

### 3 Classical Symmetries and Conservation Laws

We have used the existence of symmetries in a physical system as a guiding principle for the construction of their Lagrangians and energy functionals. We will show now that the presence of such symmetries imply the existence of conservation laws.

There are different types of symmetries which, roughly, can be classified in two classes: (a) space-time symmetries and (b) internal symmetries. Some symmetries involve discrete operations (thereby they will be called *discrete* symmetries) while others are *continuous* symmetries. Furthermore, in some cases these are *global symmetries* of the theory, while others are *local symmetries*. The latter class goes under the name of *gauge symmetries*. We will see that, in the fully quantized theory, global and local symmetries play different roles.

Examples of space-time symmetries are the most common symmetries that are encountered in Physics. They include translation invariance and rotation invariance. If the system is isolated, then time-translation is also a symmetry. A *non-relativistic* system is in general invariant under *Galilean* transformations, while *relativistic* systems, are instead *Lorentz* invariant. Other space-time symmetries include *time-reversal* (T), *parity* (P) and *charge conjugation* (C). These symmetries are *discrete*.

In classical mechanics, the existence of symmetries has important consequences. Thus, *translation invariance* (or uniformity of space) implies the *conservation* of the *total momentum*  $\vec{P}$  of the system. Similarly *isotropy* implies the conservation of the *total angular momentum*  $\vec{L}$  and *time translation invariance* implies the conservation of the *total energy*  $E$ .

All of these concepts have analogs in field theory. However new symmetries will also appear. These are the *internal symmetries* which we will discuss below.

#### 3.1 Continuous Symmetries and Noether's Theorem

We will show now that the existence of continuous symmetries has very profound implications, such as the existence of *conservation laws*. One important feature of these conservation laws is the existence of *locally conserved currents*. This is the content of the following theorem, due to Emmy Noether.

**Noether's Theorem:** *For every continuous global symmetry there exists a global conservation law.*

Before we prove this statement, let us discuss the connection that exists between *locally conserved currents* and *constants of motion*. In particular, let us show that for every locally conserved current there exist a globally conserved quantity, *i. e.* a constant of motion.

Let  $j^\mu(x)$  be some locally conserved current, *i. e.*

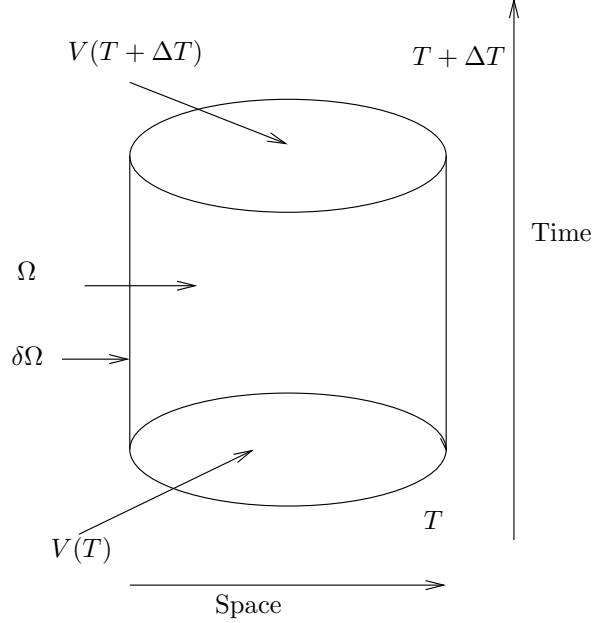
$$\partial_\mu j^\mu(x) = 0 \tag{1}$$

Let  $\Omega$  be a bounded 4-volume of space-time, with boundary  $\partial\Omega$ . The Divergence

Theorem (or Gauss' Theorem) tells us that

$$0 = \int_{\Omega} d^4x \partial_{\mu} j^{\mu}(x) = \oint_{\partial\Omega} dS_{\mu} j^{\mu}(x) \quad (2)$$

where the r. h. s. is a surface integral on the *oriented closed surface*  $\partial\Omega$  (a 3-volume). Let  $\Omega$  be a 4-volume which extends all the way to infinity in space but which has a finite extent in time  $\Delta T$ . If there are no currents at *spacial* infinity



(i. e.  $\lim_{|\vec{x}| \rightarrow \infty} j^{\mu}(\vec{x}, x_0) = 0$ ), then only the top and bottom of the boundary  $\partial\Omega$  contribute to the surface integral. Hence,

$$0 = \int_{V(T+\Delta T)} dS_0 j^0(\vec{x}, T + \Delta T) - \int_{V(T)} dS_0 j^0(\vec{x}, T) \quad (3)$$

Since  $dS_0 \equiv d^3x$ , we get

$$0 = \int_{V(T+\Delta T)} d^3x j^0(\vec{x}, T + \Delta T) - \int_{V(T)} d^3x j^0(\vec{x}, T) \quad (4)$$

Thus, we see that the quantity  $Q(T)$

$$Q(T) = \int_{V(T)} d^3x j^0(\vec{x}, T) \quad (5)$$

is a *constant of motion*, i. e.

$$Q(T + \Delta T) = Q(T) \quad \forall \Delta T \quad (6)$$

Hence, the existence of a locally conserved current ( $\partial_\mu j^\mu = 0$ ) implies the existence of a conserved *charge*  $Q$ ,  $Q = \int d^3x j^0(\vec{x}, T)$ , which is a *constant of motion*. Thus, the proof of Noether's theorem reduces to the proof of the existence of a locally conserved current.

### 3.2 Internal symmetries

Let us begin for simplicity with a *complex scalar field*  $\phi(x) \neq \phi^*(x)$ . The arguments that follow are easily generalized to other cases. Let  $\mathcal{L}(\phi, \partial_\mu \phi)$  be its Lagrangian density. This system has the *continuous global symmetry*

$$\begin{aligned}\phi(x) &\rightarrow \phi'(x) = e^{i\alpha} \phi(x) \\ \phi^*(x) &\rightarrow \phi'^*(x) = e^{-i\alpha} \phi^*(x)\end{aligned}\quad (7)$$

where  $\alpha$  is an arbitrary real *number* (not a function!). The system is *invariant* if  $\mathcal{L}$  satisfies

$$\mathcal{L}(\phi', \partial_\mu \phi') \equiv \mathcal{L}(\phi, \partial_\mu \phi) \quad (8)$$

In particular, for an infinitesimal transformation we have

$$\phi'(x) = \phi(x) + \delta\phi + \dots \quad \phi'^*(x) = \phi^*(x) + \delta\phi^* + \dots \quad (9)$$

where  $\delta\phi = i\alpha\phi(x)$ . Since  $\mathcal{L}$  is *invariant*, its variation must be identically equal to zero. The variation  $\delta\mathcal{L}$  is

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi} \delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi} \delta\partial_\mu \phi + \frac{\delta\mathcal{L}}{\delta\phi^*} \delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi^*} \delta\partial_\mu \phi^* \quad (10)$$

Using the equations of motion

$$\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \left( \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi} \right) = 0 \quad (11)$$

and its complex conjugate, we can write the variation  $\delta\mathcal{L}$  in the form of a total divergence

$$\delta\mathcal{L} = \partial_\mu \left[ \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi} \delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi^*} \delta\phi^* \right] \quad (12)$$

Thus, since  $\delta\phi = i\alpha\phi$  and  $\delta\phi^* = -i\alpha\phi^*$ , we get

$$\delta\mathcal{L} = \partial_\mu \left[ i \left( \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi} \phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi^*} \phi^* \right) \alpha \right] \quad (13)$$

Hence, since  $\alpha$  is arbitrary,  $\delta\mathcal{L}$  will vanish identically if and only if the 4-vector  $j^\mu$ , defined by

$$j^\mu = i \left( \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi} \phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu \phi^*} \phi^* \right) \quad (14)$$

is locally conserved, *i. e.*

$$\delta\mathcal{L} = 0 \Leftrightarrow \partial_\mu j^\mu = 0 \quad (15)$$

If  $\mathcal{L}$  has the form

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - V(|\phi|^2) \quad (16)$$

which is manifestly invariant, we see that  $j^\mu$  is given by

$$j^\mu = i (\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi) \equiv i \phi^* \overleftrightarrow{\partial}_\mu \phi \quad (17)$$

Thus, the presence of an continuous internal symmetry implies the existence of a locally conserved current. Furthermore, the conserved charge  $\mathcal{Q}$  is given by

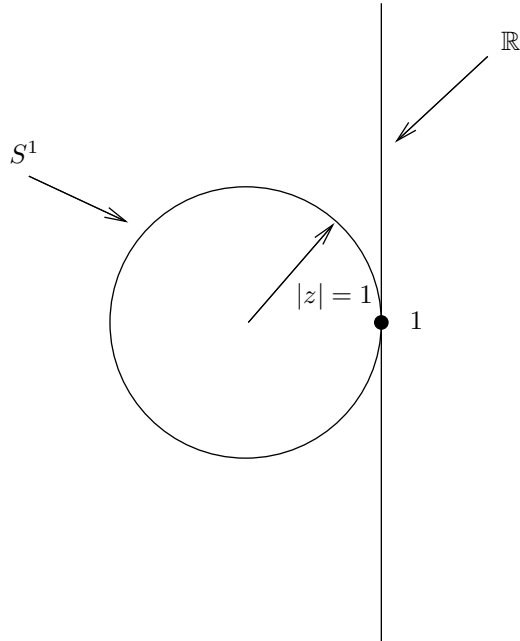
$$\mathcal{Q} = \int d^3x j^0(\vec{x}, x_0) = \int d^3x i \phi^* \overleftrightarrow{\partial}_0 \phi \quad (18)$$

In terms of the canonical momentum  $\Pi(x)$ , we can write

$$\mathcal{Q} = \int d^3x i(\phi^* \Pi - \phi \Pi^*) \quad (19)$$

### 3.3 Global symmetries and group representations

Let us generalize this last result. Let us consider a scalar field  $\phi^a$  which transforms like a certain representation of a Lie group  $G$ . In the case considered in the previous section the group  $G$  is the group of complex vectors of unit length, the group  $U(1)$ . The elements of this group,  $g \in U(1)$ , have the form  $g = e^{i\alpha}$ . They form a group in the sense that,



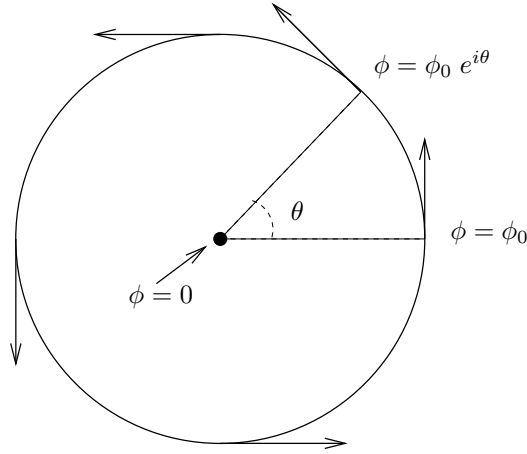
1. this set it is closed under complex multiplication *i. e.*

$$g = e^{i\alpha} \in U(1) \quad \text{and} \quad g' = e^{i\beta} \in U(1) \Rightarrow g * g' = e^{i(\alpha+\beta)} \in U(1) \quad (20)$$

2. there is an identity element, *i. e.*  $g = 1$
3. for every element  $g = e^{i\alpha}$  there is an unique inverse element  $g^{-1} = e^{-i\alpha}$ .

