to compute the determinant of the Euclidean Klein-Gordon operator. Thus, we will take the hermitian operator $\hat{A}$ to be

$$\hat{A} = -\nabla^2 + m^2 \tag{201}$$

in D Euclidean space-time dimensions. This operator has a bounded positive spectrum. Here we will be interested in a system with infinite size $L \to \infty$, and a large volume $V = L^D$. We will follow the steps outlined above.

1. We begin by constructing the heat kernel $G(x, y; \tau)$. By definition it is the solution of the partial differential equation

$$\left(-\nabla^2 + m^2\right) G(x, y; \tau) = -\partial_\tau G(x, y; \tau) \tag{202}$$

satisfying the initial condition

$$\lim_{\tau \to 0^+} G(x, y; \tau) = \delta^D(x - y) \tag{203}$$

We will find $G(x, y; \tau)$ by Fourier transforms,

$$G(x, y; \tau) = \int \frac{d^D p}{(2\pi)^D} \; G(\vec{p}, \tau) \, e^{\, i\vec{p} \cdot (\vec{x} - \vec{y})} \tag{204}$$

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We find that in order for $G(x, y; \tau)$ to satisfy Eq.(202), its Fourier transform $G(\vec{p}; \tau)$ must satisfy the differential equation

$$-\partial_\tau G(\vec{p}; \tau) = \left(\vec{p}^{\,2} + m^2\right) G(\vec{p}; \tau) \tag{205}$$

The solution of this equation, consistent with the initial condition of Eq.(203) is

$$G(\vec{p}; \tau) = e^{-\left(\vec{p}^{\,2} + m^2\right)\tau} \tag{206}$$

We can now easily find $G(x, y; \tau)$ by simply finding the anti-transform of $G(\vec{p}; \tau)$:

$$
\begin{aligned}
G(x, y; \tau) &= \int \frac{d^D p}{(2\pi)^D} \, e^{-\left(\vec{p}^{\,2} + m^2\right)\tau + i\vec{p} \cdot (\vec{x} - \vec{y})} \\
&= \frac{e^{-\left[m^2\tau + \frac{|\vec{x} - \vec{y}|^2}{4\tau}\right]}}{(4\pi\tau)^{D/2}}
\end{aligned}
\tag{207}
$$

Notice that for $m \to 0$, $G(x, y; \tau)$ reduces to the usual diffusion kernel (with unit diffusion constant.)

$$\lim_{m \to 0} G(x, y; \tau) = \frac{e^{-\frac{|x - y|^2}{4\tau}}}{(4\pi\tau)^{D/2}} \tag{208}$$

2. Next we construct the ζ-function

$$\zeta_{-\nabla^2 + m^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \tau^{s-1} \int d^D x \, \lim_{y \to x} G(x, y; \tau) \tag{209}$$

We first do the integral

$$\int_0^\infty d\tau \, \tau^{s-1} \int d^D x \, G(x, y; \tau) = \frac{V}{(4\pi)^{D/2}} \int_0^\infty d\tau \, \tau^{s-1-D/2} \, e^{-\left(m^2\tau + \frac{R^2}{4\tau}\right)} \tag{210}$$

where $R = |\vec{x} - \vec{y}|$. Upon scaling the variable $\tau = \lambda t$, with $\lambda = R/2m$, we find that

$$\int_0^\infty d\tau \, \tau^{s-1} \, G(x, y; \tau) = \frac{2}{(4\pi)^{D/2}} \left(\frac{R}{2m}\right)^{s-\frac{D}{2}} K_{\frac{D}{2}-s}(mR) \tag{211}$$

where $K_\nu(z)$,

$$K_\nu(z) = \frac{1}{2} \int_0^\infty dt \, t^{\nu-1} e^{-\frac{z}{2}\left(t + \frac{1}{t}\right)} \tag{212}$$

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is a modified Bessel function. Its short argument behavior is

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu + \dots \tag{213}$$

As a check, we notice that for $s = 1$ the integral of Eq.(210) does reproduce the Euclidean Klein-Gordon propagator that we discussed earlier in these lectures.

3. The next step is to take the short distance limit

$$
\begin{aligned}
\lim_{R\to 0}\int_0^\infty d\tau\,\tau^{s-1}G(\vec{x},\vec{y};\tau) &= \lim_{R\to 0}\frac{2^{1-s}}{(2\pi)^{D/2}}\frac{m^{D-2s}}{(mR)^{\frac{D}{2}-s}}K_{\frac{D}{2}-s}(mR) \\
&= \frac{\Gamma\left(s-\frac{D}{2}\right)}{(4\pi)^{D/2}m^{2s-D}} \tag{214}
\end{aligned}
$$

Notice that we have exchanged the order of the limit and the integration. Also, after we took the short distance limit $R \to 0$, the expression above acquired a factor of $\Gamma(s - D/2)$, which is singular as $s - D/2$ approaches zero (or any negative integer). Thus, a small but finite R smears this singularity.

4. Finally we find the ζ-function by doing the (trivial) integration over space

$$
\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty d\tau \int d^D x \lim_{\vec{y}\to\vec{x}}G(\vec{x},\vec{y};\tau) \\
&= V\mu^{-2s}\frac{m^D}{(4\pi)^{D/2}}\frac{\Gamma\left(s-\frac{D}{2}\right)}{\Gamma(s)}\left(\frac{m}{\mu}\right)^{-2s} \tag{215}
\end{aligned}
$$

where $\mu = 1/R$ plays the role of a cutoff mass (or momentum) scale that we will need to make some quantities dimensionless. The appearance of this quantity is also a consequence of the singularities.

We will now consider the specific case of $D = 4$ dimensions. For $D = 4$ the ζ-function is

$$\zeta(s) = V\frac{m^4}{16\pi^2}\frac{\mu^{-2s}}{(s-1)(s-2)}\left(\frac{m}{\mu}\right)^{-2s} \tag{216}$$

We can now compute the desired (logarithm of the) determinant for $D = 4$ dimensions:

$$\ln\text{Det}\left[-\nabla^2+m^2\right] = -\lim_{s\to 0^+}\frac{d\zeta}{ds} = \frac{m^4}{16\pi^2}\left[\ln\frac{m}{\mu}-\frac{3}{4}\right]V \tag{217}$$

where $V = L^4$. A similar calculation for $D = 2$ yields the result

$$\ln\text{Det}\left[-\nabla^2+m^2\right] = \frac{m^2}{2\pi}\left[\ln\frac{m}{\mu}-\frac{1}{2}\right]V \tag{218}$$

where $V = L^2$.

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8.7 Path integral for spin.

We will now discuss the use of path integral methods to describe a quantum mechanical spin. Consider a quantum mechanical system which consists of a spin in the spin-S representation of the group $SU(2)$. The space of states of the spin-S representation is $2S + 1$-dimensional, and it is spanned by the basis $\{|S, M\rangle\}$ which are the eigenstates of the operators $\vec{S}^2$ and S_3, *i.e.,*

$$
\begin{aligned}
\vec{S}^2 |S, M\rangle &= S(S + 1) |S, M\rangle \\
S_3 |S, M\rangle &= M |S, M\rangle
\end{aligned}
$$

(219)

with $|M| \leq S$ (in integer-spaced intervals). This set of states is complete ad it forms a basis of this Hilbert space. The operators S_1, S_2 and S_3 obey the $SU(2)$ algebra,

$$[S_a, S_b] = i\epsilon_{abc}S_c \tag{220}$$

where $a, b, c = 1, 2, 3$.

The simplest physical problem involving spin is the coupling to an external magnetic field $\vec{B}$ through the Zeeman interaction

$$H_{\text{Zeeman}} = \mu\, \vec{B} \cdot \vec{S} \tag{221}$$

where μ is the Zeeman coupling constant (*i.e.,* the product of the Bohr magneton and the gyromagnetic factor).