The elements of the group  $U(1)$  are in one-to-one correspondence with the points of the unit circle  $S_1$ . Consequently, the parameter  $\alpha$  that labels the transformation (or element of this group) is defined modulo  $2\pi$ , and it should be restricted to the interval  $(0, 2\pi]$ . However, transformations infinitesimally close to the identity 1 lie essentially on the straight line tangent to the circle at 1 and are isomorphic to the group of real numbers  $\mathbb{R}$ . The group  $U(1)$ , which is said to be compact in the sense that the length of its natural parametrization is  $2\pi$  which is finite. In contrast the group  $\mathbb{R}$  of real numbers is non-compact.

For infinitesimal transformations the groups  $U(1)$  and  $\mathbb{R}$  are essentially identical. There are, however, configurations for which they are not. A typical case is the *vortex configuration* in two dimensions. For a vortex the phase of the field on la large circle of radius  $R \rightarrow \infty$  winds by  $2\pi$ . Such configurations would not exist if the symmetry group was  $\mathbb{R}$  instead of  $U(1)$ . (Note: analyticity requires that  $\phi \rightarrow 0$  as  $x \rightarrow 0$ .)



Another example is the  $N$ -component *real* scalar field  $\phi^a(x) (a = 1, \dots, N)$ . In this case the symmetry is the group of *rotations* in  $N$ -dimensional space

$$\phi'^a(x) = R^{ab} \phi^b(x) \quad (21)$$

$\phi^a$  is said to transform like the  $N$ -dimensional (vector) representation of the Orthogonal group  $O(N)$ . The elements of the orthogonal group,  $R \in O(N)$ ,

satisfy

$$\begin{aligned}
R_1 \in O(N) \text{ and } R_2 \in O(N) &\Rightarrow R_1 R_2 \in O(N) \\
\exists I \in O(N) \text{ such that } \forall R \in O(N) &\Rightarrow RI = IR = R \\
\forall R \in O(N), \exists R^{-1} \in O(N) \text{ such that} &R^{-1} = R^T
\end{aligned}
\tag{22}$$

where  $R^T$  is the transpose of the matrix  $R$ .

Similarly, if the  $N$ -component vector  $\phi^a(x)$  is a *complex field*, it transforms under the group of  $(N \times N)$  *Unitary* transformations  $U$

$$\phi'^a(x) = U^{ab} \phi^b(x) \tag{23}$$

The complex  $N \times N$  matrices  $U$  are elements of the Unitary group  $U(N)$  and satisfy

$$\begin{aligned}
U_1 \in U(N) \text{ and } U_2 \in U(N) &\Rightarrow U_1 U_2 \in U(N) \\
\exists I \in U(N) \text{ such that } \forall U \in U(N) &\Rightarrow UI = IU = U \\
\forall U \in U(N), \exists U^{-1} \in U(N) \text{ such that} &U^{-1} = U^\dagger
\end{aligned}
\tag{24}$$

where  $U^\dagger = (U^T)^*$ .

In the particular case discussed above  $\phi^a$  transforms like the fundamental (spinor) representation of  $U(N)$ . If we impose the further restriction that  $|\det U| = 1$ , the group becomes  $SU(N)$ . For instance, if  $N = 2$ , the group is  $SU(2)$  and

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{25}$$

it transforms like the spin-1/2 representation of  $SU(2)$ .

In general, for an arbitrary continuous Lie group  $G$ , the field transforms like

$$\phi'_a(x) = (\exp [i\lambda^k \theta^k])_{ab} \phi_b(x) \tag{26}$$

where the vector  $\vec{\theta}$  is arbitrary and constant (*i. e.* independent of  $x$ ). The *matrices*  $\lambda^k$  are  $N \times N$  linearly independent matrices which span the *algebra* of the Lie group  $G$ . For a given Lie group  $G$ , the number of such matrices is  $D(G)$  and it is independent of the *dimension*  $N$  of the representation that was chosen.  $D(G)$  is called the *rank* of the group. The matrices  $\lambda_{ab}^k$  are the *generators* of the group in this representation.

In general, from a symmetry point of view, the field  $\phi$  does not have to be a vector, as it can also be a tensor or for the matter transform under any representation of the group. For simplicity, we will only consider the case of vector representation of  $O(N)$  and the fundamental (spinor) and adjoint (vector) (see below) representations of  $SU(N)$

For an arbitrary Lie group  $G$ , the generators  $\{\lambda^j\}$ ,  $j = 1, \dots, D(G)$ , are a set of hermitean ( $\lambda_j^\dagger = \lambda_j$ ) traceless matrices ( $\text{tr}\lambda_j = 0$ ) which obey the commutation relations

$$[\lambda^j, \lambda^k] = i f^{jkl} \lambda^l \quad (27)$$

The numerical constants  $f^{jkl}$  are known as the *structure constants* of the group and are the same in all representations.

In addition, the generators have to be normalized. It is standard to require the normalization condition

$$\text{tr}\lambda^a \lambda^b = \frac{1}{2} \delta^{ab} \quad (28)$$

In the case considered above, the complex scalar field  $\phi(x)$ , the symmetry group is the group of unit length complex numbers of the form  $e^{i\alpha}$ . This group is known as the group  $U(1)$ . All its representations are one-dimensional and it has only one generator.

A commonly used group is  $SU(2)$ . This group, which is familiar from non-relativistic quantum mechanics, has three generators  $J_1, J_2$  and  $J_3$ , which obey the angular momentum algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad (29)$$

with

$$\begin{aligned} \text{tr}J_i &= 0 \\ \text{tr}(J_i J_j) &= \frac{1}{2} \delta_{ij} \end{aligned} \quad (30)$$

The representations of  $SU(2)$  are labelled by the angular momentum quantum number  $J$ . Each representation  $J$  is a  $2J+1$ -fold degenerate multiplet, *i. e.* the dimension is  $2J+1$ .

The lowest non-trivial representation, *i. e.*  $J \neq 0$ , is the spinor representation which has  $J = \frac{1}{2}$  and it is two-fold degenerate. In this representation the field  $\phi_a(x)$  is a two-component complex spinor and the generators  $J_1, J_2$  and  $J_3$  are given by the set of  $2 \times 2$  Pauli matrices  $J_j, J_j = \frac{1}{2} \sigma_j$ .

The vector (or spin-1) representation is three dimensional and  $\phi_a$  is a three-component vector. In this representation, the generators are very simple

$$(J_j)_{kl} = \epsilon_{jkl} \quad (31)$$

Notice that the dimension of this representation (3) is the same as the rank (3) of the group  $SU(2)$ . In this representation, which is known as the *adjoint* representation, the matrix elements of the generators are the structure constants. This is a general property of all Lie groups. In particular, for the group  $SU(N)$ , whose rank is  $N^2 - 1$ , it has  $N^2 - 1$  infinitesimal generators, and the dimension of its adjoint (vector) representation is  $N^2 - 1$ . Thus, for  $SU(3)$  the number of generators is eight.

Another important case for the applications, is the group of rotations  $O(N)$ . In this case, the group has  $N(N-1)/2$  generators which can be labelled by the matrices  $L^{ij}(0, j=1, \dots, N)$ . The fundamental (vector) representation of  $O(N)$  is  $N$ -dimensional and, in this representation, the generators are

$$(L^{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \quad (32)$$

It is easy to see that the  $L^{ij}$ 's generate infinitesimal rotations in  $N$ -dimensional space.

Quite generally, if the ‘‘Euler angles’’  $\vec{\theta}$  are infinitesimal then the representation matrix  $\exp(i\vec{\lambda} \cdot \vec{\theta})$  is close to the identity and it can be expanded in powers of  $\vec{\theta}$ . To leading order, the change in  $\phi^a$  is

$$\delta\phi^a(x) = i(\vec{\lambda} \cdot \vec{\theta})^{ab} \phi^b(x) + \dots \quad (33)$$

If  $\phi_a$  is real, the conserved current  $j^\mu$  is

$$j_\mu^k(x) = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^a(x)} \lambda_{ab}^k \phi_b(x) \quad (34)$$

where  $k=1, \dots, D(G)$ . Here, the generators  $\lambda^k$  are real hermitean matrices. In contrast, for a complex field  $\phi_a$ , the conserved currents are

$$j_\mu^k(x) = i \left( \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^a(x)} \lambda_{ab}^k \phi_b(x) - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^a(x)^*} \lambda_{ab}^k \phi_b(x)^* \right) \quad (35)$$

are the generators  $\lambda^k$  are hermitean matrices (but are not all real).

Thus, we conclude that *the number of conserved currents is equal to the number of generators of the group*. For the particular choice

$$\mathcal{L} = (\partial_\mu\phi_a)^*(\partial^\mu\phi_a) - V(\phi_a^*, \phi_a) \quad (36)$$

the conserved current is

$$j_\mu^k = i \left( (\partial_\mu\phi_a^*) \lambda_{ab}^k \phi_b - \phi_a^* \lambda_{ab}^k (\partial_\mu\phi_b) \right) \quad (37)$$

and the conserved charges are

$$Q^k = \int d^3x i \phi_a^* \overleftrightarrow{\partial}_0 \lambda_{ab}^k \phi_b \quad (38)$$

## 4 Global and Local Symmetries: Gauge Invariance

The existence of global symmetries presupposes that, at least in principle, we can measure and change all of the components of a field  $\phi^a(x)$  at all points  $\vec{x}$  in space at the same time. Relativistic invariance tells us that, although the theory may possess this global symmetry, in principle this experiment cannot



be carried out. One is then led to consider theories which are invariant if the symmetry operations are performed *locally*. Namely, we should require that the Lagrangian be invariant under *local transformations*

$$\phi_a(x) \rightarrow \phi'_a(x) = \left( \exp \left[ i\lambda^k \theta^k(x) \right] \right)_{ab} \phi_b(x) \quad (39)$$

For instance, we can demand that the theory of a complex scalar field  $\phi(x)$  be invariant under local changes of phase

$$\phi(x) \rightarrow \phi'(x) = e^{i\theta(x)} \phi(x) \quad (40)$$

The standard local Lagrangian  $\mathcal{L}$

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - V(|\phi|^2) \quad (41)$$

is invariant under *global* transformations with  $\theta = \text{const.}$ , but it is not invariant under arbitrary smooth *local* transformations  $\theta(x)$ . The main problem is that since the derivative of the field does not transform like the field itself, the kinetic energy term is no longer invariant. Indeed, under a local transformation we find

$$\partial_\mu \phi(x) \rightarrow \partial_\mu \phi'(x) = \partial_\mu \left[ e^{i\theta(x)} \phi(x) \right] = e^{i\theta(x)} [\partial_\mu \phi + i\phi \partial_\mu \theta] \quad (42)$$

In order to make  $\mathcal{L}$  *locally* invariant we must find a new *derivative operator*  $D_\mu$ , the *covariant derivative*, which must transform like the field  $\phi(x)$  under local phase transformations, *i. e.*

$$D_\mu \phi \rightarrow D'_\mu \phi' = e^{i\theta(x)} D_\mu \phi \quad (43)$$

From a “geometric” point of view we can picture the situation as follows. In order to define the phase of  $\phi(x)$  locally, we have to define a local frame or fiducial field with respect to which the phase of the field is measured. Local invariance is then the statement that the physical properties of the system must be independent of the particular choice of frame. From this point of view, local gauge invariance is an extension of the principle of relativity to the case of internal symmetries.