Let us denote by $|0\rangle$ the *highest weight state* $|S, S\rangle$. Let us define the spin raising and lowering operators $S^\pm$,

$$S^\pm = S_1 \pm iS_2 \tag{222}$$

The highest weight state $|0\rangle$ is annihilated by S^+,

$$S^+|0\rangle = S^+|S, S\rangle = 0 \tag{223}$$

Clearly, we also have

$$
\begin{aligned}
\vec{S}^2|0\rangle &= S(S + 1)|0\rangle \\
S_3|0\rangle &= S|0\rangle
\end{aligned}
$$

(224)

Let us consider now the state $|\vec{n}\rangle$,

$$|\vec{n}\rangle = e^{i\theta(\vec{n}_0 \times \vec{n} \cdot \vec{S}}\, |0\rangle \tag{225}$$

where $\vec{n}$ is a three-dimensional unit vector ($\vec{n}^2 = 1$), $\vec{n}_0$ is a unit vector pointing along the direction of the quantization axis (*i.e.,* the "North Pole" of the unit sphere) and θ is the *colatitude*, (see Fig. 2)

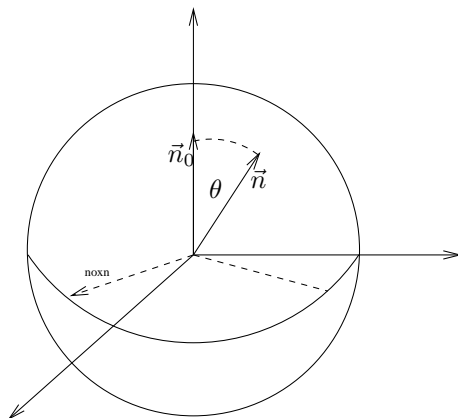$$\vec{n} \cdot \vec{n}_0 = \cos\theta \tag{226}$$

Figure 1:

As we will see the state $|\vec{n}\rangle$ is a coherent spin state which represents a spin polarized along the $\vec{n}$ axis. The state $|\vec{n}\rangle$ can be expanded in the basis $|S, M\rangle$,

$$|\vec{n}\rangle = \sum_{M=-S}^{S} D_{MS}^{(S)}(\vec{n}) \, |S, M\rangle \qquad (227)$$

Here $D_{MS}^{(S)}(\vec{n})$ are the representation matrices in the spin-S representation.

It is important to note that there are many rotations that lead to the same state $|\vec{n}\rangle$ from the highest weight $|0\rangle$. For example any rotation along the direction $\vec{n}$ results only in a change in the phase of the state $|\vec{n}\rangle$. These rotations are equivalent to a multiplication on the right by a rotation about the z axis. However, in Quantum Mechanics this phase has no physically observable consequence. Hence we will regard all of these states as being physically equivalent. In other terms, the states for *equivalence classes* (or *rays*) and we must pick one and only one state from each class. These rotations are generated by S_3, the (only) diagonal generator of $SU(2)$. Hence, the physical states are not in one-to-one correspondence with the elements of $SU(2)$ but instead with the elements of the right *coset $SU(2)/U(1)$*, with the $U(1)$ generated by S_3. (In the case of a more general group we must divide out the Maximal Torus generated by all the diagonal generators of the group.) In mathematical language, if we consider all the rotations at once , the spin coherent states are said to form a Hermitian line bundle.

A consequence of these observations is that the D matrices do not form a group under matrix multiplication. Instead they satisfy

$$D^{(S)}(\vec{n}_1)D^{(S)}(\vec{n}_2) = D^{(S)}(\vec{n}_3) \, e^{i\Phi(\vec{n}_1, \vec{n}_2, \vec{n}_3)S_3} \qquad (228)$$

where the phase factor is usually called a *cocycle*. Here $\Phi(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ is the (oriented) area of the spherical triangle with vertices at $\vec{n}_1, \vec{n}_2, \vec{n}_3$. However,

since the sphere is a closed surface, which area do we actually mean? "Inside" or "outsider"? Thus, the phase factor is ambiguous by an amount determined by 4π, the total area of the sphere,