Now, if we wish to make phase transformations which differ from point to point, we have to specify how the phase changes as we go from one point  $x$  in space-time to another one  $y$ . In other words, we have to establish a *connection* which will tell us how we are suppose to *transport* the phase of  $\phi$  from  $x$  to  $y$  as we travel along some path  $\Gamma$ . Let us consider the situation in which  $x$  and  $y$  are arbitrarily close to each other, *i. e.*  $y_\mu = x_\mu + dx_\mu$  where  $dx_\mu$  is an infinitesimal 4-vector. The change in  $\phi$  is

$$\phi(x + dx) - \phi(x) = \delta\phi(x) \quad (44)$$

If the *transport* of  $\phi$  along some path going from  $x$  to  $x + dx$  is to *correspond* to a *phase transformation*, then  $\delta\phi$  must be proportional to  $\phi$ . So we are led to *define*

$$\delta\phi(x) = iA_\mu(x) dx^\mu \phi(x) \quad (45)$$

where  $A_\mu(x)$  is a suitably chosen *vector field*. Clearly, this implies that the *covariant derivative*  $D_\mu$  must be defined to be

$$D_\mu\phi \equiv \partial_\mu\phi(x) - ieA_\mu(x)\phi(x) \equiv (\partial_\mu - ieA_\mu)\phi \quad (46)$$

where  $e$  is a parameter which we will give the physical interpretation of a coupling constant.

How should  $A_\mu(x)$  transform? We must choose its transformation law in such a way that  $D_\mu\phi$  transforms like  $\phi(x)$  itself. Thus, if  $\phi \rightarrow e^{i\theta}\phi$  we have

$$D'\phi' = (\partial_\mu - ieA'_\mu)(e^{i\theta}\phi) \equiv e^{i\theta}D_\mu\phi \quad (47)$$

This requirement can be met if

$$i\partial_\mu\theta - ieA'_\mu = -ieA_\mu \quad (48)$$

Hence,  $A_\mu$  should transform like

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta \quad (49)$$

But this is nothing but a gauge transformation! Indeed if we define the gauge transformation  $\Lambda(x)$

$$\Lambda(x) \equiv \frac{1}{e}\theta(x) \quad (50)$$

(not to be confused with a Lorentz transformation!) we see that the vector field  $A_\mu$  transforms like the vector potential of Maxwell's electromagnetism.

We conclude that we can promote a global symmetry to a local (*i. e. gauge*) symmetry by replacing the derivative operator by the *covariant derivative*. Thus, we can make a system invariant under local gauge transformations at the expense of introducing a *vector field*  $A_\mu$  (the gauge field) which plays the role of a connection. From a physical point of view, this result means that the impossibility of making a comparison at a distance of the phase of the field  $\phi(x)$  requires that a physical gauge field  $A_\mu(x)$  must be present. This procedure, which relates the matter and gauge fields through the covariant derivative, is known as *minimal coupling*.

There is a set of configurations of  $\phi(x)$  which changes only because of the presence of the gauge field. They are the *geodesic* configurations  $\phi_c(x)$ . They satisfy the equation

$$D_\mu\phi_c = (\partial_\mu - ieA_\mu)\phi_c \equiv 0 \quad (51)$$

which is equivalent to the linear equation (see Eq. 45)

$$\partial_\mu\phi_c = ieA_\mu\phi_c \quad (52)$$

Let us consider, for example, two points  $x$  and  $y$  in space-time at the end point of a path  $\Gamma(x, y)$ . For a *given* path  $\Gamma(x, y)$ , the solution of the equation is the path-ordered exponential of a line integral

$$\phi_c(x) = \phi_c(y)e^{-ie\int_{\Gamma(x,y)} dz_\mu A^\mu(z)} \quad (53)$$

Indeed, under a gauge transformation, the line integral transforms like

$$\begin{aligned} e \int_{\Gamma(x,y)} dz_\mu A^\mu &\rightarrow e \int_{\Gamma(x,y)} dz_\mu A^\mu + e \int_{\Gamma(x,y)} dz_\mu \frac{1}{e} \partial^\mu \theta \\ &= e \int_{\Gamma(x,y)} dz_\mu A^\mu(z) + \theta(y) - \theta(x) \end{aligned} \quad (54)$$

Hence, we get

$$\begin{aligned} \phi_c(y) e^{-ie \int_\Gamma dz_\mu A^\mu} &\rightarrow \phi_c(y) e^{-ie \int_\Gamma dz_\mu A^\mu} e^{-i\theta(y)} e^{i\theta(x)} \\ &\equiv e^{i\theta(x)} \phi_c(x) \end{aligned} \quad (55)$$

as it should be.

However, we may now want to ask how does the change of phase of  $\phi_c$  depend on the choice of the path  $\Gamma$ . Thus, let  $\phi_c^{\Gamma_1}(y)$  and  $\phi_c^{\Gamma_2}(y)$  be solutions of the geodesic equations for two different paths  $\Gamma_1$  and  $\Gamma_2$  with the same end points. Clearly, we have that the change of phase  $\Delta\gamma$  is

$$e^{i\Delta\gamma} \equiv \frac{\phi_c^{\Gamma_1}(y)}{\phi_c^{\Gamma_2}(y)} = \frac{e^{ie \int_{\Gamma_1} dz_\mu A^\mu} \phi_c(x)}{e^{ie \int_{\Gamma_2} dz_\mu A^\mu} \phi_c(x)} \quad (56)$$

where  $\Delta\gamma$  is given by

$$\Delta\gamma = -e \int_{\Gamma_1} dz_\mu A^\mu + e \int_{\Gamma_2} dz_\mu A^\mu \equiv -e \oint_{\Gamma^+} dz_\mu A^\mu \quad (57)$$

Here  $\Gamma^+$  is the *closed oriented path*

$$\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^- \quad (58)$$

and

$$\int_{\Gamma_2^-} dz_\mu A_\mu = - \int_{\Gamma_2^+} dz_\mu A_\mu \quad (59)$$

Using Stokes theorem we see that, if  $\Sigma^+$  is an *oriented surface* whose *boundary* is the *closed path*  $\Gamma^+$  (*i. e.*  $\partial\Sigma^+ \equiv \Gamma^+$ ), then  $\Delta\gamma$  is given by the flux of the curl of  $A_\mu$  through the surface  $\Sigma^+$ , *i. e.*

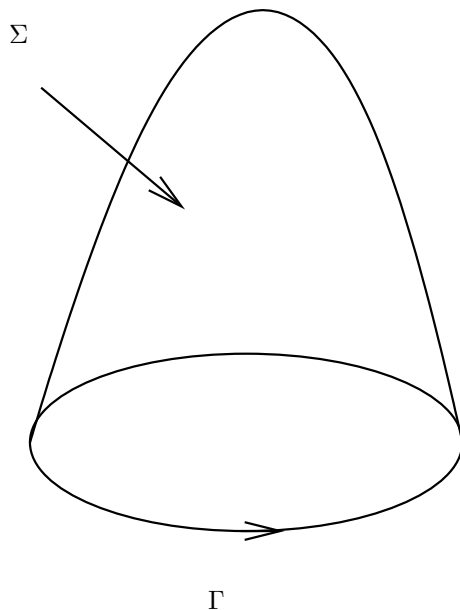
$$\Delta\gamma = -\frac{e}{2} \int_{\Sigma^+} ds_{\mu\nu} F^{\mu\nu} = -e \Phi(\Sigma) \quad (60)$$

where  $F^{\mu\nu}$  is the field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (61)$$

$dS_{\mu\nu}$  is the *oriented area element*, and  $\Phi(\Sigma)$  is the flux through the surface  $\Sigma$ . Both  $F^{\mu\nu}$  and  $dS_{\mu\nu}$  are antisymmetric in their space-time indices. In particular,  $F^{\mu\nu}$  can also be written as a commutator of covariant derivatives

$$F^{\mu\nu} = \frac{i}{e} [D^\mu, D^\nu] \quad (62)$$



Thus,  $F^{\mu\nu}$  measures the (infinitesimal) incompatibility of displacements along two independent directions. In other words,  $F^{\mu\nu}$  is a *curvature*. These results show very clearly that if  $F^{\mu\nu}$  is non-zero on some region of space-time, then the *phase* of  $\phi$  cannot be uniquely determined: the *phase* of  $\phi_c$  depends on the *path*  $\Gamma$  along which it is measured.

#### 4.1 The Aharonov-Bohm Effect

The path dependence of the phase of  $\phi_c$  is known as the *Aharonov-Bohm Effect*. This is a very subtle effect which was first discovered in the context of elementary quantum mechanics. However it plays a fundamental role in field theory as well.

Consider a quantum mechanical particle of charge  $e$  and mass  $m$  moving on a plane. The particle is coupled to an external electromagnetic field  $A_\mu$  (here  $\mu = 0, 1, 2$  only, since there is no motion out of the plane). Let us consider the following geometry in which an infinitesimally thin solenoid is piercing the plane at some point  $\vec{r} = 0$ . The Schrödinger Equation for this problem is

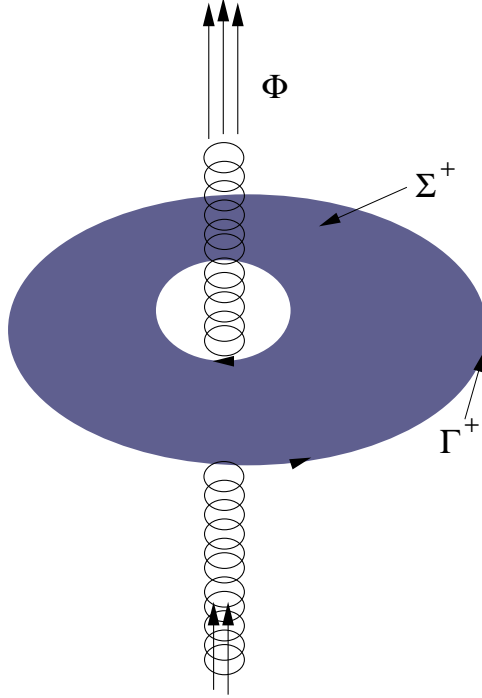
$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t} \quad (63)$$

where

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2\Psi \quad (64)$$

is the Hamiltonian. The magnetic field  $\vec{B} = B\hat{z}$  is non zero only at  $\vec{r} = 0$

$$B = \Phi_0 \delta(\vec{r}) \quad (65)$$



Thus the flux of  $\vec{B}$  through an arbitrary region  $\Sigma^+$  with boundary  $\Gamma^+$  is

$$\Phi = \int_{\Sigma^+} d\vec{S} \cdot \vec{B} = \oint_{\Gamma^+} d\vec{\ell} \cdot \vec{A} \quad (66)$$

Hence,  $\Phi = \Phi_0$  for all  $\Sigma^+$  enclosing the point  $\vec{r} = 0$  and it is equal to zero otherwise.

Although the magnetic field is zero for  $\vec{r} \neq 0$ , the vector potential is not. Indeed, Stokes theorem implies that

$$\Phi_0 = \int_{\Sigma^+} d\vec{S} \cdot \vec{B} = \oint_{\Gamma^+} d\vec{\ell} \cdot \vec{A} \quad (67)$$

and the vector potential  $\vec{A}(\vec{r})$  cannot vanish.

The wave function  $\Psi(\vec{r})$  can be calculated in a very simple way. Let us define

$$\Psi(\vec{r}) = e^{i\theta(\vec{r})} \Psi_0(\vec{r}) \quad (68)$$

where  $\Psi_0(\vec{r})$  satisfies the Schrödinger Equation *in the absence* of the field, *i. e.*

$$H_0 \Psi_0 = i\hbar \frac{\partial \Psi_0}{\partial t} \quad (69)$$

with

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 \quad (70)$$

Since the wave function  $\Psi$  has to be differentiable and  $\Psi_0$  is single valued, we must also demand the boundary condition that

$$\lim_{\vec{r} \rightarrow 0} \Psi_0(\vec{r}, t) = 0 \quad (71)$$

The wave function  $\Psi = \Psi_0 e^{i\theta}$  looks like a gauge transformation. But we will discover that there is a subtlety here. Indeed,  $\theta$  can be determined as follows. By direct substitution we get

$$\left(\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\vec{A}\right)(e^{i\theta}\Psi_0) = e^{i\theta}\left(\hbar\vec{\nabla}\theta + \frac{e}{c}\vec{A} + \frac{\hbar}{i}\vec{\nabla}\right)\Psi_0 \quad (72)$$

Thus, in order to succeed in our task, we only have to require that  $\vec{A}$  and  $\theta$  must obey the relation

$$\hbar\vec{\nabla}\theta + \frac{e}{c}\vec{A} \equiv 0 \quad (73)$$

Or, equivalently,

$$\vec{\nabla}\theta(\vec{r}) = -\frac{e}{\hbar c}\vec{A}(\vec{r}) \quad (74)$$

However, if this relation holds,  $\theta$  cannot be a smooth function of  $\vec{r}$ . In fact, the line integral of  $\vec{\nabla}\theta$  on an arbitrary *closed* path  $\Gamma^+$  is given by

$$\int_{\Gamma^+} d\vec{\ell} \cdot \vec{\nabla}\theta = \Delta\theta \quad (75)$$

where  $\Delta\theta$  is the total change of  $\theta$  in one full counterclockwise turn around the path  $\Gamma$ . It is immediate to see that  $\Delta\theta$  is given by

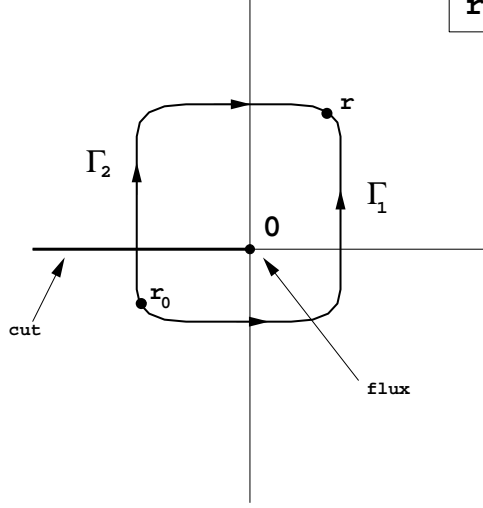
$$\Delta\theta = -\frac{e}{\hbar c} \oint_{\Gamma^+} d\vec{\ell} \cdot \vec{A} \quad (76)$$

We must conclude that, in general,  $\theta(\vec{r})$  is a *multivalued* function of  $\vec{r}$  which has a branch cut going from  $\vec{r} = 0$  out to some arbitrary point at infinity. The actual position and shape of the branch cut is irrelevant but the *discontinuity*  $\Delta\theta$  of  $\theta$  across the cut is not.

Hence, if  $\bar{\Psi}_0$  is chosen to be a smooth, single valued, solution of the Schrödinger equation in the *absence* of the solenoid, satisfying the boundary condition of Eq. 71. Such wave functions are (almost) plane waves.

Since the function  $\theta(\vec{r})$  is multivalued and, hence, path-dependent, the wave function  $\Psi$  is also multivalued and path-dependent. In particular, let  $\vec{r}_0$  be some arbitrary point on the plane and  $\Gamma(\vec{r}_0, \vec{r})$  is a path that begins in  $\vec{r}_0$  and ends at  $\vec{r}$ . The phase  $\theta(\vec{r})$  is, for that choice of path, given by

$$\theta(\vec{r}) = \theta(\vec{r}_0) - \frac{e}{\hbar c} \int_{\Gamma(\vec{r}_0, \vec{r})} d\vec{x} \cdot \vec{A}(\vec{x}) \quad (77)$$



The overlap of two wave functions which are defined by two different paths  $\Gamma_1(\vec{r}_0, \vec{r})$  and  $\Gamma_2(\vec{r}_0, \vec{r})$  is (with  $\vec{r}_0$  fixed)

$$\begin{aligned} \langle \Gamma_1 | \Gamma_2 \rangle &= \int d^2\vec{r} \Psi_{\Gamma_1}^*(\vec{r}) \Psi_{\Gamma_2}(\vec{r}) \\ &\equiv \int d^2\vec{r} |\Psi_0(\vec{r})|^2 \exp\left\{ +\frac{ie}{\hbar c} \left( \int_{\Gamma_1(\vec{r}_0, \vec{r})} d\vec{\ell} \cdot \vec{A} - \int_{\Gamma_2(\vec{r}_0, \vec{r})} d\vec{\ell} \cdot \vec{A} \right) \right\} \end{aligned} \quad (78)$$

If  $\Gamma_1$  and  $\Gamma_2$  are chosen in such a way that the origin (where the solenoid is piercing the plane) is always to the left of  $\Gamma_1$  but it is also always to the right of  $\Gamma_2$ , the difference of the two line integrals is the circulation of  $\vec{A}$

$$\int_{\Gamma_1(\vec{r}_0, \vec{r})} d\vec{\ell} \cdot \vec{A} - \int_{\Gamma_2(\vec{r}_0, \vec{r})} d\vec{\ell} \cdot \vec{A} \equiv \oint_{\Gamma_{(\vec{r}_0)}^+} d\vec{\ell} \cdot \vec{A} \quad (79)$$

on the closed, positively oriented, contour  $\Gamma^+ = \Gamma_1(\vec{r}_0, \vec{r}) \cup \Gamma_2(\vec{r}, \vec{r}_0)$ . Since this circulation is constant, and equal to  $\Phi$ , we find that the overlap  $\langle \Gamma_1 | \Gamma_2 \rangle$  is

$$\langle \Gamma_1 | \Gamma_2 \rangle = \exp\left\{ \frac{ie}{\hbar c} \Phi \right\} \quad (80)$$

where we have taken  $\Psi_0$  to be normalized to unity. The result of Eq. 80 is known as the *Aharonov-Bohm Effect*.

We find that the overlap is a pure phase factor which, in general, is different from one. Notice that, although the wave function is always defined up to a

constant arbitrary phase factor, *phase changes are physical effects*. For some special choices of  $\Phi$  the wave function becomes single valued. They correspond to the choice

$$\frac{e}{\hbar c}\Phi = 2\pi n \quad (81)$$

where  $n$  is an arbitrary integer. This requirement amounts to a quantization condition for the flux  $\Phi$ , *i. e.*

$$\Phi = n \left( \frac{\hbar c}{e} \right) \equiv n\Phi_0 \quad (82)$$

where  $\Phi_0$  is the *flux quantum*,  $\Phi_0 = \frac{\hbar c}{e}$ .

## 4.2 Non-Abelian Gauge Invariance

Let us now consider systems which have a non-abelian global symmetry. This means that the field  $\phi$  transforms like some representation of a Lie group  $G$ , *i. e.*

$$\phi'_a(x) = U_{ab}\phi_b(x) \quad (83)$$

where  $U$  is a matrix which represents the action of a group element. The local Lagrangian density

$$\mathcal{L} = \partial_\mu \phi_a^* \partial^\mu \phi^a - V(|\phi|^2) \quad (84)$$

is invariant under *global* transformations. Suppose now that we want to promote this *global* symmetry to a *local* one. While the potential term  $V(|\phi|^2)$  is invariant even under local transformations  $U(x)$ , the first term is not.

Indeed, the gradient of  $\phi$  does not transform properly (*i. e.* covariantly) under  $G$ , *i. e.*

$$\begin{aligned} \partial_\mu \phi'(x) &= \partial_\mu [U(x)\phi(x)] = (\partial_\mu U(x))\phi(x) + U(x)\partial_\mu \phi(x) \\ &= U(x)[\partial_\mu \phi(x) + U^{-1}(x)\partial_\mu U(x)\phi(x)] \end{aligned} \quad (85)$$

Hence  $\partial_\mu \phi$  does not transform like  $\phi$  itself. We can now follow the same approach that we used in the abelian case and *define* a *covariant derivative* operator  $D_\mu$  which should have the property that  $D_\mu \phi$  should transform like  $\phi$ , *i. e.*

$$(D_\mu \phi(x))' = U(x)(D_\mu \phi(x)) \quad (86)$$

It is clear that  $D_\mu$  is both a differential operator as well as a matrix acting on the field  $\phi$ . Thus,  $D_\mu$  depends on the representation that was chosen. We can now proceed in analogy with electrodynamics and guess that the covariant derivative  $D_\mu$  should be of the form

$$D_\mu = \partial_\mu - igA_\mu(x) \quad (87)$$

where  $g$  is a coupling constant,  $I$  is the identity matrix and  $A_\mu$  is a matrix-valued vector field. If  $\phi$  is an  $N$ -component vector,  $A_\mu(x)$  is an  $N \times N$  matrix which



can be expanded in the basis of group generators  $\lambda_{ab}^k$  ( $k = 1, \dots, D(G); a, b = 1, \dots, N$ )

$$(A(x))_{ab} = A_\mu^k(x) \lambda_{ab}^k \quad (88)$$

Thus, the vector field  $A_\mu(x)$  is parametrized by the  $D(G)$ -component 4-vectors  $A_\mu^k(x)$ . We will choose the transformation properties of  $A_\mu(x)$  in such a way that  $D_\mu\phi$  transforms covariantly under gauge transformations. Namely,

$$\begin{aligned} D'_\mu\phi(x)' &\equiv D'_\mu(U(x)\phi(x)) = (\partial_\mu - igA'_\mu(x))(U\phi(x)) \\ &= U(x)[\partial_\mu\phi(x) + U^{-1}(x)\partial_\mu U(x)\phi(x) - igU^{-1}(x)A'_\mu(x)U(x)\phi(x)] \\ &\equiv U(x) D_\mu\phi(x) \end{aligned} \quad (89)$$

Hence, it is sufficient to require that

$$U^{-1}(x)igA'_\mu(x)U(x) = igA_\mu(x) + U^{-1}(x)\partial_\mu U(x) \quad (90)$$

Or, equivalently, that  $A_\mu$  should transform as follows

$$A'_\mu(x) = U(x)A_\mu(x)U^{-1}(x) - \frac{i}{g}(\partial_\mu U(x))U^{-1}(x) \quad (91)$$

Since the matrices  $U(x)$  are unitary and invertible, we have

$$U^{-1}(x)U(x) = I \quad (92)$$

which allows us also to write  $A'_\mu$  in the form

$$A'_\mu(x) = U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g}U(x)(\partial_\mu U^{-1}(x)) \quad (93)$$

In the case of an abelian symmetry group, such as  $(U(1))$ , then  $U(x) = e^{i\theta(x)}$  and  $A_\mu(x)$  is a real number-valued vector field. It is easy to check that, in this case,  $A_\mu$  transforms as follows

$$\begin{aligned} A'_\mu(x) &= e^{i\theta(x)}A_\mu(x)e^{-i\theta(x)} + \frac{i}{g}e^{i\theta(x)}\partial_\mu(e^{-i\theta(x)}) \\ &\equiv A_\mu(x) + \frac{1}{g}\partial_\mu\theta(x) \end{aligned} \quad (94)$$

which is the correct form for an abelian gauge transformation such as in Maxwell's electrodynamics.