$$e^{i4\pi M} \tag{229}$$

However, since M is either an integer or a half-integer this ambiguity in Φ has no consequence whatsoever,

$$e^{i4\pi M} = 1 \tag{230}$$

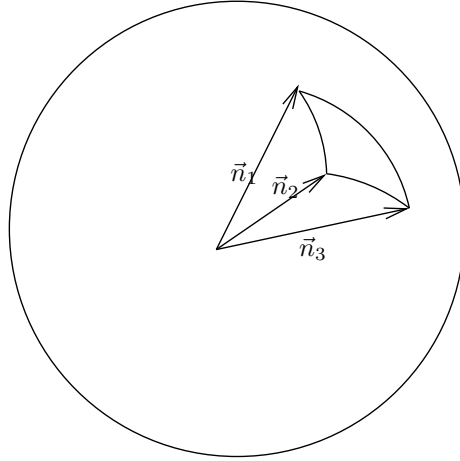(we can also regard this result as a requirement that M be quantized).

Figure 2:

The states $|\vec{n}\rangle$ are coherent states which satisfy the following properties (see Perelomov's book *Coherent States*). The overlap of two coherent states $|\vec{n}_1\rangle$ and $|\vec{n}_2\rangle$ is

$$\begin{aligned}
\langle \vec{n}_1 | \vec{n}_2 \rangle &= \langle 0 | D^{(S)}(\vec{n}_1)^\dagger D^{(S)}(\vec{n}_2) | 0 \rangle \\
&= \langle 0 | D^{(S)}(\vec{n}_0) e^{i\Phi(\vec{n}_1, \vec{n}_2, \vec{n}_0) S_3} | 0 \rangle \\
&= \left(\frac{1 + \vec{n}_1 \cdot \vec{n}_2}{2} \right)^S e^{i\Phi(\vec{n}_1, \vec{n}_2, \vec{n}_0) S}
\end{aligned} \tag{231}$$

The (diagonal) matrix element of the spin operator is

$$\langle \vec{n} | \vec{S} | \vec{n} \rangle = S \, \vec{n} \tag{232}$$

Finally, the (over-complete) set of coherent states $\{|\vec{n}\rangle\}$ have a resolution of the identity of the form

$$\hat{I} = \int d\mu(\vec{n}) \, |\vec{n}\rangle \langle \vec{n}| \tag{233}$$

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where the integration measure $d\mu(\vec{n})$ is

$$d\mu(\vec{n}) = \left(\frac{2S+1}{4\pi}\right)\delta(\vec{n}^2 - 1)d^3n \tag{234}$$

Let us now use the coherent states $\{|\vec{n}\rangle\}$ to find the path integral for a spin. In imaginary time τ (and with periodic boundary conditions) the path integral is simply the partition function

$$Z = \mathrm{tr}\,e^{-\beta H} \tag{235}$$

where $\beta = 1/T$ (T is the temperature) and H is the Hamiltonian. As usual the path integral form of the partition function is found by splitting up the imaginary time interval $0 \leq \tau \leq \beta$ in N_τ steps each of length $\delta\tau$ such that $N_\tau\delta\tau = \beta$. Hence we have

$$Z = \lim_{N_\tau \to \infty, \delta\tau \to 0} \mathrm{tr}\left(e^{-\delta\tau H}\right)^{N_\tau} \tag{236}$$

and insert the resolution of the identity at every intermediate time step,

$$Z = \lim_{N_\tau \to \infty, \delta\tau \to 0}\left(\prod_{j=1}^{N_\tau}\int d\mu(\vec{n}_j)\right)\left(\prod_{j=1}^{N_\tau}\langle\vec{n}(\tau_j)|e^{-\delta\tau H}|\vec{n}(\tau_{j+1})\rangle\right)$$

$$\simeq \lim_{N_\tau \to \infty, \delta\tau \to 0}\left(\prod_{j=1}^{N_\tau}\int d\mu(\vec{n}_j)\right)\left(\prod_{j=1}^{N_\tau}[\langle\vec{n}(\tau_j)|\vec{n}(\tau_{j+1})\rangle - \delta\tau\langle\vec{n}(\tau_j)|H|\vec{n}(\tau_{j+1})\rangle]\right) \tag{237}$$