Under an infinitesimal transformation  $U(x)$

$$(U(x))_{ab} = [\exp(i\lambda^k\theta^k(x))]_{ab} \cong \delta_{ab} + i\lambda_{ab}^k\theta^k(x) \quad (95)$$

the scalar field  $\phi(x)$  transforms like

$$\delta\phi_a(x) \cong i\lambda_{ab}^k\phi_b(x)\theta^k(x) \quad (96)$$

while the vector field  $A^k{}_\mu$  transforms like

$$\delta A^k{}_\mu(x) \cong i f^{ksj} A^j{}_\mu(x) \theta^s(x) + \frac{1}{g} \partial_\mu \theta^k(x) \quad (97)$$

Thus,  $A^k{}_\mu(x)$  transforms like a vector in the *adjoint representation* of  $G$  since, in that representation, the matrix elements of the generators are the group structure constants  $f^{ksj}$ . Notice that  $A^k{}_\mu$  is *always* in the adjoint representation, independently of the representation in which  $\phi(x)$  happens to be in.

From the discussion given above, it is clear that the field  $A_\mu(x)$  can be interpreted as a generalization of the vector potential of electromagnetism. Furthermore,  $A_\mu$  provides for a natural *connection* which tell us how does the “internal coordinate system”, in reference to which the field  $\phi(x)$  is defined, change from one point  $x_\mu$  to a neighboring point  $x_\mu + dx_\mu$ . In particular, the configurations  $\phi^a(x)$  which are solutions of the geodesic equation

$$D_\mu^{ab} \phi_b(x) = 0 \quad (98)$$

correspond to the *parallel transport* of  $\phi$  from some point  $x$  to some point  $y$ . In fact, this equation can be written in the form

$$\partial_\mu \phi_a(x) = ig A^k{}_\mu(x) \lambda_{ab}^k \phi_b(x) \quad (99)$$

This linear partial differential equation can be solved as follows. Let  $x_\mu$  and  $y_\mu$  be two arbitrary points in space-time and  $\Gamma(x, y)$  a fixed path with endpoints at  $x$  and  $y$ . This path is parametrized by a mapping  $z_\mu$  from the real interval  $[0, 1]$  to Minkowski space  $M$  (or any other space for the present matter),  $z_\mu : [0, 1] \rightarrow M$ , of the form

$$z_\mu = z_\mu(t), \quad t \in [0, 1] \quad (100)$$

with the boundary conditions

$$z_\mu(0) = x_\mu \quad \text{and} \quad z_\mu(1) = y_\mu \quad (101)$$

By integrating the geodesic equation along the path  $\Gamma$  we get

$$\int_{\Gamma(x,y)} dz_\mu \frac{\partial \phi_a(z)}{\partial z_\mu} = ig \int_{\Gamma(x,y)} dz_\mu A^{\mu}_{ab}(z) \phi_b(z) \quad (102)$$

Hence, we get the integral equation

$$\phi(y) = \phi(x) + ig \int_{\Gamma(x,y)} dz_\mu A^\mu(z) \phi(z) \quad (103)$$

where I have omitted all indices. In terms of the parametrization  $z_\mu(t)$  of the path  $\Gamma(x, y)$  we can write

$$\phi(y) = \phi(x) + ig \int_0^1 dt \frac{dz_\mu}{dt}(t) A^\mu(z(t)) \phi(z(t)) \quad (104)$$

We will solve this equation by means of an iterative procedure. (We will use a similar approach for the study of the evolution operator in QFT). By substituting the l.h.s. of this equation into its r. h. s. we get the series

$$\begin{aligned}
\phi(y) &= \phi(x) + ig \int_0^1 dt \frac{dz_\mu}{dt}(t) A^\mu(z(t)) \phi(x) + \\
&+ (ig)^2 \int_0^1 dt_1 \int_0^{t_1} dt_2 \frac{dz_{\mu_1}}{dt_1}(t_1) \frac{dz_{\mu_2}}{dt_2}(t_2) A^{\mu_1}(z(t_1)) A^{\mu_2}(z(t_2)) \phi(x) \\
&+ \dots + (ig)^n \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \prod_{j=1}^n \left( \frac{dz_{\mu_j}}{dt_j}(t_j); A^{\mu_j}(z(t_j)) \right) \phi(x) \\
&+ \dots
\end{aligned} \tag{105}$$

(keep in mind that the  $A^\mu$ 's are matrices and are ordered from left to right!). The nested integrals  $I_n$  can be written in the form

$$\begin{aligned}
I_n &= (ig)^n \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n F(t_1) \dots F(t_n) \\
&\equiv \frac{(ig)^n}{n!} \hat{P} \left[ \left( \int_0^1 dt F(t) \right)^n \right]
\end{aligned} \tag{106}$$

where the  $F$ 's are matrices and the operator  $\hat{P}$  means the (path) ordered product of the objects sitting to its right. If we *formally* define the exponential of an operator to be equal to its power series expansion, *i. e.*

$$e^A \equiv \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{107}$$

where  $A$  is some arbitrary matrix, we see that the geodesic equation has the formal solution

$$\phi(y) = \hat{P} \left[ \exp \left( +ig \int_0^1 dt \frac{dz_\mu}{dt}(t) A^\mu(z(t)) \right) \right] \phi(x) \tag{108}$$

or, what is the same

$$\phi(y) = \hat{P} \left[ \exp \left( ig \int_{\Gamma(x,y)} dz_\mu A^\mu(z) \right) \right] \phi(x) \tag{109}$$

Thus,  $\phi(y)$  is given by an operator, the *path-ordered* exponential of the line integral of the vector potential  $A^\mu$ , acting on  $\phi(x)$ . Also, by expanding the exponential in a power series, it is easy to check that, under an arbitrary local

gauge transformation  $U(z)$ , the path ordered exponential transforms like

$$\begin{aligned} \hat{P} \left[ \exp \left( ig \int_{\Gamma(x,y)} dz_\mu A'^\mu(t) \right) \right] \\ \equiv U(y) \hat{P} \left[ \exp \left( ig \int_{\Gamma(x,y)} dz_\mu A^\mu(z) \right) \right] U^{-1}(x) \end{aligned} \quad (110)$$

In particular we can consider the case of a *closed path*  $\Gamma(x, x)$  (where  $x$  is an *arbitrary* point on  $\Gamma$ ). The path-ordered exponential  $\hat{W}_{\Gamma(x,x)}$  is

$$\hat{W}_{\Gamma(x,x)} = \hat{P} \left[ \exp \left( ig \int_{\Gamma(x,x)} dz_\mu A^\mu(z) \right) \right] \quad (111)$$

is *not* gauge invariant since, under a gauge transformation it transforms like

$$\begin{aligned} \hat{W}'_{\Gamma(x,x)} &= \hat{P} \left[ \exp \left( ig \int_{\Gamma(x,x)} dz_\mu A'^\mu(z) \right) \right] \\ &= U(x) \hat{P} \left[ \exp \left( ig \int_{\Gamma(x,x)} dz_\mu A^\mu(t) \right) \right] U^{-1}(x) \end{aligned} \quad (112)$$

Therefore  $\hat{W}_{\Gamma(x,x)}$  transforms like a *group element*, *i. e.*

$$\hat{W}_{\Gamma(x,x)} = U(x) \hat{W}_{\Gamma(x,x)} U^{-1}(x) \quad (113)$$

However, the *trace* of  $\hat{W}_{\Gamma(x,x)}$ , which we denote by

$$W_\Gamma = \text{tr} \hat{W}_{\Gamma(x,x)} \equiv \text{tr} \hat{P} \left[ \exp \left( ig \int_{\Gamma(x,x)} dz_\mu A^\mu(z) \right) \right] \quad (114)$$

not only *is gauge-invariant* but it is also independent of the choice of the point  $x$ .  $W_\Gamma$  is known as the *Wilson loop* and it plays a crucial role in gauge theories.

Let us now consider the case of a *small* closed path  $\Gamma(x, x)$ . If  $\Gamma$  is small, then the normal sectional area  $a(\Gamma)$  enclosed by it and its length  $L(\Gamma)$  are both infinitesimal. In this case, we can expand the exponential in powers and retain only the leading terms. We get

$$\hat{W}_{\Gamma(x,x)} \approx I + ig \hat{P} \oint_{\Gamma(x,x)} dz_\mu A^\mu(z) + \frac{(ig)^2}{2!} \hat{P} \left( \oint_{\Gamma(x,x)} dz_\mu A^\mu(z) \right)^2 + \dots \quad (115)$$

Stokes' theorem says that the first integral (*i. e.* the circulation of  $A_\mu$ ) is

$$\oint_\Gamma dz_\mu A^\mu(z) = \int \int_\Sigma dx^\mu \wedge dx^\nu \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \quad (116)$$

where  $\partial\Sigma = \Gamma$ , and

$$\frac{1}{2!} \hat{P} \left( \oint_{\Gamma} dt_{\mu} A^{\mu}(z) \right)^2 \equiv \frac{1}{2} \int \int_{\Sigma} dx^{\mu} \wedge dx^{\nu} (-[A_{\mu}, A_{\nu}]) + \dots \quad (117)$$

Therefore we get

$$\hat{W}_{\Gamma(x,x')} \approx I + \frac{ig}{2} \int \int_{\Sigma} dx^{\mu} \wedge dx^{\nu} F_{\mu\nu} + O(a(\Sigma)^2) \quad (118)$$

where  $F_{\mu\nu}$  is the field tensor defined by

$$F_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ig[A_{\mu}, A_{\nu}] = i[D_{\mu}, D_{\nu}] \quad (119)$$

Keep in mind that since the fields  $A_{\mu}$  are matrices, the field tensor  $F_{\mu\nu}$  is also a matrix. Notice also that now  $F_{\mu\nu}$  is *not* gauge invariant. Indeed, under a local gauge transformation  $U(x)$ ,  $F_{\mu\nu}$  transforms like

$$F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^{-1}(x) \quad (120)$$

This property follows from the transformation properties of  $A_{\mu}$ . However, although  $F_{\mu\nu}$  itself is not gauge invariant, other quantities such as  $\text{tr}F_{\mu\nu}F^{\mu\nu}$  are invariant.

Let us finally note the form of  $F_{\mu\nu}$  on components. By expanding  $F_{\mu\nu}$  in the basis of generators  $\lambda^k$

$$F_{\mu\nu} = F_{\mu\nu}^k \lambda^k \quad (121)$$

we find that the components,  $F_{\mu\nu}^k$  are

$$F_{\mu\nu}^k = \partial_{\mu} A_{\nu}^k - \partial_{\nu} A_{\mu}^k + g f^{k\ell m} A_{\mu}^{\ell} A_{\nu}^m \quad (122)$$

### 4.3 Gauge Invariance and Minimal Coupling

We are now in a position to give a general prescription for the coupling of matter and gauge fields. Since the issue here is local gauge invariance, this prescription is valid for both relativistic and non-relativistic systems.

So far, we have considered two cases: (a) fields which describe the dynamics of matter and (b) gauge fields which describe electromagnetism. In our description of Maxwell's electrodynamics we saw that, if the Lagrangian is required to respect local gauge invariance, then only conserved currents can couple to the gauge field. However, we have also seen that the presence of a global symmetry is a sufficient condition for the existence of a locally conserved current. This is not a necessary condition since a local symmetry also requires the existence of a conserved current.

We will now consider more general Lagrangians which will include both matter and gauge fields. In the last sections we saw that if a system with Lagrangian  $\mathcal{L}(\phi, \partial_{\mu}\phi)$  has a global symmetry  $\phi \rightarrow U\phi$ , then by replacing all derivatives by covariant derivatives we get a local (or gauge) symmetry. We will proceed with our general philosophy and write down gauge-invariant Lagrangians for systems which contain both matter and gauge fields. I will give a few explicit examples:

1. Quantum Electrodynamics (QED)

In this case we have a theory of electrons and photons. The electrons are described by *Dirac fields*  $\psi_\alpha(x)$  (the reason for this choice will become clear when we discuss the quantum theory and the Spin-Statistics theorem). Photons are described by a  $U(1)$  gauge field  $A_\mu$ . The Lagrangian for *free electrons* is just the *free Dirac Lagrangian*  $\mathcal{L}_{\text{matter}}$

$$\mathcal{L}_{\text{matter}}(\psi, \bar{\psi}) = \bar{\psi}(i\cancel{D} - m)\psi \quad (123)$$

The Lagrangian for the gauge field is the free Maxwell Lagrangian  $\mathcal{L}_{\text{gauge}}(A)$

$$\mathcal{L}_{\text{gauge}}(A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv -\frac{1}{4}F^2 \quad (124)$$

The prescription that we will adopt, which is known as *minimal coupling*, consists in requiring that the *total* Lagrangian be invariant under *local* gauge transformations. The free Dirac Lagrangian is invariant under the *global* phase transformation

$$\psi_\alpha(x) \rightarrow \psi'_\alpha(x) = e^{i\theta} \psi_\alpha(x) \quad (125)$$

(*i. e.* same phase factor for *all* the Dirac components) when  $\theta$  is a constant, arbitrary phase, but it is not invariant not under the *local phase* transformation

$$\psi_\alpha(x) \rightarrow \psi'_\alpha(x) = e^{i\theta(x)} \psi_\alpha(x) \quad (126)$$

As we saw before, the matter part of the Lagrangian can be made invariant under the local transformations

$$\begin{aligned} \psi_\alpha(x) \rightarrow \psi'_\alpha(x) &= e^{i\theta(x)} \psi_\alpha(x) \\ A_\mu(x) \rightarrow A'_\mu(x) &= A_\mu(x) + \frac{1}{e} \partial_\mu \theta(x) \end{aligned} \quad (127)$$

if the derivatives  $\partial_\mu \psi$  are replaced by *covariant* derivatives  $D_\mu$

$$D_\mu = \partial_\mu - ieA_\mu(x) \quad (128)$$

The total Lagrangian is now given by the sum of two terms

$$\mathcal{L} = \mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) + \mathcal{L}_{\text{gauge}}(A) \quad (129)$$

where  $\mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A)$  is the gauge-invariant extension of the Dirac Lagrangian, *i. e.*

$$\begin{aligned} \mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) &= \bar{\psi}(i\cancel{D} - m)\psi \\ &= \bar{\psi}(i\cancel{D} - m)\psi + e\bar{\psi}\gamma_\mu\psi A^\mu \end{aligned} \quad (130)$$

$\mathcal{L}_{\text{gauge}}(A)$  is the usual Maxwell term and  $\cancel{D}$  is a shorthand for  $D_\mu\gamma^\mu$ . Thus, the total Lagrangian for QED is

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\cancel{D} - m)\psi - \frac{1}{4}F^2 \quad (131)$$

Notice that now both matter and gauge fields are dynamical degrees of freedom.

The QED Lagrangian has a local gauge invariance. Hence, it also has a locally conserved current. In fact the argument that we used above to show that there are conserved (Noether) currents if there is a continuous global symmetry, is also applicable to gauge invariant Lagrangians. As a matter of fact, under an arbitrary infinitesimal gauge transformation

$$\delta\psi = i\theta\psi \quad \delta\bar{\psi} = -i\theta\bar{\psi} \quad \delta A_\mu = \frac{1}{e}\partial_\mu\theta \quad (132)$$

the QED Lagrangian remains invariant, *i. e.*  $\delta\mathcal{L} = 0$ . An arbitrary variation of  $\mathcal{L}$  is

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta\partial_\mu\psi + (\psi \leftrightarrow \bar{\psi}) + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta\partial_\mu A_\nu + \frac{\delta\mathcal{L}}{\delta A_\mu}\delta A_\mu \quad (133)$$

After using the equations of motion and the form of the gauge transformation,  $\delta\mathcal{L}$  can be written in the form

$$\delta\mathcal{L} = \partial_\mu[j^\mu(x)\theta(x)] - \frac{1}{e}F^{\mu\nu}(x)\partial_\mu\partial_\nu\theta(x) + \frac{\delta\mathcal{L}}{\delta A_\mu}\frac{1}{e}\partial_\mu\theta(x) \quad (134)$$

where  $j^\mu(x)$  is the *electron number current*

$$j^\mu = i\left(\frac{\partial\mathcal{L}}{\delta\partial_\mu\psi}\psi - \bar{\psi}\frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\psi}}\right) \quad (135)$$

The term  $F^{\mu\nu}\partial_\mu\partial_\nu\theta$  vanishes because of the antisymmetry of  $F$ . Hence we can write

$$\delta\mathcal{L} = \theta(x)\partial_\mu j^\mu(x) + \partial_\mu\theta(x)\left[j^\mu(x) + \frac{1}{e}\frac{\delta\mathcal{L}}{\delta A_\mu(x)}\right] \quad (136)$$

The first term tell us that the *number current*  $j^\mu(x)$  *conserved*. Let us define the *charge* ( or *gauge*) current  $J^\mu(x)$  by the relation

$$J^\mu(x) \equiv \frac{\delta\mathcal{L}}{\delta A_\mu(x)} \quad (137)$$

which is the current that enters in the Equation of Motion for the gauge field  $A_\mu$ , *i. e.* Maxwell's equations.

The vanishing of the second term of Eq. 136 tells us that the charge current and the number current are related by

$$J_\mu(x) = -ej_\mu(x) = -e\bar{\psi}\gamma_\mu\psi \quad (138)$$

This relation tells us that since  $j^\mu(x)$  is locally conserved, then the global conservation of particle number  $Q_0$

$$Q_0 = \int d^3x j_0(x) \equiv \int d^3x \psi^\dagger(x)\psi(x) \quad (139)$$

implies the global conservation of the electric charge  $Q$

$$Q \equiv -eQ_0 = -e \int d^3x \psi^\dagger(x)\psi(x) \quad (140)$$

This property justifies the interpretation of the *coupling constant*  $e$  as the *electric charge*. In particular the gauge transformation of Eq. 127, tells us that the matter field  $\psi(x)$  represents excitations which carry the unit of charge,  $\pm e$ . From this point of view, the electric charge can be regarded as a *quantum number*. This point of view becomes very useful in the quantum theory in the strong coupling limit. In this case, under special circumstances, the excitations may acquire unusual quantum numbers. This is *not* the case of Quantum Electrodynamics, but it is the case of a number of theories in one-space dimensions (with applications in Condensed Matter systems such as polyacetylene or the two-dimensional electron gas in high magnetic fields (*i. e.* the fractional quantum Hall effect), or in gauge theories with magnetic monopoles).

## 2. Quantum Chromodynamics (QCD)

QCD is gauge field theory of strong interactions for hadron physics. In this theory the elementary constituents of the hadrons, the *quarks*, are represented by the Dirac field  $\psi_\alpha^i(x)$ . The theory also contains a set of gauge fields  $A_\mu^a(x)$ , which represent the *gluons*. The quark fields have both *Dirac* indices  $\alpha = 1, \dots, 4$  and *color* indices  $i = 1, \dots, N_c$ ;  $N_c$  is called the number of colors. In QCD the quarks also carry flavor quantum numbers which we will ignore for the sake of simplicity. (In the Standard Model, and in QCD, there are  $N_f = 6$  flavors of quarks.)

Quarks are assumed to transform like the *fundamental representation* of the gauge (or color) group  $G$ , say  $SU(N_c)$ . The theory is invariant under the group of gauge transformations. In QCD, the color group is  $SU(3)$  and so  $N_c = 3$ . The color symmetry is a non-abelian gauge symmetry. The gauge field  $A_\mu$  is needed in order to enforce the local invariance. In components we get  $A_\mu = A_\mu^a \lambda^a$ , where  $\lambda^a$  are the generators of  $SU(N_c)$ . Thus,  $a = 1, \dots, D(SU(N_c))$ , and  $D(SU(N_c)) = N_c^2 - 1$ . Thus,  $A_\mu$  is a  $N_c^2 - 1$  dimensional vector in the adjoint representation of  $G$ . For  $SU(3)$ ,  $N_c^2 - 1 = 8$  and there are eight generators.