However, since

$$\frac{\langle\vec{n}(\tau_j)|H|\vec{n}(\tau_{j+1})\rangle}{\langle\vec{n}(\tau_j)|\vec{n}(\tau_{j+1})\rangle} \simeq \langle\vec{n}(\tau_j)|H|\vec{n}(\tau_j)\rangle = \mu S\vec{B}\cdot\vec{n}(\tau_j) \tag{238}$$

and

$$\langle\vec{n}(\tau_j)|\vec{n}(\tau_{j+1})\rangle = \left(\frac{1 + \vec{n}(\tau_j)\cdot\vec{n}(\tau_{j+1})}{2}\right)^S e^{i\Phi(\vec{n}(\tau_j),\vec{n}(\tau_{j+1}),\vec{n}_0)S} \tag{239}$$

we can write the partition function in the form

$$Z = \lim_{N_\tau \to \infty, \delta\tau \to 0}\int \mathcal{D}\vec{n}\, e^{-S_E[\vec{n}]} \tag{240}$$

where $S_E[\vec{n}]$ is given by

$$-S_E[\vec{n}] = iS\sum_{j=1}^{N_\tau}\Phi(\vec{n}(\tau_j),\vec{n}(\tau_{j+1}),\vec{n}_0)$$

$$+S\sum_{j=1}^{N_\tau}\ln\left(\frac{1 + \vec{n}(\tau_j)\cdot\vec{n}(\tau_{j+1})}{2}\right) - \sum_{j=1}^{N_\tau}(\delta\tau)\mu S\,\vec{n}(\tau_j)\cdot\vec{B} \tag{241}$$

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The first term of the r. h. s. of Eq. 251 contains the expression $\Phi(\vec{n}(\tau_j), \vec{n}(\tau_{j+1}), \vec{n}_0)$ which has a simple geometric interpretation: it is the sum of the areas of the N_τ contiguous spherical triangles. These triangles have the pole $\vec{n}_0$ as a common vertex, and their other pairs of vertices trace a spherical polygon with vertices at $\{\vec{n}(\tau_j)\}$. In the time continuum limit this spherical polygon becomes the *history* of the spin, which traces a closed oriented curve $\Gamma = \{\vec{n}(\tau)\}$ (with $0 \leq \tau \leq \beta$). Let us denote by Ω^+ the region of the sphere whose boundary is Γ and which contains the pole $\vec{n}_0$. The complement of this region is Ω^- and it contains the opposite pole $-\vec{n}_0$. Hence we find that

$$\lim_{N_\tau \to \infty, \delta\tau \to 0} \Phi(\vec{n}(\tau_j), \vec{n}(\tau_{j+1}), \vec{n}_0) = \mathcal{A}[\Omega^+] = 4\pi - \mathcal{A}[\Omega^-] \qquad (242)$$

where $\mathcal{A}[\Omega]$ is the area of the region Ω. Once again, the ambiguity of the area leads to the requirement that S should be an integer or a half-integer.
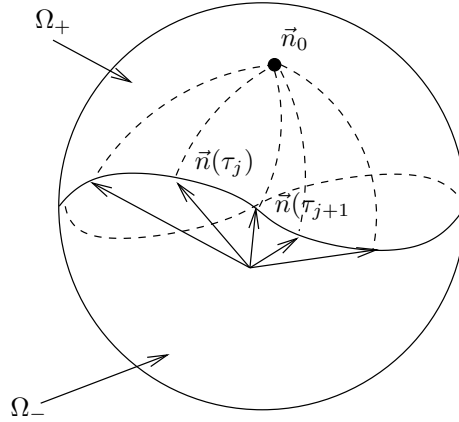

Figure 3:

There is a simple an elegant way to write the area enclosed by Γ. Let $\vec{n}(\tau)$ be a history and Γ be the set of points o the 2-sphere traced by $\vec{n}(\tau)$ for $0 \leq \tau \leq \beta$. Let us define $\vec{n}(\tau, s)$ (with $0 \leq s \leq 1$) to be an *arbitrary* extension of $\vec{n}(\tau)$ from the curve Γ to the interior of the upper cap Ω^+, such that

$$\begin{aligned}
\vec{n}(\tau, 0) &= \vec{n}(\tau) \\
\vec{n}(\tau, 1) &= \vec{n}_0 \\
\vec{n}(\tau, 0) &= \vec{n}(\tau + \beta, 0)
\end{aligned}$$
$$(243)$$

Then the area can be written in the compact form

$$\mathcal{A}[\Omega^+] = \int_0^1 ds \int_0^\beta d\tau \, \vec{n}(\tau, s) \cdot \partial_\tau \vec{n}(\tau, s) \times \partial_s \vec{n}(\tau, s) \equiv S_{\text{WZ}}[\vec{n}] \qquad (244)$$

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In Mathematics this expression for the area is called the (simplectic) 2-form, and in the Physics literature is usually called a Wess-Zumino action, S_{WZ}, or *Berry's Phase*.
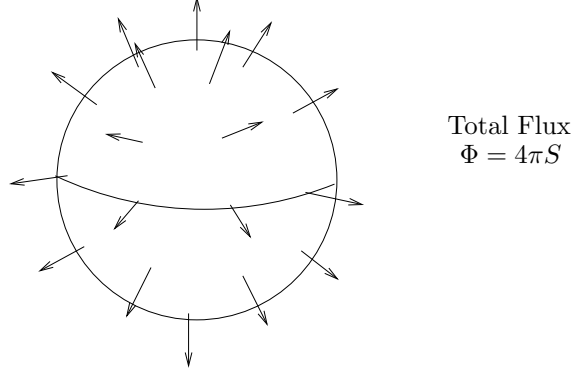

Total Flux
$$\Phi = 4\pi S$$

Figure 4: A hairy ball or monopole

Thus, in the (formal) time continuum limit, the action S_E becomes

$$\mathcal{S}_E = -iS\,\mathcal{S}_{\mathrm{WZ}}[\vec{n}] + \frac{S\delta\tau}{2}\int_0^\beta d\tau\,(\partial_\tau \vec{n}(\tau))^2 + \int_0^\beta d\tau\,\mu S\,\vec{B}\cdot\vec{n}(\tau) \qquad (245)$$

Notice that we have kept (temporarily) a term of order $\delta\tau$, which we will drop shortly.

How do we interpret Eq. 245 ? Since $\vec{n}(\tau)$ is constrained to be a point on the surface of the unit sphere, *i.e.,* $\vec{n}^2 = 1$, the action $S_E[\vec{n}]$ can be interpreted as the action of a particle of mass $M = S\delta\tau \to 0$ and $\vec{n}(\tau)$ is the position vector of the particle at (imaginary) time τ. Thus, the second term is a (vanishingly small) kinetic energy term, and the last term of Eq. (245) is a potential energy term.

What is the meaning of the first term? In Eq. (244) we saw that $\mathcal{S}_{\mathrm{WZ}}[\vec{n}]$, the the so-called Wess-Zumino or Berry phase term in the action, is the *area* of the (positively oriented) region $\mathcal{A}[\Omega_+]$ "enclosed" by the "path" $\vec{n}(\tau)$. In fact,

$$\mathcal{S}_{\mathrm{WZ}}[\vec{n}] = \int_0^1 ds \int_0^\beta d\tau \vec{n} \cdot \partial_\tau \vec{n} \times \partial_s \vec{n} \qquad (246)$$

is the area of the oriented surface Ω^+ whose boundary is the oriented path $\Gamma = \partial\Omega^+$ (see Fig. 3). Using Stokes Theorem we can write the the expression $S\mathcal{A}[\vec{n}]$ as the circulation of a vector field $\vec{A}[\vec{n}]$,