The gauge-invariant matter term of the Lagrangian,  $\mathcal{L}_{\text{matter}}$  is

$$\mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) = \bar{\psi}(i\mathcal{D} - m)\psi \quad (141)$$

where  $\mathcal{D} = \not{\partial} - ig\not{A} \equiv \not{\partial} - igA^a \lambda^a$  is the covariant derivative. The gauge field term of the Lagrangian  $\mathcal{L}_{\text{gauge}}$  is

$$\mathcal{L}_{\text{gauge}}(A) = -\frac{1}{4}\text{tr}F_{\mu\nu}F^{\mu\nu} \equiv -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} \quad (142)$$

The total Lagrangian for QCD is  $\mathcal{L}_{\text{QCD}}$

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) + \mathcal{L}_{\text{gauge}}(A) \quad (143)$$



Can we define a *color charge*? Since the color group is non-abelian it has more than one generator. We showed before that there are as many conserved currents as generators are in the group. Now, in general, the group generators do not commute with each other. For instance, in  $SU(2)$  there is only *one* diagonal generator ( $J_3$ ) while in  $SU(3)$  there are only *two* diagonal generators. Can all the global *charges*  $Q^a$

$$Q^a \equiv \int d^3x \psi^\dagger(x) \lambda^a \psi(x) \quad (144)$$

be defined simultaneously? It is straightforward to show that the Poisson Brackets of any pair of charges are, in general, different from zero. We will see below, when we *quantize* the theory, that the charges  $Q^a$  obey the same commutation relations as the group generators themselves do. So, in the quantum theory, the only charges that can be assigned to *states* are precisely the same as the quantum numbers which label the representations. Thus, if the group is  $SU(2)$ , we can only give the values of the Casimir operator  $\vec{J}^2$  and of the projection  $J_3$ . Similar restrictions apply to the case of  $SU(3)$  and to other groups.

#### 4.4 Space-Time Symmetries and the Energy-Momentum Tensor

Thus far we have considered only the role of internal symmetries. We now turn the case of space-time symmetries.

There are three continuous space-time symmetries which will be important to us: A) Translation Invariance, B) Rotation Invariance and C) Homogeneity of Time. While rotation invariance is a special Lorentz transformation, space and time translations are examples of inhomogeneous Lorentz transformations (in the relativistic case) and of Galilean transformations (in the non-relativistic case). Inhomogeneous Lorentz transformations also form a group: the Poincaré group. Note that the transformations discussed above are particular cases of more general coordinate transformations. However, it is important to keep in mind that in most cases general coordinate transformations are not symmetries of an arbitrary system. They are the symmetries of General Relativity.

In what follows we are going to consider how does a system respond to infinitesimal coordinate transformations of the form

$$x'_\mu = x_\mu + \delta x_\mu \quad (145)$$

where  $\delta x_\mu$  may be a function of  $x$ . The fields themselves also change

$$\phi(x) \rightarrow \phi'(x') = \phi(x) + \delta\phi(x) + \partial_\mu \phi \delta x^\mu \quad (146)$$

where  $\delta\phi$  is the variation of  $\phi$  in the *absence* of a change of coordinates, *i. e.* a functional change. In this notation, a uniform infinitesimal translation by a constant vector  $a_\mu$  has  $\delta x_\mu = a_\mu$  and an infinitesimal rotation of the space axes is  $\delta x_0 = 0$  and  $\delta x_i = \epsilon_{ijk} \theta_j x_k$ .

In general, the action of the system is not invariant under arbitrary changes in both coordinates and fields. Indeed, under an arbitrary change of coordinates  $x_\mu \rightarrow x'_\mu(x_\mu)$ , the volume element  $d^4x$  is not invariant but it changes by a Jacobian factor of the form

$$d^4x' = d^4x J \quad (147)$$

where  $J$  is the Jacobian

$$J = \frac{\partial x'_1 \cdots \partial x'_4}{\partial x_1 \cdots \partial x_4} \equiv \left| \det \left( \frac{\partial x'_\mu}{\partial x_j} \right) \right| \quad (148)$$

For an infinitesimal transformation,  $x'_\mu = x_\mu + \delta x_\mu(x)$ , we get

$$\frac{\partial x'_\mu}{\partial x_\nu} = g_\mu^\nu + \partial^\nu \delta x_\mu \quad (149)$$

Since  $\delta x_\mu$  is small, the Jacobian can be approximated by

$$J = \left| \det \left( \frac{\partial x'_\mu}{\partial x_\nu} \right) \right| = \left| \det (g_\mu^\nu + \partial^\nu \delta x_\mu) \right| \approx 1 + \text{tr}(\partial^\nu \delta x_\mu) + O(\delta x^2) \quad (150)$$

Thus

$$J \approx 1 + \partial^\mu \delta x_\mu + O(\delta x^2) \quad (151)$$

The Lagrangian itself, in general, it is not invariant. For instance, it may be an explicit function of  $x$  (although this is not the usual case) or it may not be invariant under the given transformation of coordinates. Also, under a coordinate change the fields may also transform. Thus, in general,  $\delta \mathcal{L}$  does not vanish. The most general variation of  $\mathcal{L}$  is

$$\delta \mathcal{L} = \partial_\mu \mathcal{L} \delta x^\mu + \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \partial_\mu \phi \quad (152)$$

If  $\phi$  obeys the equations of motion, *i. e.*

$$\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \quad (153)$$

the general change  $\delta \mathcal{L}$  obeyed by the solutions of the equations of motion is

$$\delta \mathcal{L} = \delta x^\mu \partial_\mu \mathcal{L} + \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \phi \right) \quad (154)$$

The total change in the action is a sum of two terms

$$\delta S = \delta \int d^4x \mathcal{L} = \int \delta d^4x \mathcal{L} + \int d^4x \delta \mathcal{L} \quad (155)$$

where the change in the integration measure is due to the Jacobian factor, *i. e.*

$$\delta d^4x = d^4x \partial_\mu \delta x^\mu \quad (156)$$

Hence,  $\delta S$  is given by

$$\delta S = \int d^4x \left[ (\partial_\mu \delta x^\mu) \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} + \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \phi \right) \right] \quad (157)$$

Since the *total* variation of  $\phi$ ,  $\delta_T \phi$ , is the sum of the functional change of the fields plus the changes in the fields caused by the coordinate transformation, *i. e.*

$$\delta_T \phi \equiv \delta \phi + \partial_\mu \phi \delta x^\mu \quad (158)$$

we can write  $\delta S$  as a sum of two contributions: one due to change of coordinates and another one due to functional changes of the fields:

$$\delta S = \int d^4x \left[ (\partial_\mu \delta x^\mu) \mathcal{L} + \delta x^\mu \mathcal{L}_\mu + \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} (\delta_T \phi - \partial_\nu \phi \delta x^\nu) \right) \right] \quad (159)$$

Therefore, we get

$$\delta S = \int d^4x \left\{ \partial_\mu \left[ \left( g_\nu^\mu \mathcal{L} - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\nu \phi \right) \delta x^\nu \right] + \partial_\mu \left[ \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta_T \phi \right] \right\} \quad (160)$$

We have already encountered the 2nd term when we discussed the case of internal symmetries. The first term represents the change of the action  $S$  as a result of a change of coordinates. We will now consider a few explicit examples. To simplify matters we will consider the effects of coordinate transformations alone. For simplicity we will restrict our discussion to the case of the scalar field.

1. *Space-Time Translations:*

under a uniform infinitesimal translation  $\delta x_\mu = a_\mu$ , the field  $\phi$  does not change

$$\delta_T \phi = 0 \quad (161)$$

The change of the action is

$$\delta S = \int d^4x \partial_\mu \left( g^{\mu\nu} \mathcal{L} - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial^\nu \phi \right) a^\nu \quad (162)$$

For a system which is *isolated* and *translationally invariant* the action must not change under a redefinition of the origin of the coordinate system. Thus,  $\delta S = 0$ . Since  $a_\mu$  is *arbitrary*, it follows that the tensor  $T^{\mu\nu}$

$$T^{\mu\nu} \equiv -g^{\mu\nu} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial^\nu \phi \quad (163)$$

is conserved, *i. e.*

$$\partial_\mu T^{\mu\nu} = 0 \quad (164)$$

The tensor  $T^{\mu\nu}$  is known as the *energy-momentum tensor*. The reason for this name is as follows. Given that  $T^{\mu\nu}$  is conserved, we can define the 4-vector  $P^\nu$

$$P^\nu = \int d^3x T^{0\nu}(\vec{x}, x_0) \quad (165)$$

which is a *constant of motion*, since  $T^{\mu\nu}$  is a locally conserved current. In particular,  $P^0$  is given by

$$P^0 = \int d^3x T^{00}(\vec{x}, x_0) \equiv \int d^3x \left[ -\mathcal{L} + \frac{\delta\mathcal{L}}{\delta\partial_0\phi} \partial^0\phi \right] \quad (166)$$

But  $\frac{\delta\mathcal{L}}{\delta\partial_0\phi}$  is just the canonical momentum  $\Pi(x)$ , *i. e.*

$$\Pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\phi} \quad (167)$$

We easily recognize that the quantity in brackets in the definition of  $P^0$  is just the Hamiltonian density  $\mathcal{H}$

$$\mathcal{H} = \Pi \partial^0\phi - \mathcal{L} \quad (168)$$

Therefore, we find that the time component of  $P^\nu$  is just the *total energy* of the system

$$P^0 = \int d^3x \mathcal{H} \quad (169)$$

The space components of  $P^\nu$  are

$$P^j = \int d^3x T^{0j} = \int d^3x \left[ -g^{0j}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta\partial_j\phi} \partial^j\phi \right] \quad (170)$$

Thus, since  $g^{0j} = 0$ , we get

$$P^j = \int d^3x \Pi(x) \partial^j\phi(x) \quad (171)$$

which is identified with the *total linear momentum* since (a) it is a constant of motion, and (b) it is the generator of infinitesimal space translations. For the same reasons we will denote the component  $T^{0j}(x)$  with the *linear momentum density*  $\mathcal{P}^j(x)$ . It is important to stress that the *canonical momentum*  $\Pi(x)$  and the *total momentum*  $\mathcal{P}^j$  are completely different physical quantities. While the canonical momentum is a field which is canonically conjugate to the field  $\phi$ , the total momentum is the linear momentum stored in the field, *i. e.* the linear momentum of the center of mass.

## 2. Rotations:

If the action is invariant under global infinitesimal Lorentz transformations (and spacial rotations in particular)

$$\delta x_\mu = \omega_\mu^\nu x_\nu \quad (172)$$

where  $\omega^{\mu\nu}$  is infinitesimal, and antisymmetric, the variation of the action is zero,  $\delta S = 0$ . If  $\phi$  is a *scalar field*, then  $\delta_T\phi$  is also zero. But this is not the case for spinors or vectors. Because of this transformation property

of the field, the angular momentum tensor that we define below will be missing a contribution which represents the spin of the field. Here we will consider the case of scalars.