$$\oint_{\partial\Omega} d\vec{n} \cdot \vec{A}[\vec{n}(\tau)] = \iint_{\Omega^+} d\vec{S} \cdot \vec{\nabla}_{\vec{n}} \times \vec{A}[\vec{n}(\tau)] \qquad (247)$$

provided the "magnetic field" $\vec{\nabla}_{\vec{n}} \times \vec{A}$ is "constant", namely

$$\vec{B} = \vec{\nabla}_{\vec{n}} \times \vec{A}[\vec{(\tau)}] = S\,\vec{n}(\tau) \tag{248}$$

What is the total flux Φ of this magnetic field?

$$
\begin{aligned}
\Phi &= \int_{\text{sphere}} d\vec{S} \cdot \vec{\nabla}_{\vec{n}} \times \vec{A}[\vec{n}] \\
&= S \int d\vec{S} \cdot \vec{n} \equiv 4\pi S
\end{aligned} \tag{249}
$$

Thus, the total number of flux quanta N_ϕ piercing the unit sphere is

$$N_\phi = \frac{\Phi}{2\pi} = 2S = \text{magnetic charge} \tag{250}$$

We reach the condition that the magnetic charge is *quantized*, a result known as the *Dirac quantization condition*.

Is this result consistent with what we know about charged particles in magnetic fields? In particular, how is this result related to the physics of spin? To answer these questions we will go back to real time and write the action

$$\mathcal{S}[\vec{n}] = \int_0^T dt\ \left[\frac{M}{2} \left(\frac{d\vec{n}}{dt} \right)^2 + \vec{A}[\vec{n}(t)] \cdot \frac{d\vec{n}}{dt} - \mu S \vec{n}(t) \cdot \vec{B} \right] \tag{251}$$

with the constraint $\vec{n}^{\,2} = 1$ and where the limit $M \to 0$ is implied.

The classical hamiltonian associated to the action of Eq. (251) is

$$H = \frac{1}{2M} \left[\vec{n} \times \left(\vec{p} - \vec{A}[\vec{n}] \right) \right]^2 + \mu S \vec{n} \cdot \vec{B} \equiv H_0 + \mu S \vec{n} \cdot \vec{B} \tag{252}$$

It is easy to check that the vector $\vec{\Lambda}$,

$$\vec{\Lambda} = \vec{n} \times \left(\vec{p} - \vec{A} \right) \tag{253}$$

satisfies the algebra

$$[\Lambda_a, \Lambda_b] = i\hbar \epsilon_{abc} \left(\Lambda_c - \hbar S n_c \right) \tag{254}$$

where $a, b, c = 1, 2, 3$, ϵ_{abc} is the (third rank) Levi-Civita tensor, and with

$$\vec{\Lambda} \cdot \vec{n} = \vec{n} \cdot \vec{\Lambda} = 0 \tag{255}$$

the generators of rotations for this system are

$$\vec{L} = \vec{\Lambda} + \hbar S \vec{n} \tag{256}$$

The operators $\vec{L}$ and $\vec{\Lambda}$ satisfy the (joint) algebra

$$
\begin{gathered}
[L_a, L_b] = -i\hbar \epsilon_{abc} L_s \qquad \left[L_a, \vec{L}^2 \right] = 0 \\
[L_a, n_b] = i\hbar \epsilon_{abc} n_c \qquad [L_a, \Lambda_b] = i\hbar \epsilon_{abc} \Lambda_c
\end{gathered} \tag{257}
$$