For a scalar field  $\phi$ ,  $\delta_T \phi = 0$  and thus

$$\delta S = 0 = \int d^4x \partial_\mu \left[ \left( g^{\mu\nu} \mathcal{L} - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial^\nu \phi \right) \omega^{\nu\rho} x_\rho \right] \quad (173)$$

Since  $\omega^{\nu\rho}$  is constant and arbitrary, the magnitude in brackets must also be a conserved current. Let us define the tensor  $M^{\mu\nu\rho}$  by

$$M^{\mu\nu\rho} \equiv T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu \quad (174)$$

in terms of which, the quantity in brackets in Eq. 173 is equal to  $\frac{1}{2} \omega_{\nu\rho} M^{\mu\nu\rho}$ . Thus, for arbitrary constant  $\omega$ , we find that the tensor  $M^{\mu\nu\rho}$  is *conserved*, *i. e.*

$$\partial_\mu M^{\mu\nu\rho} = 0 \quad (175)$$

In particular, the transformation

$$\begin{aligned} \delta x_0 &= 0 \\ \delta x_j &= \omega_{jk} x_k \end{aligned} \quad (176)$$

represents an infinitesimal rotation of the spacial axes with

$$\omega_{jk} = \epsilon_{jkl} \theta_l \quad (177)$$

where  $\theta_l$  are three infinitesimal Euler angles. Thus, we suspect that  $M^{\mu\nu\rho}$  must be related with the total angular momentum. Indeed, the local conservation of the current  $M^{\mu\nu\rho}$  leads to the global conservation of the tensor quantity  $L^{\nu\rho}$

$$L^{\nu\rho} \equiv \int d^3x M^{0\nu\rho}(\vec{x}, x_0) \quad (178)$$

In particular, the space components of  $L_{\nu\rho}$  are

$$\begin{aligned} L_{jk} &= \int d^3x (T_{0j}(x) x_k - T_{0k}(x) x_j) \\ &= \int d^3x (P_j(x) x_j - P_k(x) x_j) \end{aligned} \quad (179)$$

If we denote by  $L_j$  the (pseudo) vector

$$L_j \equiv \frac{1}{2} \epsilon_{jkl} L_{kl} \quad (180)$$

we get

$$L_j \equiv \int d^3x \epsilon_{jkl} x_k P_l(x) \equiv \int d^3x \ell_j(x) \quad (181)$$

The vector  $L_j$  is the generator of infinitesimal rotations and is thus identified with the total *angular momentum*, whereas  $\ell_j(x)$  is the corresponding (spacial) angular momentum density.

As a side product, we find that if the angular momentum tensor  $M^{\mu\nu\lambda}$  has the form

$$M^{\mu\nu\lambda} = T^{\mu\nu} x^\lambda - T^{\mu\lambda} x^\nu \quad (182)$$

then, the conservation of  $T^{\mu\nu}$  and of  $M^{\mu\nu\lambda}$  together lead to the condition that the energy-momentum tensor should be a symmetric second rank tensor, *i. e.*

$$T_{\mu\nu} = T_{\nu\mu} \quad (183)$$

Thus, we conclude that the conservation of angular momentum requires that the energy momentum tensor  $T^{\mu\nu}$  for a *scalar field* be symmetric. The expression for  $T_{\mu\nu}$  that we have derived is not manifestly symmetric. However, if  $T_{\mu\nu}$  is conserved then the modified tensor  $\tilde{T}_{\mu\nu}$

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \partial_\lambda K^{\mu\nu\lambda} \quad (184)$$

is also conserved, provided the tensor  $K^{\mu\nu\lambda}$  is anti-symmetric. It is always possible to find such a tensor  $K^{\mu\nu\lambda}$  to make  $\tilde{T}_{\mu\nu}$  symmetric. In particular, for the Lagrangian density  $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \quad (185)$$

the energy-momentum tensor  $T_{\mu\nu}$  is

$$T_{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\nu \phi \equiv -g^{\mu\nu} \mathcal{L} + \partial_\mu \phi \partial_\nu \phi \quad (186)$$

which is symmetric. The energy-momentum vector is

$$P^\mu = \int d^3x (-g^{0\mu} \mathcal{L} + \partial^0 \phi \partial^\mu \phi) \quad (187)$$

Thus, we get

$$P^0 = \int d^3x (\Pi \partial_0 \phi - \mathcal{L}) = \int d^3x \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + V(\phi) \right] \quad (188)$$

is the total energy and the linear momentum  $\vec{P}$  is

$$\vec{P} = \int d^3x \Pi(x) \vec{\nabla} \phi(x) \quad (189)$$

## 4.5 The energy-momentum tensor for the electromagnetic field

For the case of the Maxwell field  $A_\mu$ , a straightforward application of these methods yields an energy-momentum tensor  $T^{\mu\nu}$  of the form

$$\begin{aligned} T^{\mu\nu} &= -g^{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta\partial_\nu A_\lambda}\partial^\mu A_\lambda \\ &= \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\nu\lambda}\partial^\mu A_\lambda \end{aligned} \quad (190)$$

which obeys  $\partial_\mu T^{\mu\nu} = 0$  and is conserved, but it is not gauge invariant. We can construct a gauge-invariant and conserved energy momentum tensor by exploiting the ambiguity in the definition of  $T^{\mu\nu}$ . Thus, if we define  $K_{\mu\nu\lambda} = F_{\nu\lambda}A_\mu$ , which is anti-symmetric in  $\nu$  and  $\lambda$ , we can construct the gauge invariant and conserved energy momentum-tensor

$$\tilde{T}^{\mu\nu} = \frac{1}{4}g^{\mu\nu}F^2 - F_\lambda^\nu F^{\mu\lambda} \quad (191)$$

which is also conserved. From here we find that

$$P^\mu = \int_{x_0 \text{ fixed}} d^3x \tilde{T}^{\mu 0} \quad (192)$$

is a constant of motion. Thus, we identify

$$P^0 = \int_{x_0 \text{ fixed}} d^3x \tilde{T}^{00} = \int_{x_0 \text{ fixed}} d^3x \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \quad (193)$$

as the Hamiltonian, and

$$P^i = \int_{x_0 \text{ fixed}} d^3x \tilde{T}^{i0} = \int_{x_0 \text{ fixed}} d^3x (\vec{E} \times \vec{B})_i \quad (194)$$

is the linear momentum or Poynting vector.

## 4.6 The Energy-Momentum Tensor and changes in the geometry

The energy momentum tensor  $T^{\mu\nu}$  appears in classical field theory as a result of the translation invariance (both in space and time) of the physical system. We have seen in the previous section that for a scalar field  $T^{\mu\nu}$  is a symmetric tensor as a consequence of the conservation of angular momentum. The definition that we found does not require that  $T^{\mu\nu}$  should have any definite symmetry. However we found that it is always possible to modify  $T^{\mu\nu}$ , by adding a suitably chosen antisymmetric conserved tensor, and to find a symmetric version of  $T^{\mu\nu}$ . Given this fact, it is natural to ask if there is a way to define the energy momentum tensor in a manner that it is always symmetric. This issue becomes important

if we want to consider theories for systems which contain fields which are not scalars.

It turns out that it is possible to regard  $T^{\mu\nu}$  as the change in the *action* due to a change of the *geometry* in which the system lives. From classical physics, we are familiar with the fact that when a body is distorted in some manner, in general its energy increases since we have to perform some work against the body in order to deform it. A deformation of a body is a change of the *geometry* in which its component parts evolve. Examples of such changes of geometry are shear distortions, dilatations, bendings and twists. On the other hand, there are changes that do not cost any energy since they are symmetry operations. Examples of symmetry operations are translations and rotations. These symmetry operations can be viewed as simple changes in the *coordinates* of the parts of the body which do not change its geometrical properties, *i. e.* the distances between points. Thus, coordinate transformations do not alter the energy of the system. The same type of arguments apply to *any* dynamical system. In the most general case, we have to consider transformations which leave the *action* invariant. This leads us to consider how changes in the geometry affect the action of a dynamical system.

The information about the geometry in which a system evolves is encoded in the *metric tensor* of the space. The metric tensor is a *symmetric* tensor which tells us how to measure the distance  $|ds|$  between a pair of *nearby* points  $x$  and  $x + dx$ , namely

$$|ds|^2 = g^{\mu\nu}(x) dx_\mu dx_\nu \quad (195)$$

Under an arbitrary local *change of coordinates*  $x_\mu \rightarrow x_\mu + \delta x_\mu$ , the metric tensor changes by

$$g'_{\mu\nu}(x') = g_{\lambda\rho}(x) \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} \quad (196)$$

which, for an infinitesimal change, means that the functional change in the metric tensor  $\delta g_{\mu\nu}$  is

$$\delta g_{\mu\nu} = -\frac{1}{2} (g_{\mu\lambda} \partial^\nu \delta x_\lambda + g_{\lambda\nu} \partial^\mu \delta x_\lambda + \partial_\lambda g_{\mu\nu} \delta x^\lambda) \quad (197)$$

The volume element, invariant under coordinate transformations, is  $d^4x \sqrt{g}$ , where  $g$  is the determinant of the metric tensor.

Coordinate transformations change the metric of space-time but do not change the action. *Physical* or geometric changes, are changes in the metric tensor which are not due to coordinate transformations. For a system in a space with metric tensor  $g^{\mu\nu}(x)$ , not necessarily the Minkowski ( or Euclidean) metric, the *change* of the action is linear in an the infinitesimal change in the metric  $\delta g^{\mu\nu}(x)$  (*i. e.* “Hooke’s Law”). Thus, we can write the change of the action due to an arbitrary infinitesimal change of the metric in the form

$$\delta S = \int d^4x \sqrt{g} T^{\mu\nu}(x) \delta g_{\mu\nu}(x) \quad (198)$$



We will identify the proportionality constant, the tensor  $T^{\mu\nu}$ , with the energy-momentum tensor. This definition implies that  $T^{\mu\nu}$  can be regarded as the derivative of the action with respect to the metric

$$T^{\mu\nu}(x) \equiv \frac{\delta S}{\delta g_{\mu\nu}(x)} \quad (199)$$

Since the metric tensor is symmetric, this definition always yields a symmetric energy momentum tensor.

In order to prove that this definition of the energy momentum tensor agrees with the one we obtained before (which was not unique!) we have to prove that the  $T^{\mu\nu}$  that we have just defined is a *conserved* current for coordinate transformations. Under an arbitrary local change of coordinates, which leave the distance  $|ds|$  unchanged, the metric tensor changes by the  $\delta g_{\mu\nu}$  given above. The change of the action  $\delta S$  must be zero for this case. We see that if we substitute the expression for  $\delta g_{\mu\nu}$  in  $\delta S$ , then an integration by parts will yield a conservation law. Indeed, for the particular case of a flat metric (such as Minkowski or Euclidean) the change is

$$\delta S = -\frac{1}{2} \int d^4x T^{\mu\nu}(x) (g_{\mu\lambda} \partial^\nu \delta x_\lambda + g_{\lambda\nu} \partial^\mu \delta x_\lambda) \quad (200)$$

since the Jacobian and the metric are constant. Thus, if  $\delta S$  is to vanish for an arbitrary change  $\delta x$ , the tensor  $T^{\mu\nu}(x)$  has to be a locally conserved current, *i. e.*  $\partial_\mu T^{\mu\nu}(x) = 0$ . This definition can be extended to the case of more general spaces. Finally, let us remark that this definition of the energy-momentum tensor also allows us to identify the spacial components of  $T^{\mu\nu}$  with the stress-energy tensor of the system. Indeed, for a change in geometry which is does not vary with time, the change of the action reduces to a change of the total energy of the system. Hence, the space components of the energy momentum tensor tell us how much does the total energy change for a specific deformation of the geometry. But this is precisely what the stress energy tensor is!