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Hence
$$\left[L_a, \vec{\Lambda}^2\right] = 0 \;\Rightarrow\; [L_a, H] = 0 \tag{258}$$

since the operators L_a satisfy the angular momentum algebra, we can diagonalize $\vec{L}^2$ and L_3 simultaneously. Let $|m, \ell\rangle$ be the simultaneous eigenstates of $\vec{L}^2$ and L_3,

$$\vec{L}^2|m, \ell\rangle \;=\; \hbar^2 \ell(\ell+1)|m, \ell\rangle \tag{259}$$
$$L_3|m, \ell\rangle \;=\; \hbar m|m, \ell\rangle \tag{260}$$
$$H_0|m, \ell\rangle \;=\; \frac{\hbar^2}{2MR^2}\left(\frac{\ell(\ell+1) - S}{2S}\right)|m, \ell\rangle \tag{261}$$

where $R = 1$ is the radius of the sphere. The eigenvalues ℓ are of the form $\ell = S + n$, $|m| \leq \ell$, with $n \in \mathbb{Z}^+ \cup \{0\}$ and $2S \in \mathbb{Z}^+ \cup \{0\}$. Hence each level is $2\ell + 1$-fold degenerate, or what is equivalent, $2n + 1 + 2S$-fold degenerate. Then, we get

$$\vec{\Lambda}^2 = \vec{L}^2 - \vec{n}^2\hbar^2 S^2 = \vec{L}^2 - \hbar^2 S^2 \tag{262}$$

Since $M = S\delta t \to 0$, the *lowest* energy in the spectrum of H_0 are those with the *smallest* value of ℓ, *i. e.* states with $n = 0$ and $\ell = S$. The degeneracy of this "Landau" level is $2S + 1$, and the gap to the next excited states diverges as $M \to 0$. Thus, in the $M \to 0$ limit, the *lowest* energy states have the same degeneracy as the spin-S representation. Moreover, the operators $\vec{L}^2$ and L_3 become the corresponding spin operators. thus, the equivalency found is indeed correct.

Thus, we have shown that the quantum states of a scalar (non-relativistic) particle bound to a magnetic monopole of magnetic charge $2S$, obeying the Dirac quantization condition, are identical to those of those of a spinning particle!

We close this section with some observations on the semi-classical motion. From the (real time) action (already in the $M \to 0$ limit)

$$\mathcal{S} = -\int_0^T dt\, \mu S\, \vec{n} \cdot \vec{B} + S \int_0^T dt \int_0^1 ds\, \vec{n} \cdot \partial_t \vec{n} \times \partial_s \vec{n} \tag{263}$$

we can derive a Classical Equation of Motion by looking at the stationary configurations. The variation of the second term in Eq. (263) is

$$\delta\mathcal{S} = S\,\delta \int_0^T dt \int_0^1 ds\, \vec{n} \cdot \partial_t \vec{n} \times \partial_s \vec{n} = S \int_0^T dt\, \delta\vec{n}(t) \cdot \vec{n}(t) \times \partial_t \vec{n}(t) \tag{264}$$

the variation of the first term in Eq. (263) is

$$\delta \int_0^T dt\, \mu S \vec{n}(t) \cdot \vec{B} = \int_0^T dt\, \delta n(t) \cdot \mu S \vec{B} \tag{265}$$

Hence,

$$\delta\mathcal{S} = \int_0^T dt\, \delta\vec{n}(t) \cdot \left(-\mu S\vec{B} + S\vec{n}(t) \times \partial_t \vec{n}(t)\right) \tag{266}$$

which implies that the classical trajectories must satisfy the equation of motion

$$\mu\vec{B} = \vec{n} \times \partial_t\vec{n} \tag{267}$$

If we now use the vector identity

$$\vec{n} \times \vec{n} \times \partial_t\vec{n} = (\vec{n} \cdot \partial_t\vec{n})\,\vec{n} - \vec{n}^{\,2}\partial_t\vec{n} \tag{268}$$

and

$$\vec{n} \cdot \partial_t\vec{n} = 0, \qquad \text{and} \qquad \vec{n}^{\,2} = 1 \tag{269}$$

we get the classical equation of motion

$$\partial_t\vec{n} = \mu\vec{B} \times \vec{n} \tag{270}$$

Therefore, the classical motion is *precessional* with an angular velocity $\vec{\Omega}_{\mathrm{pr}} = \mu\vec{B}$.